Canonical Hamiltonian understanding of stratified shear flows

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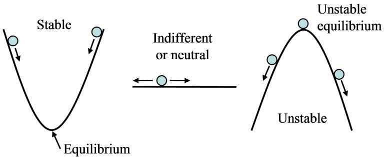
Outline

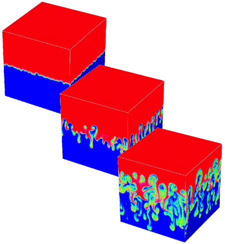
1. Introduction to stratified shear instabilities
    - Instability basics
    - Minimal model - the need for a mechanistic picture
2. Pseudo-momentum and Pseudo-energy
3. Canonical Hamiltonian formalism

Intuitive understanding of instabilities

When buoyancy drives the flow:



(a) Basics of instability



(b) Rayleigh Taylor instability (Source: K. Kadau)

What happens when density stratification is stable, and there is shear?

Shear instabilities not as intuitive as buoyancy driven ones

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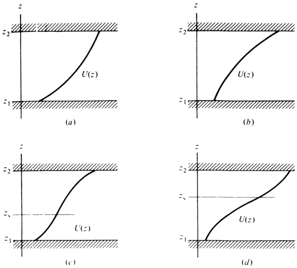

Fig. 4.2. (*a*) Stable: $U'' < 0$; (*b*) stable: $U'' > 0$; (*c*) stable: $U'' = 0$ at z_s but $U''(U - U_s) \geqslant 0$; (*d*) possibly unstable: $U'' = 0$ at z_s but $U''(U - U_s) \leqslant 0$. (From Drazin & Howard 1966.)

Figure 1: Different shear flows (Drazin & Reid (2004)).

Mathematical theorems give us necessary conditions for instabilities

For unstratified flows there are two theorems: 1) Rayleigh's inflexion point theorem, and 2) Fjørtoft's criterion $\bar{U}_{zz}(\bar{U}(z) - \bar{U}_{\mathrm{inf-point}}) \leq 0$. For stratified flows there is Miles'-Howard criterion: $Ri(z) = N^2/\bar{U}_z^2 < 1/4$.

Do shear instabilities always yield turbulence?

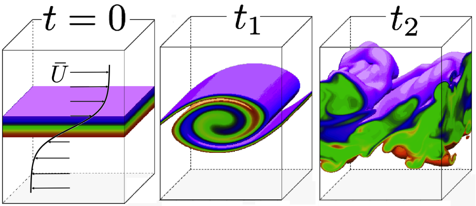

Figure 2: Kelvin-Helmholtz instability.

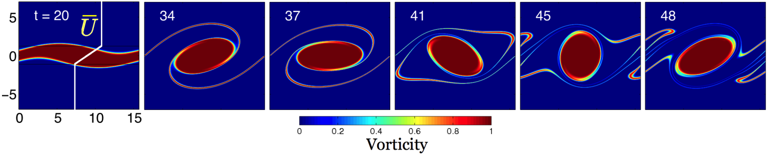

Figure 3: Barotropic instability.

Source: Guha et al. (PRE 2013)

Kinematics and dynamics of an interface in density stratified shear flows (inviscid, incompressible flow)

Vorticity evolution equation ($x - z$ plane)

$$\frac{Dq}{Dt} = w\frac{d^2\bar{U}}{dz^2} - N^2\frac{\partial\zeta}{\partial x}.$$

$D/Dt \equiv \partial/\partial t + \bar{U}\partial/\partial x$ is the *linearized* material derivative, $-\bar{U}_{zz}$ analogous to β effect for Rossby waves, N is the buoyancy frequency, $q = w_x - u_z$ is the perturbation vorticity.

Kinematic condition

$$\frac{D\zeta}{Dt} = w.$$

For now, let's make life even simpler. Assume $N = 0$. The above two equations can then be combined to yield

$$q = \zeta\bar{U}_{zz}.$$

A lonely vorticity/Rossby-edge wave

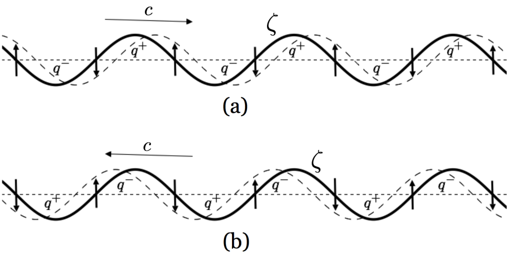

Figure 4: The mechanism why a wave (a) moves to the right, and (b) moves to the left.

Crucial point: $\zeta q > 0$ implies a right moving wave, while $\zeta q < 0$ implies a left moving wave.
But how will this motion sustain? For that we need a relation between q and ζ. Recall from the last slide

$$q = \zeta \bar{U}_{zz}.$$

The same principles work for interfacial gravity waves

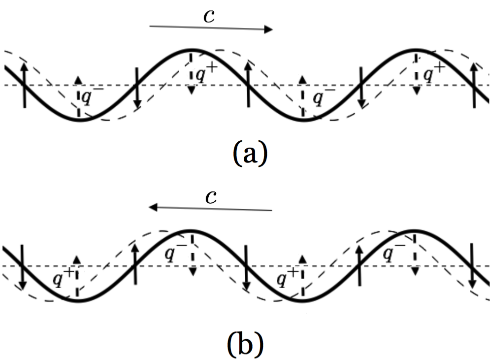

Figure 5: The mechanism why a gravity wave (a) moves to the right, and (b) moves to the left.

$$\frac{Dq}{Dt} = -N^2 \frac{\partial \zeta}{\partial x}.$$

A tale of two vorticity/Rossby-edge waves

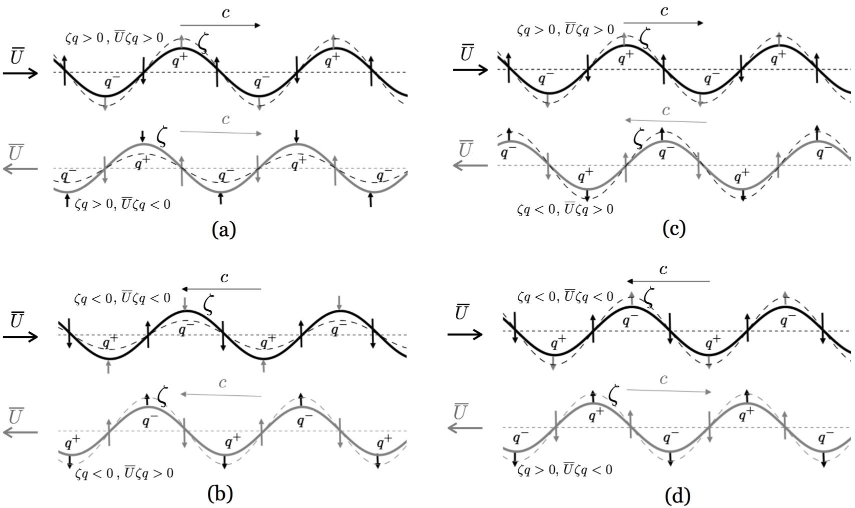

Figure 6: Two interfacial waves in presence of a background velocity shear. (a) Pro-counter, (b) counter-pro, (c) pro-pro, and (d) counter-counter.

Inferences only based on inspection

- Only *counter-propagating waves* can lead to sustained mutual growth, like what observed in normal-mode instabilities.
- $\langle \zeta q \rangle$ is small (or even zero due to symmetry between mutually amplifying pair of waves).
- $\left\langle \bar{U} \zeta q \right\rangle < 0$.

Two new concepts: Pseudo-momentum and Pseudo-energy

- In unstable shear flows, we saw that the amplitude of the waves are increasing $\implies$ wave energy is not conserved (energy exchanges occur between waves and mean flow).
- **Pseudo-momentum** and **pseudo-energy** are conserved wave activities in shear flows.
- Before our work, expressions for pseudo-momentum and pseudo-energy were known only for *unstratified* flows. We extended it to general Boussinesq stratified flows.

Two new concepts: Pseudo-momentum and pseudo-energy

Theorem

1. Pseudo-momentum theorem: *In 2D inviscid, non-diffusive, Boussinesq flows, the domain integrated pseudo-momentum*

$$\mathscr{P}_b = \left\langle \zeta q - \bar{U}_{zz}\frac{\zeta^2}{2} \right\rangle$$

is a conserved quantity. Hence $\boxed{\dfrac{\partial \mathscr{P}_b}{\partial t} = 0}$.

2. Pseudo-energy theorem: *In 2D inviscid, non-diffusive, Boussinesq flows, the domain integrated pseudo-energy*

$$\mathscr{H}_b = \left\langle \frac{1}{2}\left(u^2 + w^2 + N^2\zeta^2\right) + \bar{U}\mathscr{P}_b \right\rangle$$

is a conserved quantity. Hence $\boxed{\dfrac{\partial \mathscr{H}_b}{\partial t} = 0}$.

What is pseudo-momentum for unstratified/homogeneous flows?

$$\mathscr{P}_b = \left\langle \zeta q - \bar{U}_{zz} \frac{\zeta^2}{2} \right\rangle . \tag{1}$$

For homogeneous flows

$$q = \zeta \bar{U}_{zz} . \tag{2}$$

Combining (1) and (2) we obtain

Pseudo-momentum for homogeneous flows

$$\mathscr{P}_h = -\frac{1}{2} \langle \zeta q \rangle = -\frac{1}{2} \left\langle \zeta^2 \bar{U}_{zz} \right\rangle . \tag{3}$$

Pseudo-momentum and pseudo-energy for normal-mode perturbations

$$\mathscr{P}_h = -\frac{1}{2}\int_X\int_Z \zeta q dxdz = -\frac{1}{2}\int_X\int_Z \zeta^2 \bar{U}_{zz} dxdz = \text{constant}.$$

Assume exponentially growing perturbations: $\zeta(x,z,t) = \zeta(x,z,0)e^{\omega_i t}$.
This occurs for normal-mode instabilities. Hence $\mathscr{P}_h(t) = \mathscr{P}_h(0)e^{2\omega_i t}$.
Only possible constant that satisfies is $\boxed{\mathscr{P}_h = 0}$.

*** Pseudo-momentum and pseudo-energy are zero for normal-mode instabilities. The flow can be homogeneous or stratified.**

Lest we forget...we hypothesized $\boxed{\langle \zeta q \rangle = 0}$ ONLY by inspecting the counter-propagating waves.

Rayleigh's inflexion point theorem:

$$\mathscr{P}_h = -\frac{1}{2}\int_X\int_Z \zeta q dxdz = -\frac{1}{2}\int_X\int_Z \zeta^2 \bar{U}_{zz} dxdz = 0 \implies$$

$\bar{U}_{zz} = 0$ somewhere in the domain.

Pseudo-momentum and pseudo-energy for normal-mode perturbations

Similarly for Pseudo-energy $\mathscr{H}_h$ one can show

$$\mathscr{H}_h = \int_X \int_Z \left(E + \frac{1}{2}\bar{U}\zeta q \right) dxdz = 0,$$

where $E = \frac{1}{2}(u^2 + w^2)$ is positive definite. This implies $\left\langle \bar{U}\zeta q \right\rangle < 0$.

Again...we hypothesized $\boxed{\left\langle \bar{U}\zeta q \right\rangle < 0}$ ONLY by inspecting the counter-propagating waves.

Recall

$$q = \zeta \bar{U}_{zz}.$$

Substitute for q to obtain

Fjøroft's criterion:

$$\int_X \int_Z \bar{U}\bar{U}_{zz}\zeta^2 dxdz < 0 \implies \bar{U}\bar{U}_{zz} < 0.$$

Summary till this point

- Shear instabilities are apparently not physically intuitive (unlike buoyancy driven ones).
- Whether a shear flow is stable/unstable is guided by mathematical theorems (Rayleigh, Fjørtoft).
- *Counter-propagating waves* perspective offers physical insight into normal-mode instabilities in shear flows. Furthermore it predicts
    - $\langle \zeta q \rangle = 0$ (from which comes Rayleigh's inflexion point theorem).
    - $\left\langle \bar{U} \zeta q \right\rangle < 0$ (from which comes Fjørtoft's criterion).

Summary till this point

- Two conserved wave activities, pseudo-momentum (PM) and pseudo-energy (PE) for general (Boussinesq stratified) shear flow are stated. They are 0 for normal-mode instabilities.
- **Rayleigh's and Fjørtoft's criteria are imprinted in PM and PE respectively**.
  - The condition for mutual wave amplification (hence Rayleigh's theorem) comes from the vanishing of PM for normal mode instability.
  - The condition for counter-propagation and hence phase-locking (and hence Fjørtoft's criterion) is derived from the vanishing of PE.

Canonical Hamiltonian formalism - single interface

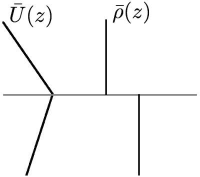

Figure 7: A vorticity-density interface

$$\bar{q}_z = -\bar{U}_{zz} = \Delta \bar{q}_0 \delta(z - z_0)\,, \qquad N^2 = \Delta N_0^2 \delta(z - z_0).$$

Interfacial Rossby/vorticity-gravity wave dispersion relation

$$c_0^{\pm} = U_0 - \frac{\Delta \bar{q}_0}{4k} \pm \beta_0, \quad \text{where} \quad \beta_0 \equiv \sqrt{\left(\frac{\Delta \bar{q}_0}{4k}\right)^2 + \frac{\Delta N_0^2}{2k}}.$$

Canonical Hamiltonian formalism - single interface

$\zeta - q$ eigen-structure of the normal modes is determined by the requirement that their spatial structure should not change with time. $\zeta - q$ can be divided into kernels, each kernel should conserve its structure.

$$q = e^{\mathrm{i}kx}[\tilde{q}_0^+(t) + \tilde{q}_0^-(t)]\delta(z - z_0)\,, \quad \zeta = e^{\mathrm{i}kx}[\zeta_0^+(t) + \zeta_0^-(t)], \tag{4}$$

$$\tilde{q}_0^{\pm} = \alpha_0^{\pm}\zeta_0^{\pm}, \qquad \alpha_0^{\pm} \equiv 2k(c_0^{\pm} - U_0). \tag{5}$$

Expressing waves' displacements in terms of their amplitudes and phases:

$$\zeta_0^{\pm} \equiv Z_0^{\pm} e^{\mathrm{i}k\chi_0^{\pm}}, \qquad \chi_0^{\pm}(t) = (\phi_0^{\pm} - c_0^{\pm} t), \tag{6}$$

where $k\phi_0^{\pm}$, are the phases of the waves at $t = 0$.

Substitute (4) and (5) in PM and PE expressions to obtain:

$$\mathscr{P}_0 = -k\beta_0(Z_0^+)^2 + k\beta_0(Z_0^-)^2 \equiv P_0^+ + P_0^-, \tag{7}$$

and

$$\mathscr{H}_0 = -(cP)_0^+ - (cP)_0^- = (\dot{\chi}P)_0^+ + (\dot{\chi}P)_0^-. \tag{8}$$

Canonical Hamiltonian formalism

$\mathscr{H}_0$ is not a function of the wave phases. Furthermore, the waves are neutral with constant amplitude. Thus (7)-(8) yield the canonical Hamilton's equations:

$$\frac{\partial \mathscr{H}_0}{\partial P_0^{\pm}} = \dot{\chi}_0^{\pm}, \qquad \frac{\partial \mathscr{H}_0}{\partial \chi_0^{\pm}} = -\dot{P}_0^{\pm} = 0$$

The idea can be extended to two (and also multiple) interfaces:

$$\mathscr{H} = \sum_{n=1}^{2} \left[(\dot{\chi} P)^{+} + (\dot{\chi} P)^{-} \right]_n .$$

$$\frac{\partial \mathscr{H}}{\partial P_n^{\pm}} = \dot{\chi}_n^{\pm}, \qquad n = 1, 2$$

$$\frac{\partial \mathscr{H}}{\partial \chi_n^{\pm}} = \sum_{j=1}^{2} \left[\left(\frac{\partial \dot{\chi}^{+}}{\partial \chi_n^{\pm}} \right) P^{+} + \left(\frac{\partial \dot{\chi}^{-}}{\partial \chi_n^{\pm}} \right) P^{-} \right]_j = -\dot{P}_n^{\pm}. \qquad n = 1, 2$$

Summary of the Hamiltonian formalism

- Equations of interacting waves in stratified shear flows become the canonical Hamilton equations.
- Pseudo-energy serves as the Hamiltonian of the system.
- The contributions of each wave to the pseudo-momentum are the generalized momenta.
- The the waves' phases, scaled by the wavenumber, are the generalized coordinates.

Thank you!