Analysis of an instability in stratified fluid flow

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Buoyancy driven fluid flow, Bangalore

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Acknowledgements

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Boussinesq equations

- Assume a two-dimensional, viscous, incompressible fluid flow
- Density perturbations only relevant when multiplied by g (acceleration due to gravity)
- Fluid is bounded by two solid walls at $z = 0$ and $z = -h$ with no-slip boundary condition (Couette type)
- Mean density profile linear in z

$$
\begin{aligned}
\partial_t u + u\partial_x u + w\partial_z u &= -\frac{1}{\rho_0}\,\partial_x p + \nu(\partial_{xx} u + \partial_{zz} u),\\
\partial_t w + u\partial_x w + w\partial_z w &= -\frac{1}{\rho_0}\,\partial_z p - g\frac{\rho}{\rho_0} + \nu(\partial_{xx} w + \partial_{zz} w),\\
\partial_x u + \partial_z w &= 0,\\
\partial_t \rho + u\partial_x \rho + w\partial_z \rho &= wN + \kappa(\partial_{xx}\rho + \partial_{zz}\rho)
\end{aligned}
$$

Boussinesq equations

Exact solution

$$
\begin{aligned}
u = U(z) &= \frac{zU_0}{h} + U_0, \\
w &= 0, \\
\rho &= 0
\end{aligned}
$$

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Linear stability

Linearise about shear flow

$$u = U(z) + \tilde{u}, \quad w = \tilde{w}, \quad \rho = \tilde{\rho}$$

Introduce a stream function ψ: $\tilde{u} = -\partial_z \psi$, $\tilde{w} = \partial_x \psi$
Hat's denote Fourier transform with horizontal wave number k

$$-\partial_{tz}\hat{\psi} - ikU\partial_z\hat{\psi} + ik\hat{\psi}\frac{U_0}{h} = -\frac{ik}{\rho_0}\hat{p} - \nu(-k^2\partial_z\hat{\psi} + \partial_{zzz}\hat{\psi}),$$

$$ik\partial_t\hat{\psi} + (ik)^2 U\hat{\psi} = -\frac{1}{\rho_0}\partial_z\hat{p} - g\frac{\hat{\rho}}{\rho_0} + ik\nu(-k^2\hat{\psi} + \partial_{zz}\hat{\psi}),$$

$$\partial_t\hat{\rho} + ikU\hat{\rho} = ik\hat{\psi}N + \kappa(-k^2\hat{\rho} + \partial_{zz}\hat{\rho})$$

Linear stability

Orr-Sommerfeld equation

Combine momentum equations for single function $\hat{\psi}$ coupled with density equation

$$\hat{\psi} = \partial_z \hat{\psi} = 0, \quad z = 0, -h$$

$$\hat{\rho} = 0, \quad z = 0, -h$$

$$U = \frac{zU_0}{h} + U_0$$

$$\partial_t(-\partial_{zz} + k^2)\hat{\psi} + ikU(-\partial_{zz} + k^2)\hat{\psi} = gik\frac{\hat{\rho}}{\rho_0} - \nu(-\partial_{zz} + k^2)^2\hat{\psi}$$

$$\partial_t \hat{\rho} + ikU\hat{\rho} = ik\hat{\psi}N + \kappa(-k^2\hat{\rho} + \partial_{zz}\hat{\rho})$$

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Proceed with eigenvalue analysis: $\hat{\psi} = e^{\lambda t} f(z)$ $\hat{\rho} = e^{\lambda t} g(z)$

$$\partial_t(-\partial_{zz} + k^2)\hat{\psi} + ikU(-\partial_{zz} + k^2)\hat{\psi} = gik\frac{\hat{\rho}}{\rho_0} - \nu(-\partial_{zz} + k^2)^2\hat{\psi}$$

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Proceed with eigenvalue analysis: $\hat{\psi} = e^{\lambda t} f(z)$ $\hat{\rho} = e^{\lambda t} g(z)$
Non-normal operator $\Rightarrow$ cannot preclude short time growth even when all eigenvalues are stable

$$\partial_t(-\partial_{zz} + k^2)\hat{\psi} + ikU(-\partial_{zz} + k^2)\hat{\psi} = gik\frac{\hat{\rho}}{\rho_0} - \nu(-\partial_{zz} + k^2)^2\hat{\psi}$$

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An alternative viewpoint

Caution: new ideas ahead!

- How much progress can be made without resorting to numerics?
- Step one: reformulate problem (H Segur)
- Step two: simplify the problem
- Step three: analyse the problem

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Disclaimer!

Reformulation

recall

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Reformulation

$\hat{w} - \hat{p} - \hat{\rho}$ equations

The two momentum equations are equivalent iff

$$(-\partial_{zz} + k^2)\hat{p} = 2\rho_0 (ik)^2 \hat{\psi} \frac{U_0}{h} - g\partial_z \rho$$

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To keep things clean, re-introduce $\hat{w} = ik\hat{\psi}$

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$$U = \frac{zU_0}{h} + U_0$$

Boundary conditions

$$\hat{w} = 0, \partial_z \hat{w} = 0, \quad z = 0, -h$$

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No boundary conditions for $\hat{p}$!

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General solution to second order ODE

$$\hat{p} = \rho_0 \int_{-h}^{0} G(z-y) \left(2ik\frac{U_0}{h}\hat{w} - g\frac{\partial_z \rho}{\rho_0} \right) dy + A(t)\rho_0 e^{-|k|z} + B(t)\rho_0 e^{|k|z}$$

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No boundary conditions to apply $\Rightarrow$ retain A and B as unknowns.

$$(-\partial_{zz} + k^2)\hat{p} = 2\rho_0 ik\frac{U_0}{h}\hat{w} - g\partial_z \rho$$

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$$\partial_t \begin{pmatrix} \hat{w} \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} -ikU & 0 \\ 0 & -ikU \end{pmatrix} \begin{pmatrix} \hat{w} \\ \hat{\rho} \end{pmatrix} + \mathcal{L} \begin{pmatrix} \hat{w} \\ \hat{\rho} \end{pmatrix}$$

A toy problem

Consider

$$\partial_t q = i f(x) q + L q,$$

where L is some operator (e.g. $L = \partial_{xx}$) posed on finite interval $(0, 1)$ with homogeneous BC's

- If eigenvalues of L are known, then perturbation theory can be used to find eigenvalues for $if(x) + L$
- Perturbation theory usually requires $f(x)$ be small
- Energy integral

$$\partial_t \int \frac{|q|^2}{2} dx = \mathrm{Re} \int q^* L q \, dx$$

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$$\partial_t \int \frac{|q|^2}{2} dx = \mathrm{Re} \int q^* L q \, dx$$

$$\|q\| = \|q_0\| \exp\left(\int_0^t \mathrm{Re} \frac{\int q^* L q \, dx}{\|q\|^2} \right)$$

where

$$\|q\|^2 = \int_0^1 |q|^2 dx$$

A toy problem

$$\partial_t q = if(x)q + Lq,$$
$$\Rightarrow \|q\| = \|q_0\| \exp\left(\int_0^t \mathrm{Re}\, \frac{\int q^* Lq\, dx}{\|q\|^2}\right)$$

- $\mathrm{Num}(L) = \left\{ z \in \mathbb{C} \,|\, z = \int \phi^* L\phi\, dx \;\; \forall \|\phi\| = 1 \right\}$
- If $L\phi + if(x)\phi = \lambda\phi$ then $\lambda \in \mathrm{Num}(L)$
- $\mathrm{Num}(P + Q) \subseteq \mathrm{Num}(P) + \mathrm{Num}(Q)$
- If $\mathrm{Re}\,\mathrm{Num}(L) < 0$ then q is asymptotically stable $\Rightarrow$ all solutions decay to zero
- $\|q\| \leq \|q_0\| e^{Mt}$, $M = \sup\{\mathrm{Re}\,\mathrm{Num}(L)\}$
- If all eigenvalues of $L + if(x)$ are stable, then M gives the short time growth rate and we can estimate greatest possible algebraic growth rate
- Long time behavior is dictated by eigenvalues

A toy problem

Moral of the story...

Consider

$$\partial_t q = if(x)q + Lq,$$

where L is some operator (e.g. $L = \partial_{xx}$) posed on finite interval $(0,1)$ with homogeneous BC's

$$\|q\| = \|q_0\| \exp\left(\int_0^t \mathrm{Re}\, \frac{\int q^* Lq\, dx}{\|q\|^2}\right)$$

- $$\partial_t \begin{pmatrix} \hat{w} \\ \hat{\rho} \end{pmatrix} = \begin{pmatrix} -ikU & 0 \\ 0 & -ikU \end{pmatrix} \begin{pmatrix} \hat{w} \\ \hat{\rho} \end{pmatrix} + \mathcal{L} \begin{pmatrix} \hat{w} \\ \hat{\rho} \end{pmatrix}$$

  is the system equivalent of the toy problem and all the claims follow.

The effective boundary-value problem

$$\hat{w} = \partial_z \hat{w} = \hat{\rho} = 0, \quad z = 0, -h$$

$$\partial_t \hat{w} = -\partial_z \int_{-h}^{0} G(z-y) \left(2ik \frac{U_0}{h} \hat{w} - g \frac{\partial_z \rho}{\rho_0} \right) dy - A|k|e^{-|k|z} + B|k|e^{|k|z} - g \frac{\hat{\rho}}{\rho_0} + \nu(-k^2 \hat{w} + \partial_{zz} \hat{w}),$$

$$\partial_t \hat{\rho} = \hat{w} N + \kappa(-k^2 \hat{\rho} + \partial_{zz} \hat{\rho})$$

- We now seek as much as information about the BVP as possible.

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- We now seek as much as information about the BVP as possible. Claim: this problem can be solved explicitly!
- New method to solve boundary value problems: Unified Transform Method (UTM)
- Encompasses the standard methods.
- It is more general than the standard methods: UTM handles non self-adjoint operators (*etc.*)
- Readily obtain eigenvalues, estimate asymptotic behavior, compute numerical range.
- Efficient evaluation of the solution: contour parametrization leading to stable numerical methods, steepest descent, the residue theorem, *etc.*

Unified Transform Method

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- Developed by Fokas and collaborators (see SIAM Review Deconinck, Trogdon & Vasan 2014)
- Recently extended to problems with nonlocal operators (Vasan 2016), systems of equations (Deconinck et al 2017), interface problems (Sheils, Pelloni and others)
- Solution given as

$$\int_{\Gamma_k} \frac{e^{ikx - \omega_j t}}{\Delta(k)} \hat{q}_0(k)\, dk + \int_{\Gamma_\nu} \frac{e^{ikx - \omega_j t}}{\Delta(k)} \hat{q}_0(\nu)\, dk$$

  with $\Gamma_{\nu,k}$ as contours in complex plane. ω_j is dispersion relation
- $\Delta(k_m) = 0$ give rise to eigenfunctions and $\omega_j(k_m)$ is the eigenvalue

Concluding remarks

- Fluid shear problems in rectangular domains give rise to

$$\partial_t \bar{s} = -ikU\bar{s} + \mathcal{K}_U \bar{s} + \mathcal{L}\bar{s}$$

  $\bar{s}$ are the fluid variables of interest, $\mathcal{K}_U$ is a compact operator and $\mathcal{L}$ is an operator amenable to UTM.

- Using perturbation theory and numerical range we can estimate growth rates in terms of $\mathcal{L}$ and shear
- Currently we're studying interfacial waves with density and viscosity jump with a background time-dependent shear profile

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