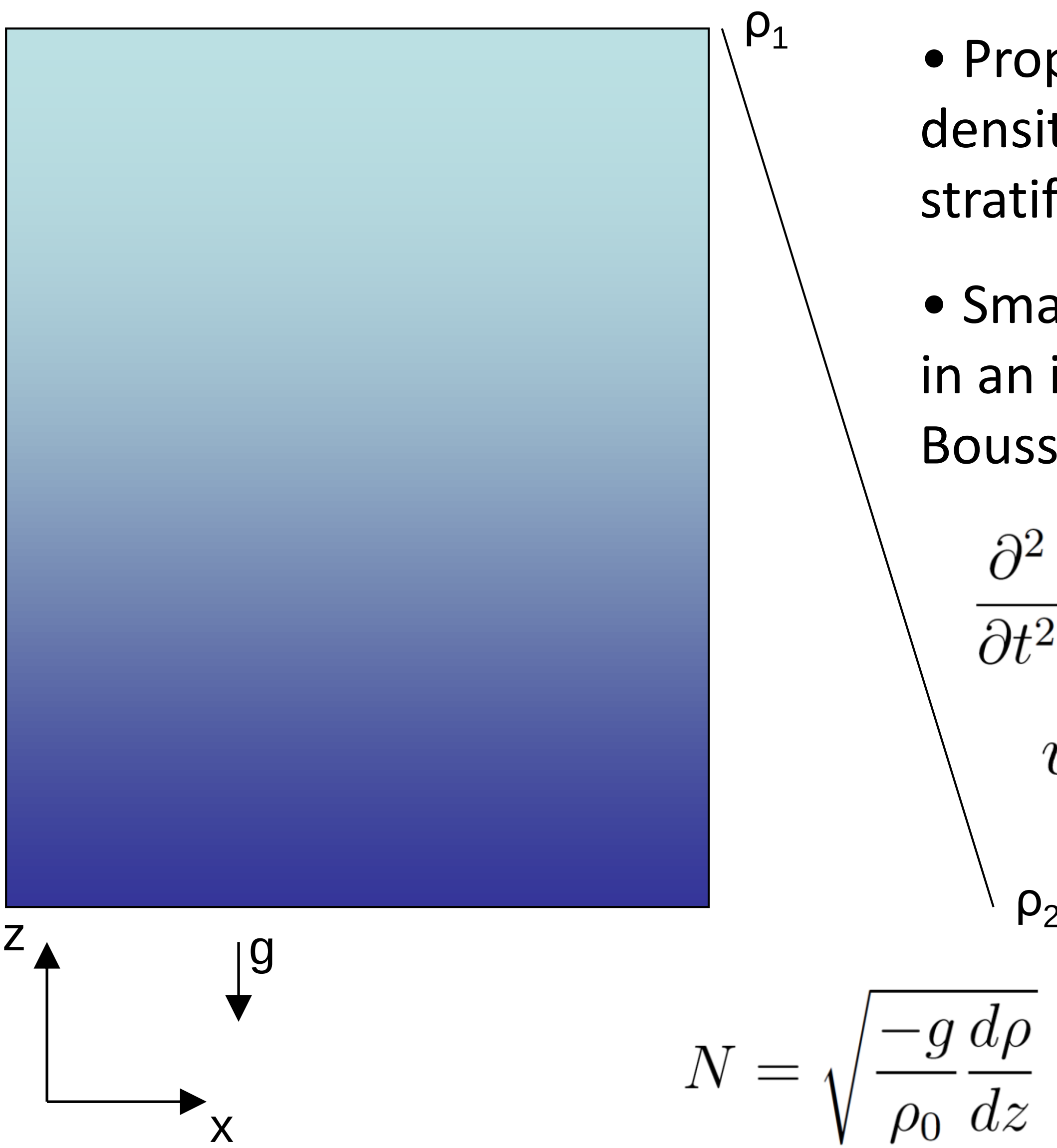
Modeling internal gravity waves for oceanic applications

Manikandan Mathur

Dept. of Aerospace Engg., IIT Madras

Collaboration: Dheeraj Varma (PhD student)

What are internal waves ?



- Propagating disturbances of the density stratification $\rho(z)$ of a stably stratified fluid

- Small two-dimensional perturbations in an incompressible, inviscid, Boussinesq fluid are governed by:

$$\frac{\partial^2}{\partial t^2}[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}]w+N^2\frac{\partial^2 w}{\partial x^2}=0$$

w – vertical velocity perturbation

$N=\sqrt{\frac{-g}{\rho_0}\frac{d\rho}{dz}}$ – Brunt Vaisala frequency

The dispersion relation

$$\frac{\partial^2}{\partial t^2}[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}]w + N^2\frac{\partial^2 w}{\partial x^2} = 0$$

- Look for plane wave solutions:

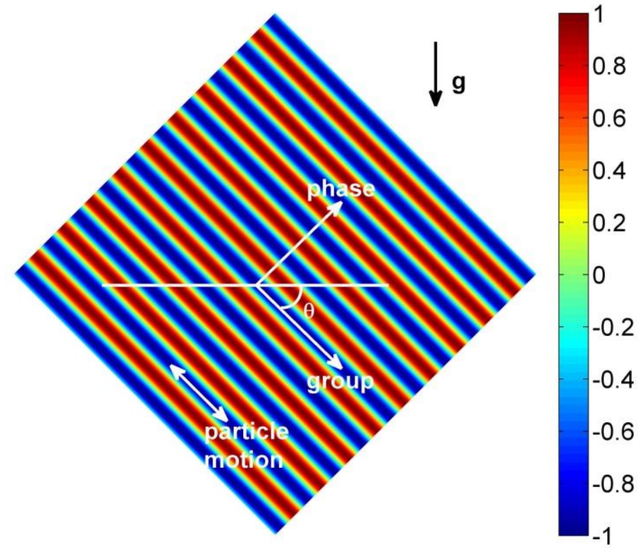
$$w = We^{i(k_x x + k_z z - \omega t)}$$

- Dispersion relation:

$$\omega^2 = N^2 \sin^2\theta, \text{ where } \tan\theta = \frac{k_x}{k_z}$$

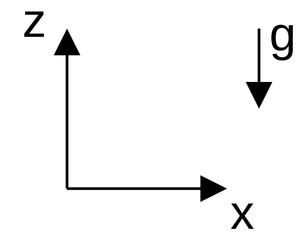
$$\cot^2\theta = \frac{N^2 - \omega^2}{\omega^2}$$

1. $|\vec{k}|$ not fixed by the dispersion relation.
2. θ goes from 0 to $\pi/2$ as ω goes from 0 to N
3. $\vec{c}_p \perp \vec{c}_g$

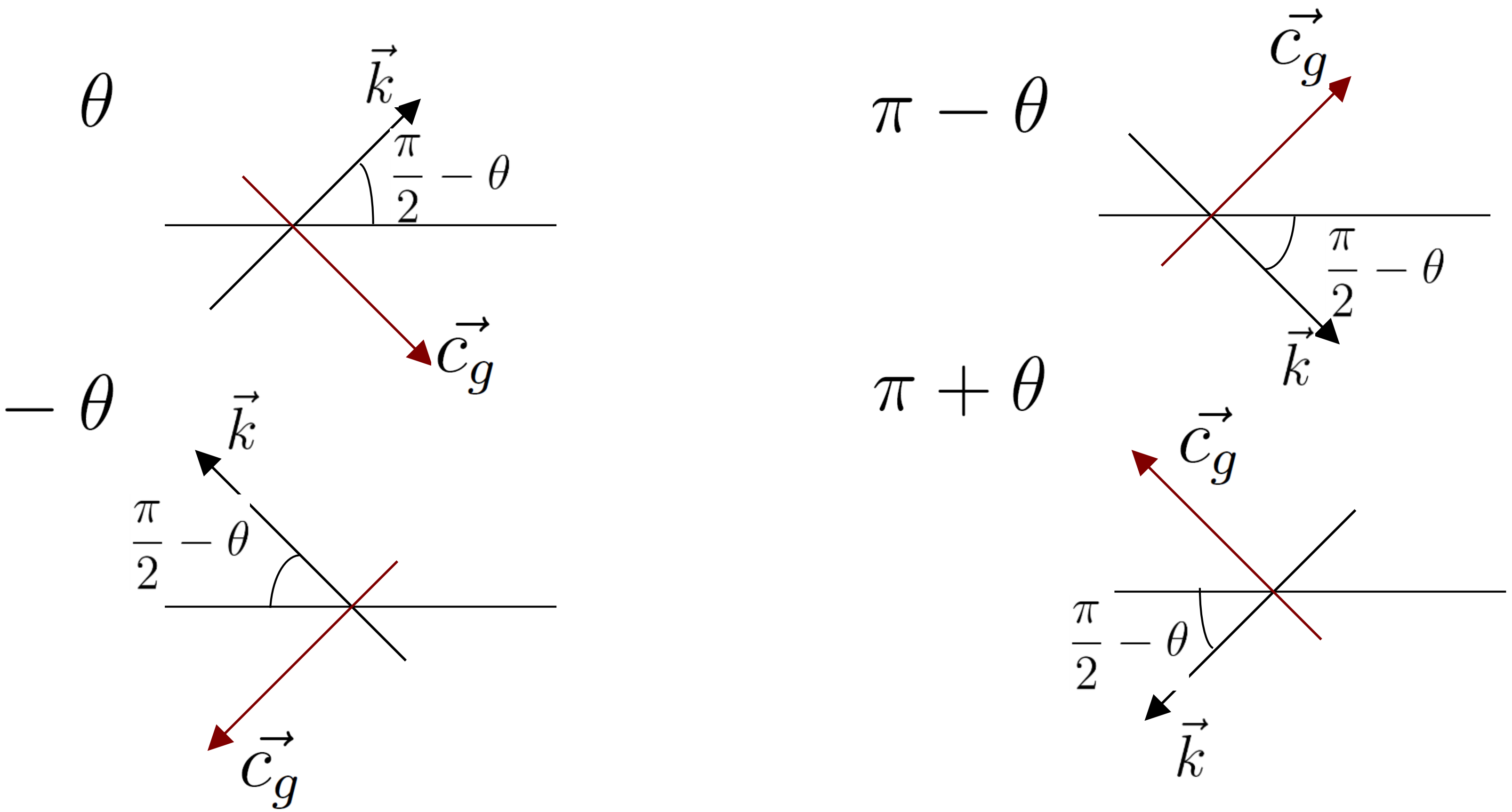


Internal Waves – Dispersion Relation

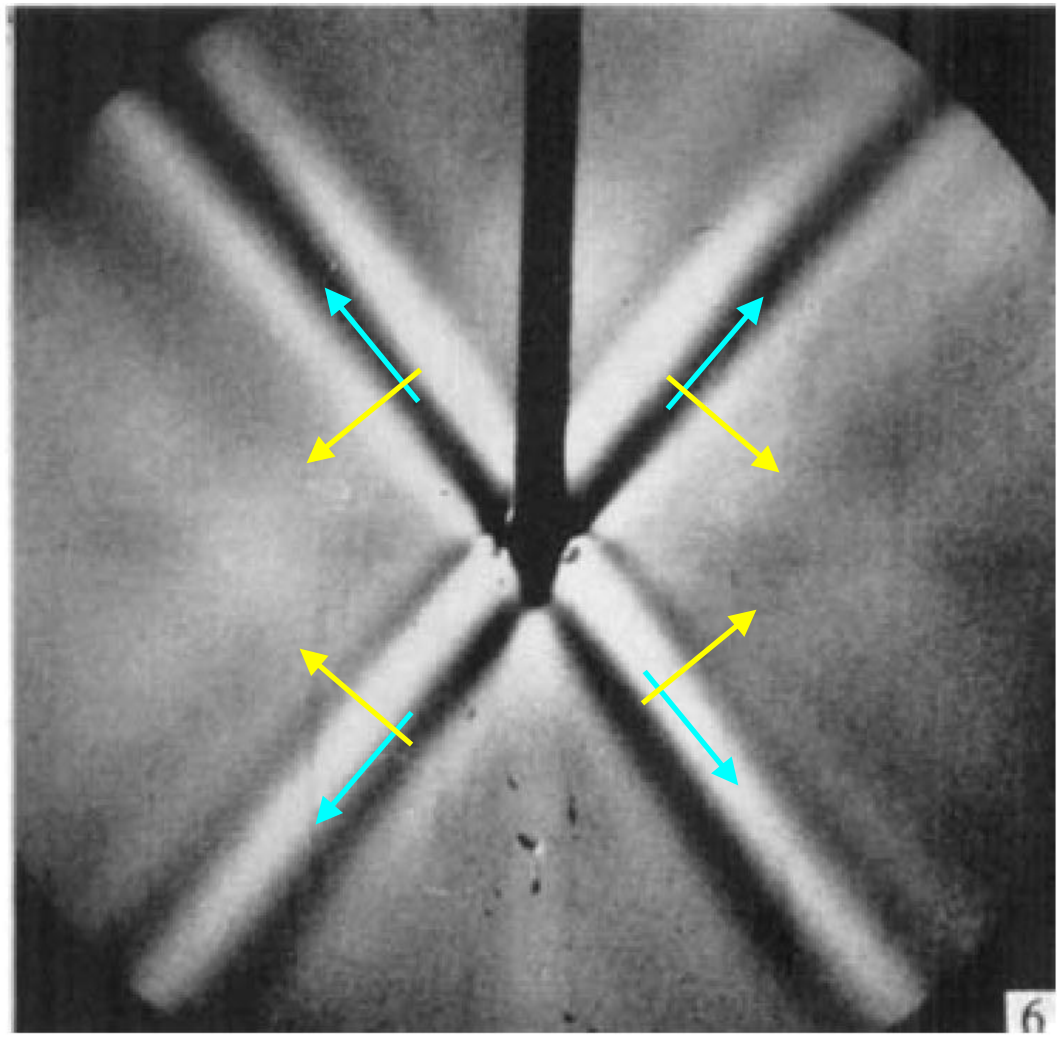
$$\omega^2 = N^2 \sin^2\theta, \ \ where \ \ \tan\theta = \frac{k_x}{k_z}$$

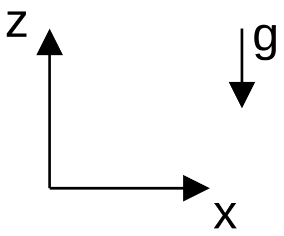


- Four possible solutions:



A simple experiment

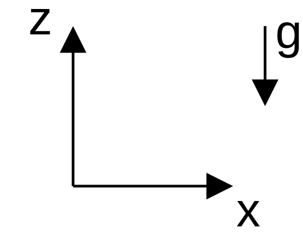


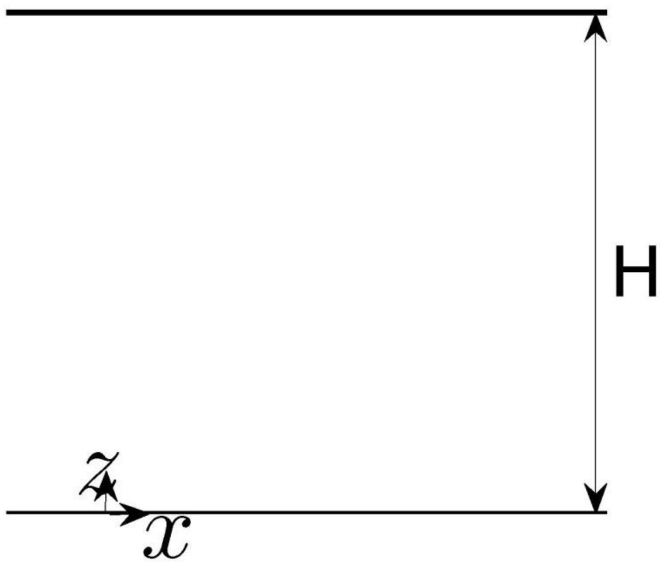


St. Andrews cross - Mowbray & Rarity (1967)

Vertical modes

$$\frac{\partial^2}{\partial t^2}[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}]w+N^2\frac{\partial^2 w}{\partial x^2}=0$$





Boundary conditions

$$w=0\ at\ z=0,H$$

$$w=W(z)e^{i(k_x x-\omega t)}$$

$$\frac{d^2W}{dz^2}+\frac{N^2-\omega^2}{\omega^2}k_x^2W=0$$

$$mode\ n\ :\ W_n(z)=\sin(\frac{n\pi z}{H}),\ U_n(z)=\cot\theta\cos(\frac{n\pi z}{H})$$

$$k_x=\frac{n\pi}{H\cot\theta}$$

Linear wave field: $u'(x,z,t)=\Sigma_{n=-\infty}^{n=\infty}U_n(z)\cos(k_nx-\omega t+\phi_n)$

Dispersion Relation with Coriolis effects

- Fully nonlinear equations – Traditional, inviscid and Boussinesq

$$\frac{\partial^2}{\partial t^2}(\nabla^2\psi) + f^2\frac{\partial^2\psi}{\partial z^2} = \frac{g}{\rho^*}\frac{\partial}{\partial x}[J(\psi,\rho)] - \frac{\partial}{\partial t}[J(\psi,\nabla^2\psi)] + f\frac{\partial}{\partial z}[J(\psi,v)]$$

$$\frac{\partial\rho}{\partial t} = -J(\psi,\rho) \qquad \frac{\partial v}{\partial t} + J(\psi,v) = f\frac{\partial\psi}{\partial z}$$

- Dispersion Relation

$$\cot^2\theta = \frac{N_0^2-\omega^2}{\omega^2-f^2}$$

- For propagation

$$f < \omega < N_0 \quad or \quad f > \omega > N_0$$

- Near-inertial waves $\omega \approx f \quad implies \quad \theta = 0$

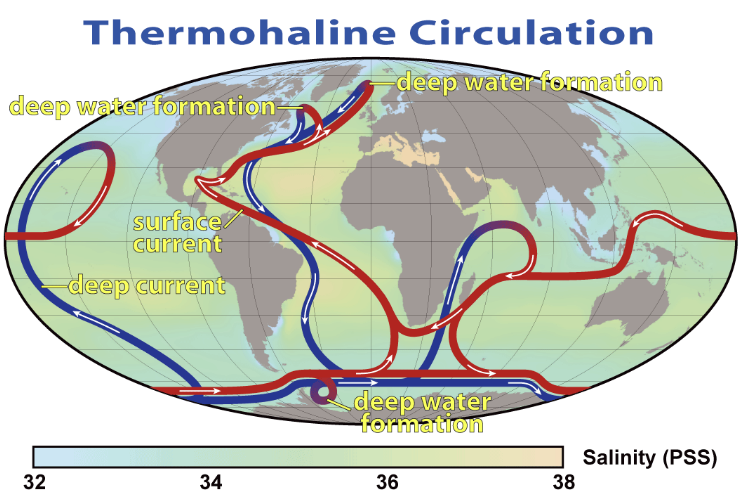
- Typical values in the ocean

$$f \approx 10^{-4} rad/s \quad \omega_{M2} = 1.4\times 10^{-4} rad/s$$

$$N_0 \approx 6\times 10^{-4} rad/s \quad \theta \approx 9.5^\circ$$

What maintains the state of the ocean ?

• "*Without deep mixing, the ocean would turn, within a few thousand years, into a stagnant pool of cold salty water with equilibrium maintained locally by near-surface mixing and with very weak convectively driven surface-intensified circulation*" – Munk & Wunsch (1998)



wikipedia.org

- Necessitates mechanical sources of deep ocean mixing:

1. Tides
2. Winds

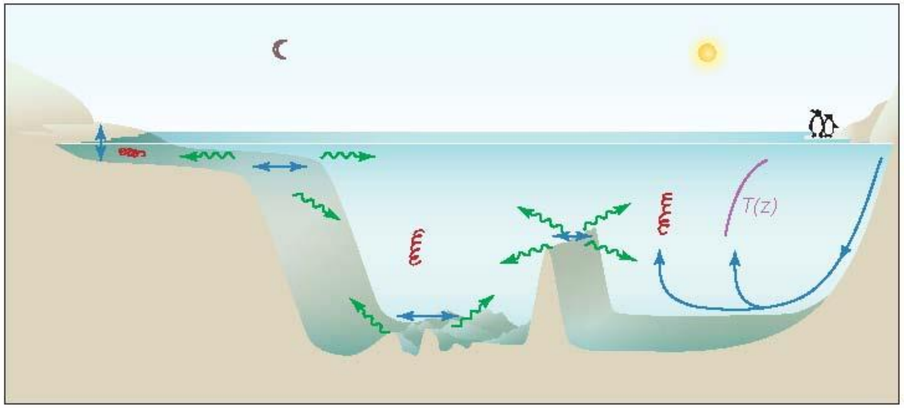
• " *In the open ocean, away from topography, th internal wave field is the only serious candidate for vertical mixing*"
– Munk & Wunsch (1998)

• In this talk, we focus on one possible nonlinear mechanism that may lead to dissipation – relevant for both tides and winds

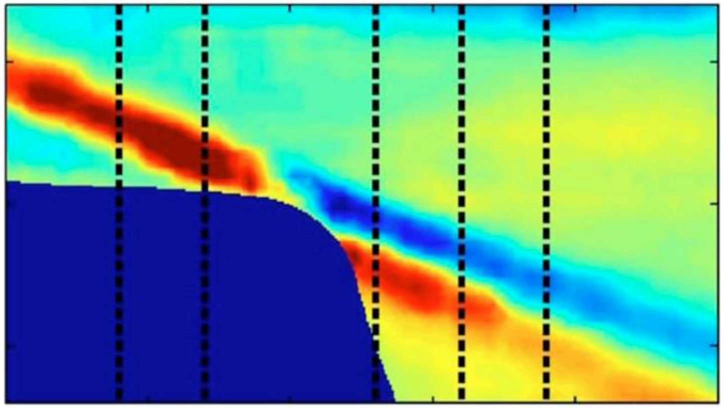
Internal tide generation in the ocean

Garret (2003)

- Internal tides are caused by tidal currents flowing over topography
- High modes dissipate near generation site
- Low modes travel far

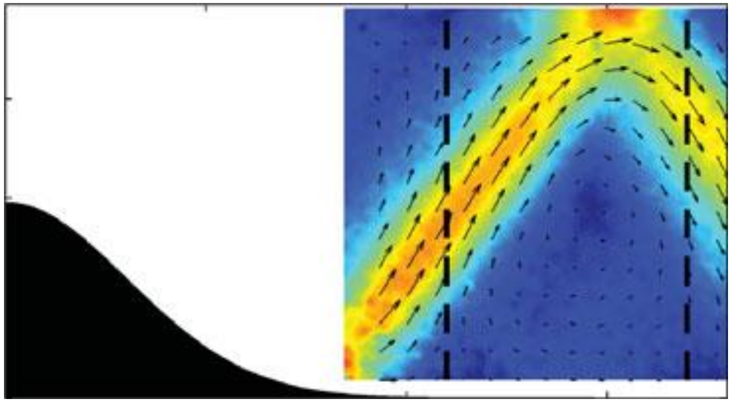


Gostiaux & Dauxois (2007)



Continental shelf generation

Echeverri et al. (2009)



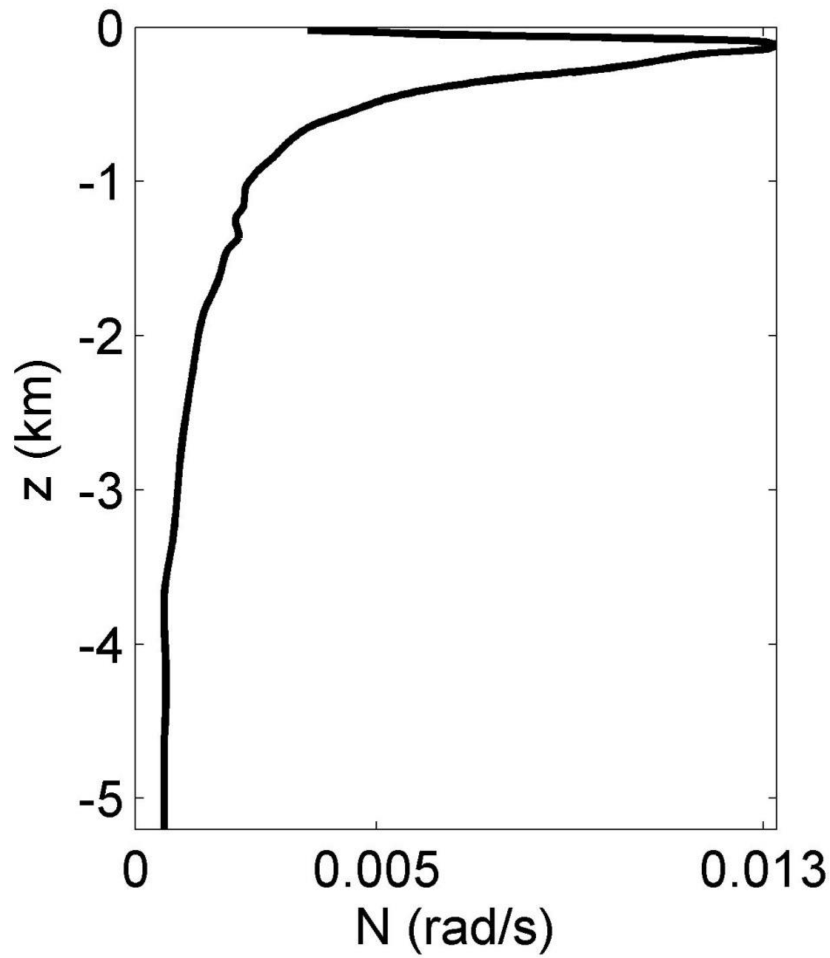
Deep ocean generation

Internal tide dissipation mechanisms

- ➢ _Scattering by supercritical slopes (Kunze & Llewelyn Smith 2004)
- ➢ Scattering by far-field topography (Johnston & Merrifield 2003)
- ➢ Parametric subharmonic instability (MacKinnon & Winters 2005)
- ➢ Interaction with mean flow and mesoscale structures (St. Laurent & Garrett (2002), Rainville & Pinkel 2006)
- ➢ Interaction of internal wave beams with nonuniformly stratified upper ocean – Generation of higher harmonics and/or solitons (New & Pingree 1992, Gerkema 2001, Diamessis *et al.* 2014)

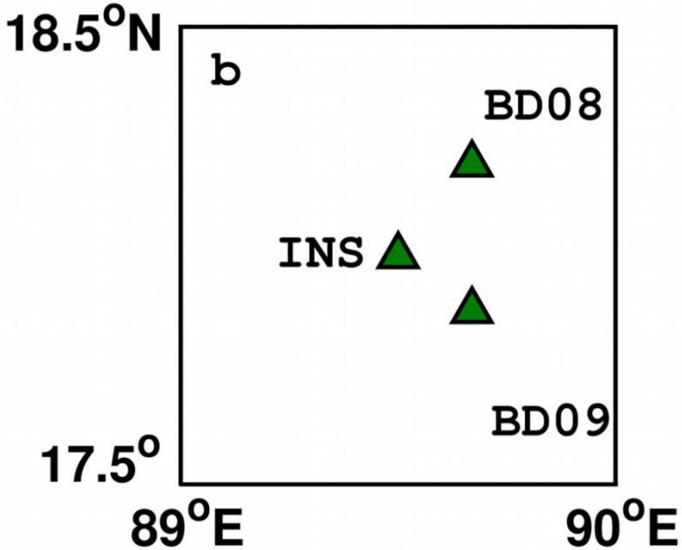
✓ Relative importance of each mechanism is unclear. In this talk, we focus on one such mechanism: nonlinear effects resulting from modal interactions

Typical N(z) in the ocean



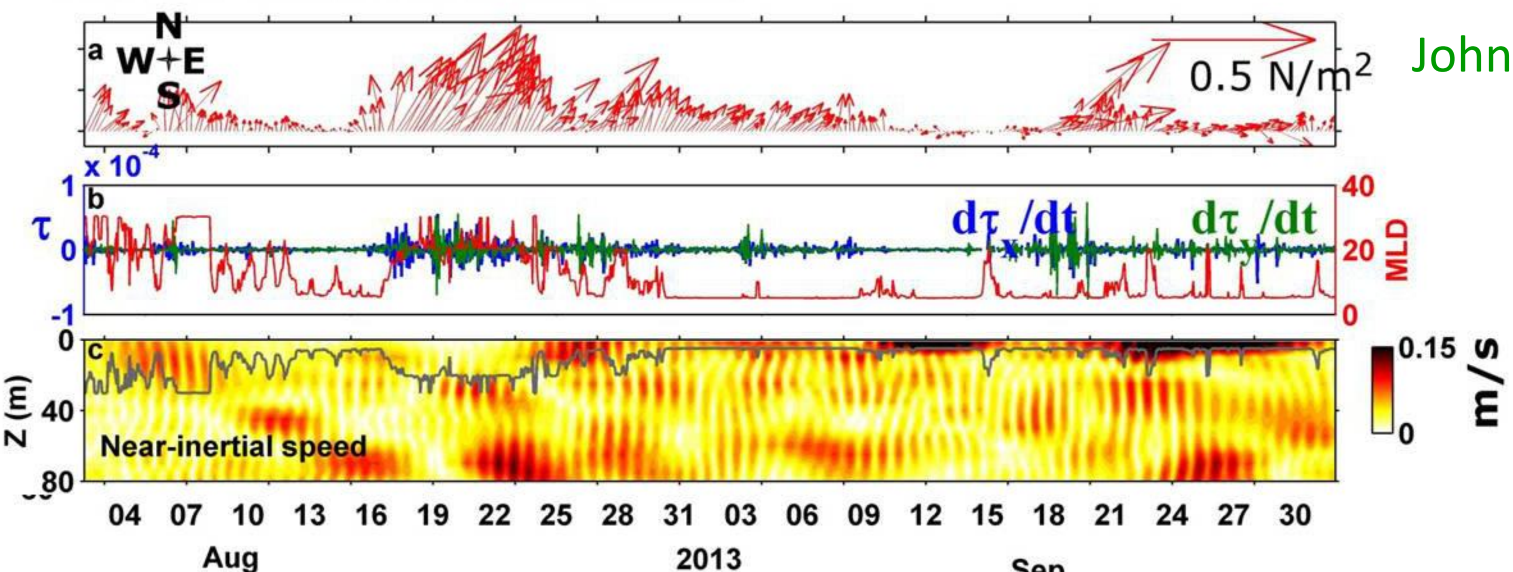
➢ Nonuniformities ubiquitous in geophysical settings. Upper ocean characterized by strongly stratified pycnocline sitting below well-mixed thin layer called the mixed layer. Deep ocean uniformly stratified.

NEAR-INERTIAL CURRENTS - GENERATION & DISSIPATION



• Ocean Moored Buoy Network in the Northern Indian Ocean (OMNI) – Venkatesan *et al.* 2013.

•We present data (1hr. resolution) obtained at 17.88deg. N, 89.67deg. E.



Johnston *et al.* (2016)

• Obvious downward propagation in early August (relatively large MLD); Fresh water influx in September – small MLD & strong stratification below – not so obvious downward propagation.

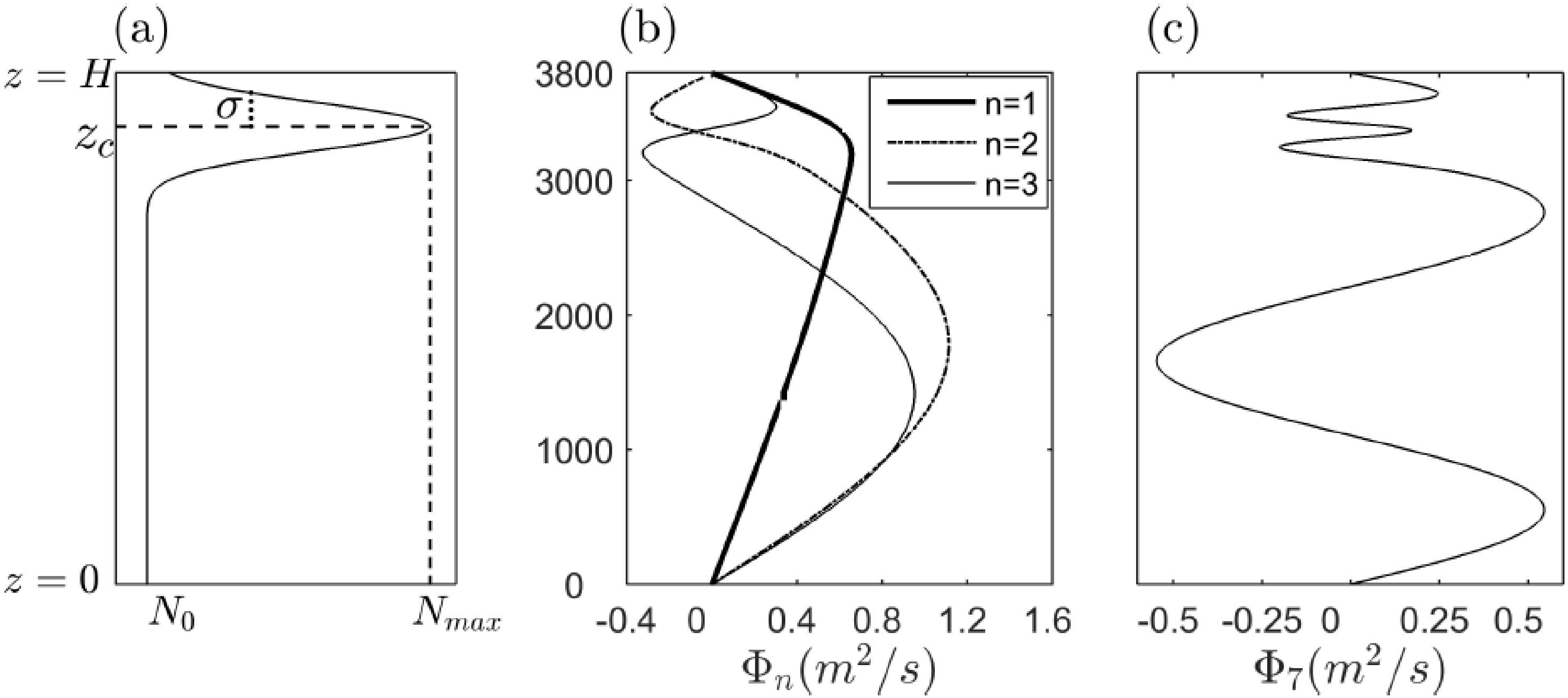
Decay mechanisms for near-inertial currents

- Radiation of downward-propagating NIOs – Gill (1984), Young & Ben Jelloul (1997), Moehlis & Llewllyn Smith (2008)

- Nonlinear Interactions transferring energy to other frequencies – Henyey *et al*. (1986)

- Turbulent Dissipation - Hebert & Moum (1993)

Nonlinear effects from modal interactions



$$\psi_1 = \sum_{n=1}^{\infty} \sqrt{\frac{2\omega k_n E_n}{\rho^*(\omega^2 - f^2)\int_0^H (\frac{d\Phi_n}{dz})^2 dz}} \Phi_n(z)\cos(k_n x - \omega t + \alpha_n)$$

$$\frac{d^2\Phi_n}{dz^2} + \frac{k_n^2(N^2 - \omega^2)}{\omega^2 - f^2}\Phi_n = 0$$

Regular perturbation expansion

$$(\psi, v, \rho) = (\psi_0, v_0, \rho_0) + \epsilon(\psi_1, v_1, \rho_1) + \epsilon^2(\psi_2, v_2, \rho_2) + \ldots$$

- Zeroth order flow described by quiescent stably stratified fluid
- First-order flow described by a linear combination of modes at a fixed frequency
- Second-order flow ?

$$\frac{\partial^2}{\partial t^2}(\nabla^2 \psi_2) + f^2 \frac{\partial^2 \psi_2}{\partial z^2} + N^2 \frac{\partial^2 \psi_2}{\partial x^2} = \frac{g}{\rho^*} \frac{\partial}{\partial x}[J(\psi_1, \rho_1)] - \frac{\partial}{\partial t}[J(\psi_1, \nabla^2 \psi_1)] + f \frac{\partial}{\partial z}[J(\psi_1, v_1)]$$

$$R = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [A_{mn} \cos((k_m + k_n)x - 2\omega t + \alpha_m + \alpha_n) + B_{mn} \cos((k_m - k_n)x + \alpha_m - \alpha_n)]$$

- Forcing at higher harmonic and zero frequency
- Generation of superharmonics and mean flow

Second-order flow

$$\psi_2 = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} [h_{mn}(z) \cos((k_m + k_n)x - 2\omega t + \alpha_m + \alpha_n) + g_{mn}(z) \cos((k_m - k_n)x + \alpha_m - \alpha_n)]$$

$$\frac{d^2 \bar{h}_{mn}}{dz^2} + (k_m + k_n)^2 \frac{N^2 - 4\omega^2}{4\omega^2 - f^2} \bar{h}_{mn} = \bar{C}_{mn}$$

$$\frac{d^2 \bar{g}_{mn}}{dz^2} - (k_m - k_n)^2 \frac{N^2}{f^2} \bar{g}_{mn} = \bar{F}_{mn}$$

- Higher harmonic at sum horizontal wave number
- Mean flow at difference horizontal wave number
- Vertical structure governed by linear internal wave-like equations with a non-zero forcing

Uniform stratification

$$\bar{h}_{mn} = \bar{I}_{mn} \sin\left(\frac{(m-n)\pi z}{H}\right); \quad \bar{g}_{mn} = \bar{J}_{mn} \sin\left(\frac{(m+n)\pi z}{H}\right),$$

where

$$\bar{I}_{mn} = \frac{4\pi\omega\sqrt{mnE_mE_n}}{\rho^* H^2(\omega^2 - f^2)}\left(\frac{(m^2-n^2)\left(\frac{N_0^2}{2\omega\cot^2\theta} + \frac{\omega(N_0^2-f^2)}{(\omega^2-f^2)\cot^2\theta} + \frac{f^2}{2\omega}\right)}{(m+n)^2(N_0^2-4\omega^2) - (m-n)^2(4\omega^2-f^2)\cot^2\theta}\right)$$

$$\bar{J}_{mn} = \frac{2\pi\sqrt{mnE_mE_n}}{\rho^* H^2(\omega^2-f^2)\cot^2\theta}.$$

- Higher harmonic at difference vertical wave number and mean flow at sum vertical wave number

- Amplitude of higher harmonic wave diverges if

$$\frac{\omega^2}{N_0^2} = \frac{(m+n)^2 - (m-n)^2(4-f^2/\omega^2)/(1-f^2/\omega^2)}{4(m+n)^2 - (m-n)^2(4-f^2/\omega^2)/(1-f^2/\omega^2)}$$

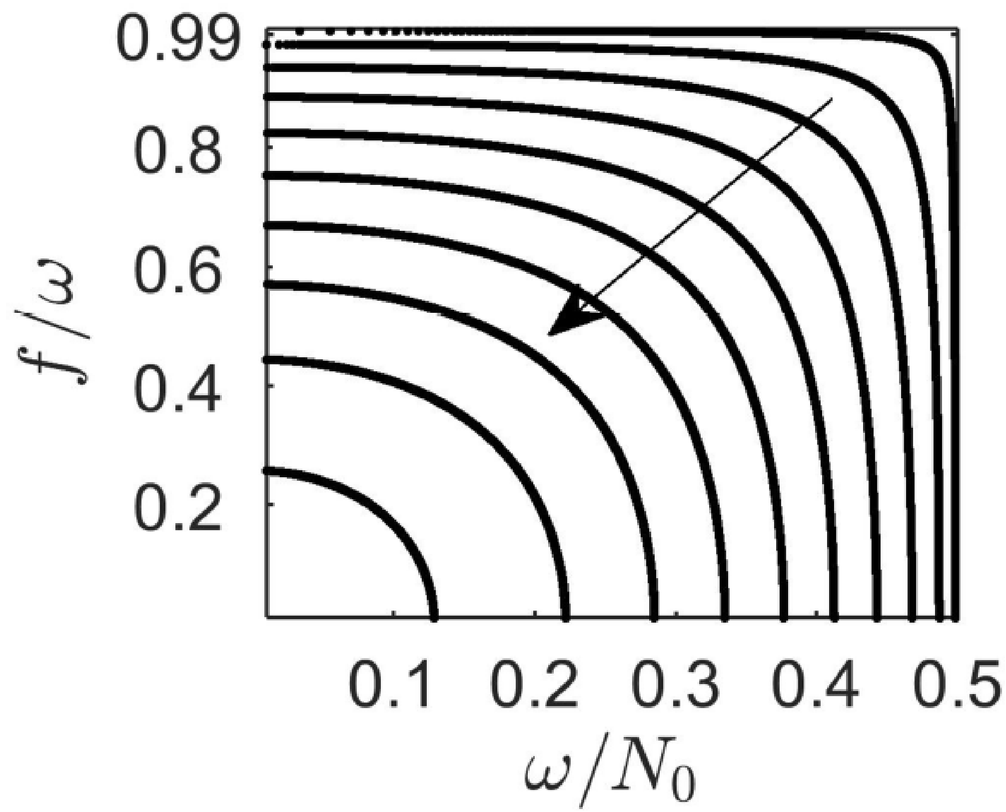
- Existence of resonance triad comprising modes m & n at fundamental frequency and mode (m-n) at higher harmonic frequency

- Inherently unstable linear internal waves irrespective of their amplitude

Uniform stratification

- Divergence of higher harmonic solution only if

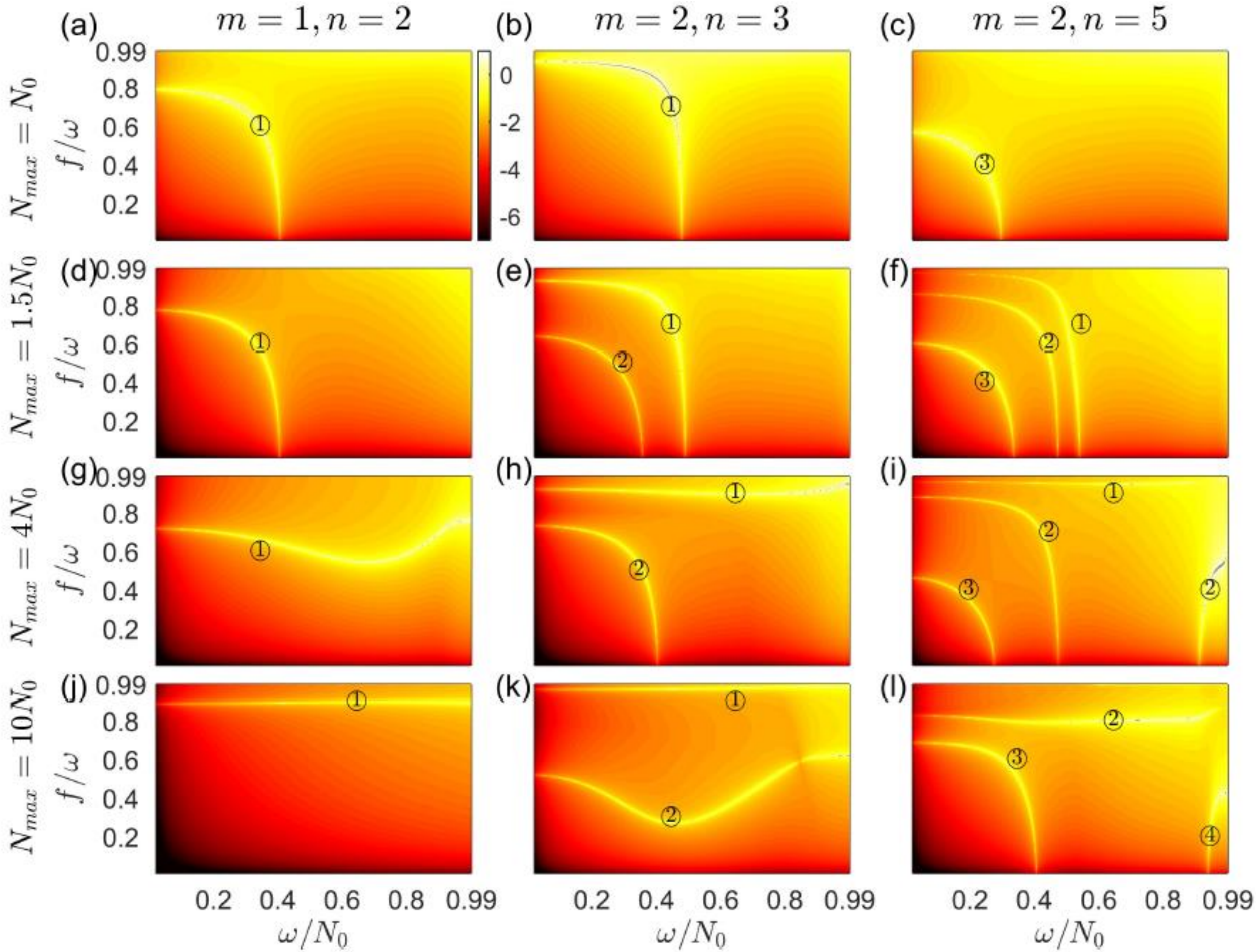
$$(m/3) < n < 3m, \; m \neq n$$



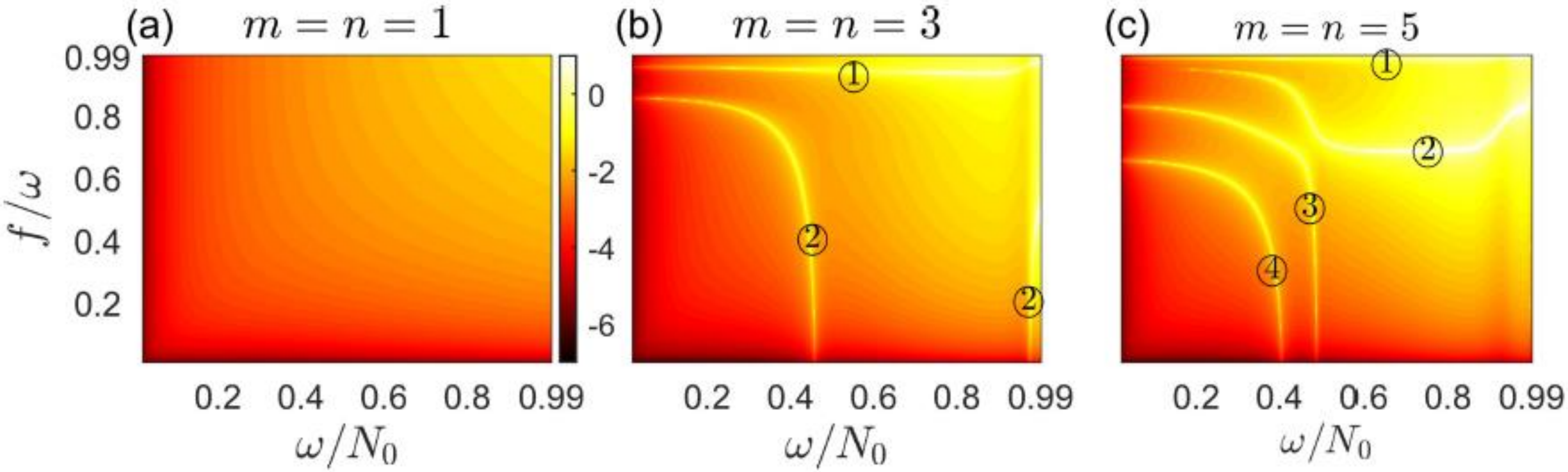
- m/n goes from 1.1 to 2.9 in the direction of the arrow

- Neighbouring high-mode interactions result in diverging weakly nonlinear solutions at near-inertial frequencies

Nonuniform stratifications

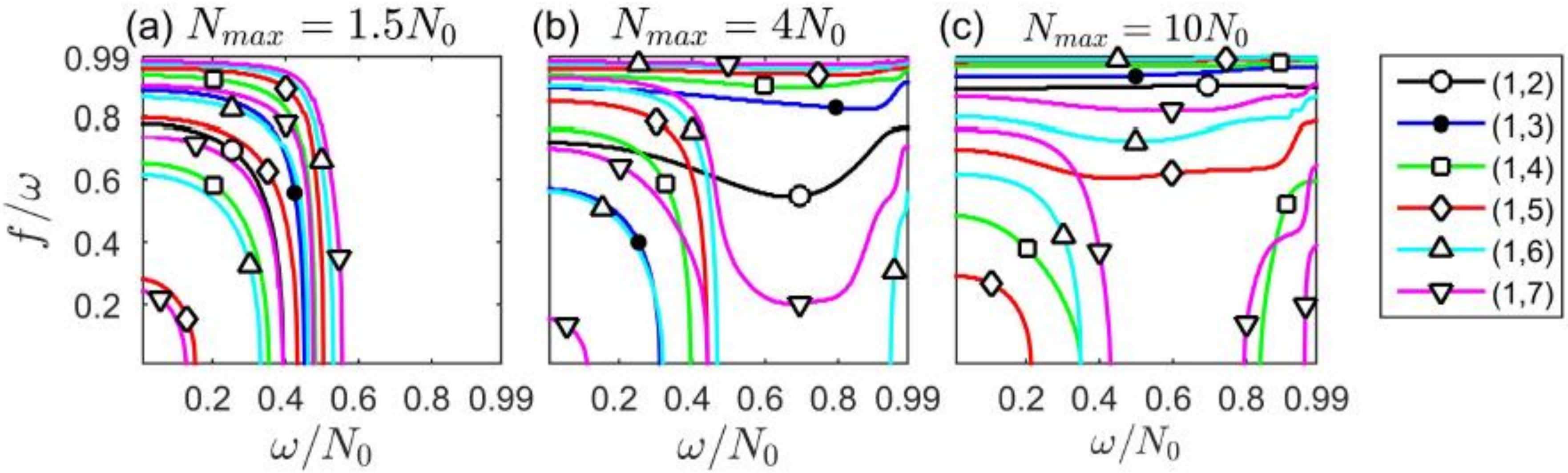


Self-interactions



- ➢ No divergence based on self-interaction in uniform stratifications
- ➢ Individual modes inherently unstable in nonuniform stratifications
- ➢ Multiple divergence curves
- ➢ Appearance of near-horizontal divergence curves at near-inertial frequencies

An application: internal tides



- ➢ Tidal internal wave beams generated by topography typically contain a range of modes
- ➢ Divergence curves corresponding to all possible modal interactions within mode-1 to mode-7 span the entire frequency plane
- ➢ Uniform stratifications have far fewer such interactions

Summary

- Modal interactions can result in strong nonlinear effects
- Nonuniform stratifications correspond to an infinitely larger number of resonant triads containing two modes at a fixed frequency when compared with a uniform stratification
- Nonlinear effects may be more likely than previously thought; raises questions about linear models that consider superposition of modes at the forcing frequency
- Existence of resonant triads understood via dispersion curves in nonuniform stratification – Varma & Mathur (JFM 2017)
- Currently setting up laboratory experiments to validate the theoretical results and perform observations in the nonlinear regime