Dynamics of quantized vortices in a wall bounded superfluid turbulence

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Pankaj Mishra[1]

in collaboration with

Dmytro Khomenko[2], Anna Pomyalov[2], L. Kondaurova[3], Victor Lvov[2], and Itamar Procaccia[2]

1)Department of Physics, IIT Guwahati, India.
2) Chemical Physics, Weizmann Institute of Science, Israel.
3) Institute of Thermophysics, Novosibirsk, Russia.

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Outline

- Motivation: recent experimental development.
- Introduction of superfluidity in Helium-4.
- Superfluid experiment and theory.
- Two fluid model.
- Closure problem.
- Joe Vinen's proposition for the dynamics of quantized vortices for homogeneous turbulence state.
- Our proposition for the tangle dynamics in inhomogeneous channel with frozen normal component.
- Coupled dynamics of superfluid.
- Conclusion and perspective.

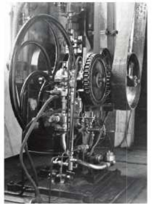
Superfluid Helium

Heike Kamerlingh-Onnes

Noble prize 1913.

“for his investigation of the properties of matter at low temperature which led to the production of liquid Helium”.

using the compressor



liquified He at T=4.2 K

in July 10, 1908.

K-O & coworkers in 1924

discovered density charge

at T=2.18 K.

Piotr Leonidovich Kapitza

Noble prize 1978.

“for his inventions and discoveries in the area of low-temperature Physics”.

P.L. Kapitza in Mascow

discovered and named in

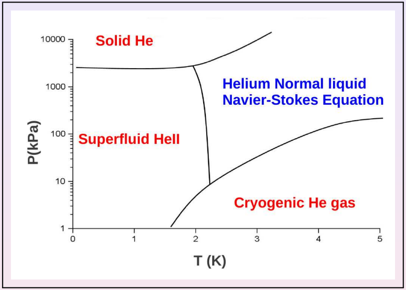
1937.

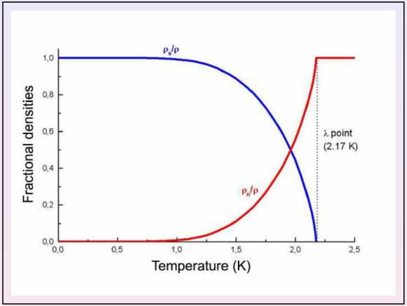
Jack Allen

and his student Donald Missener

independently discovered superfluidity in Cambridge’s lab.

Superfluid Helium (^{4}He)



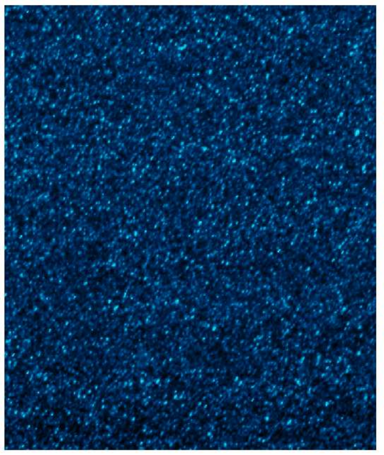


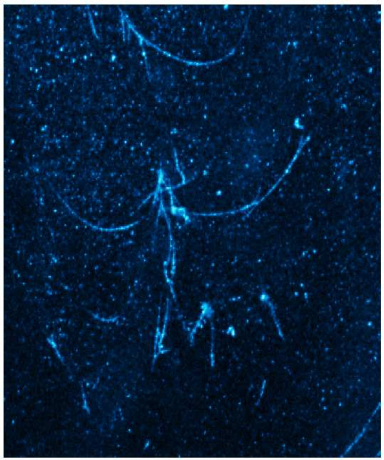
- The He II can be described as macroscopic wavefunction $\Psi = |\Psi|e^{i\phi} \Rightarrow$ **velocity**$\sim \nabla\phi$.
- The vortices are quantized

$$\oint \mathbf{V_s} \cdot dS = n\frac{h}{2m_4} = n\kappa.$$

- Every vortex has the same circulation ($\kappa = 9.97 \times 10^{-8} m^2 - sec^{-1}$).

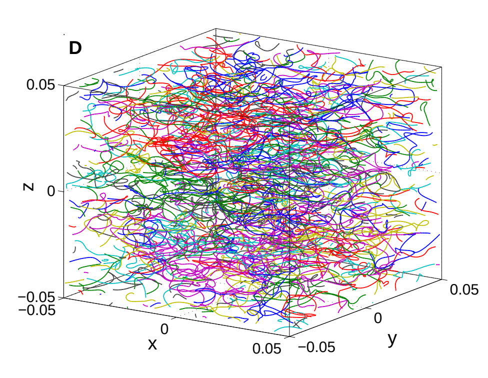
Quantum vortices in the experiments





Bewely, Lathrop, Sreenivasan, Nature 441, 558 (2006).
Fonda *et al.*, PNAS 111, 4653 (2014).
Paoletti *et al.*, J. Phys. Soc. Japan 77, (2008).

Quantum Turbulence



Turbulence in a superfluid was predicted first by Richard Feynman in 1955 and found experimentally (in counterflow ^{4}He) by Henry Hall and Joe Vinen in 1956.

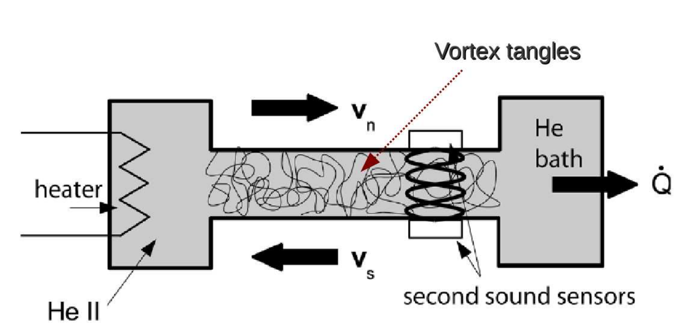
The turbulence in normal fluid and superfluid

Normal fluid

- Energy injection at large scale generates the large eddies.
- Non-linear interaction helps to generate the smaller and smaller eddies until energy dissipates at the viscous scale.
- The smaller eddies (small k) interact with one another and produces the smaller eddies (with large k) with high strain and shear that dissipates the energy into heat.

Superfluid fluid

- The vortex core is fixed in size and there is no viscous losses.
- Bundles of nearly parallel vortex lines form.
- The energy from one vortex lines to another vortex lines is transmitted through the vortex reconnections.
- The Reconnection generates the Kelvin wave which due to the nonlinear interaction generates the higher and higher wavenumbers until they loose the energy due to the emission



Two states: TI-Superfluid:turbulent; Normal:laminar.
T2-both component becomes turbulent.

Hall and Vinen Proc. Roy. Soc. A 238, 204-214 (1956).
Guo etal. Phys. Rev. Lett. 105, 045301 (2010).
Guo etal. PNAS 111, 4653 (2014).

Experimental results

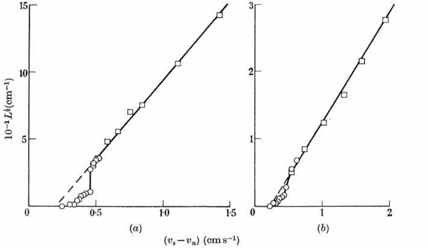


FIGURE 3. Mutual friction at small heat current densities; $L^{\frac{1}{2}}$ plotted as a function of $(v_s - v_n)$. ○, points obtained by the use of equation (4); □, points obtained from the observed attenuation of second sound in the heat current. (a) Apparatus no. 2, 1·40° K; (b) apparatus no. 1, 1·60° K.

TI state: Low vortex line density (v_n laminar, v_s turbulent); TII state: High vortex line density state (Both turbulent).

W. F. Vinen, Proc. R. Soc.. Lon. A, **243**, 400 (1958).

The Gross-Pitaevskii model

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + g\psi|\psi|^2 - \mu\psi \qquad (GPE)$$

- Macroscopic wave function: $\psi = \sqrt{\rho}exp(i\phi(r,t))$.

$$\hat{p} = i\hbar\nabla \rightarrow v_s = \frac{\hbar}{m_4}\nabla\phi \rightarrow curlv_s = 0$$

- Density $\rho = |\psi|^2$, Velocity $\mathbf{v}$=$(\hbar/m)\nabla\phi$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{v}) = 0 \qquad (continuity)$$

$$\rho\left(\frac{\partial v_j}{\partial t} + v_k\frac{\partial v_j}{\partial x_k}\right) = -\frac{\partial p}{\partial x_j} + \frac{\partial\sum_{jk}}{\partial x_k} \qquad (almostEuler)$$

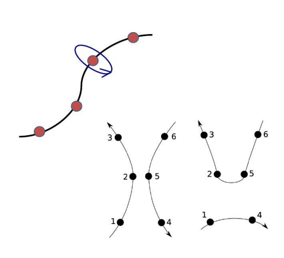
- Pressure $p = \frac{g}{2m^2}\rho^2$, Quantum stress $\sum_{jk} = \left(\frac{\hbar}{2m}\right)^2\rho\frac{\partial^2 ln\rho}{\partial x_j\partial x_k}$.
- At length scale larger than $\xi = (\hbar^2/m\mu)^{1/2}$ neglect $\sum_{jk}$ and recover compressible Euler.

The vortex Filament model

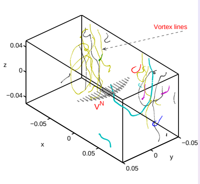
- Neglect density variation and GPE reduces to incompressible Euler.
- Assume vortex as a thin filament Euler reduces to Biot-Savart law for space curve s(t):

$$V_{s\,nolocal}(s_{(i,j,k)}) = \frac{\kappa}{4\pi}\int_C \frac{(s_1 - s_{(i,j,k)}) \times ds_1}{|s_1 - s_{(i,j,k)}|^3}$$

- Reconnections performed by using particular scheme. (Schwarz PRB 1988).



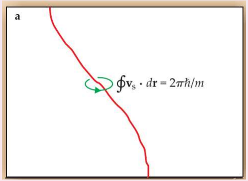
Vortex filament dynamics in presence of normal component



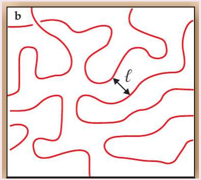
- Must take account of: two-fluid behaviour, at least at high temperature.
- Rotation of superfluid component is possible only through the presence of the quantized vortex lines, circulation.
- Mutual friction $f_D = -\gamma_0 \hat{\kappa} \times [\hat{\kappa} \times (v_n - v_L)] + \gamma_0' \hat{\kappa} \times (v_n - v_L)$.
- Magnus effect $f_m = \rho_s \kappa \hat{\kappa} \times (v_L - v_s)$.
- Balance of forces $f_D + f_m = 0$
- viscosity of normal fluid: very small for 4He.
- Hence $f_D = -\alpha \rho_s \kappa \hat{\kappa} \times [\hat{\kappa} \times (v_n - v_s)] - \alpha' \rho_s \kappa \hat{\kappa} \times (v_n - v_s)$.
- $v_L = v_s + \alpha \hat{\kappa} \times (v_n - v_s) - \alpha' \hat{\kappa} \times [\hat{\kappa} \times (v_n - v_s)]$ if $\alpha, \alpha' \ll 1$, $v_L \approx v_s$. (where $v_L = \frac{ds}{dt}$)

- Normal fluid vs. superfluid at $T \to 0$ limit:
  -Normal fluid kinematic viscosity $\nu \neq 0$ vs. $\nu \equiv 0$ in superfluids.
  -Two scales in normal fluids: Outer scale L and dissipative microscale $\eta \ll L$.
  -Two additional scales in superfluids due to quantization of vortex lines:

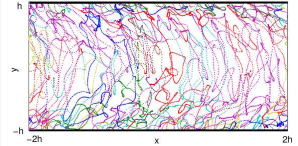
a: Vortex core diameter $a_0 \simeq 1\AA$



b:inter-vortex distance ℓ



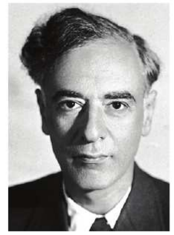
c: Outer scale $\mathcal{L}$



- Vortex tangle at large scale. Another interesting quantity vortex line density $\mathcal{L}$.

Two-fluid theoretical model of Turbulence

Lev Davidovich Landau



E. Andronikashvili

Laszlo Tizsa

Noble prize 1962.

"for his pioneering theories for condensed matter especially liquid Helium. In particular, he quantized in 1941 the Tisza-1940 two-fluid model and suggested Andronikashvilii's 1946 experiment on oscillating in HeII discs. Its period and damping measures densities of superfluid, ρ_n and normal, ρ_s, components:

$$
\begin{aligned}
\rho_{\rm n}\frac{\partial u_{\rm n}}{\partial t} + \rho_{\rm n}(u_{\rm n}\cdot\nabla)u_{\rm n} &= -\frac{\rho_{\rm n}}{\rho}\nabla p_{\rm n} - \rho_{\rm s} S\nabla T + \mathcal{F}_{\rm ns} + \eta\Delta u_{\rm n}\,,\\
\rho_{\rm s}\frac{\partial u_{\rm s}}{\partial t} + \rho_s(u_{\rm s}\cdot\nabla)u_{\rm s} &= -\frac{\rho_s}{\rho}\nabla p_{\rm s} + \rho_{\rm s} S\nabla T - \mathcal{F}_{\rm ns}\,.
\end{aligned}
$$

S-entropy, T-temperature and F_{ns}- mutual friction.

Hall-Vinen-Bekarevich-Khalatnikov Coarse-grained equations

In hydrodynamic region of the scales $R \gg \ell$ one can neglect the quantization of vortex lines and make use of the coarse-grained two fluid equation for velocities of the superfluid and normal components u_s and u_n with the densities ρ_s and ρ_n and pressures p_s and p_n:

$$\rho_s \left[\frac{\partial u_s}{\partial t} + (u_s \cdot \nabla) u_s \right] - \nabla p_s = -F_{ns}, \qquad p_s = \frac{\rho_s}{\rho}[p - \rho_n |u_s - u_n|^2],$$

$$\rho_n \left[\frac{\partial u_n}{\partial t} + (u_n \cdot \nabla) u_n \right] - \nabla p_n = \rho_n \nu \triangle u_n + F_{ns}, \qquad p_n = \frac{\rho_n}{\rho}[p + \rho_s |u_s - u_n|^2],$$

The above equations are coupled by the mutual friction between the superlfuid and normal fluid components:

$$F_{ns} = -\rho_s \alpha' (u_s - u_n) \times \omega_s + \alpha \hat{\omega}_s \times [\omega_s \times (u_s - u_n)] \approx \alpha \rho_s \kappa \mathcal{L} (u_s - u_n).$$

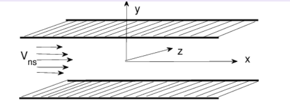
- $\mathcal{L}$: Average vortex length per unit volume.
- The dynamical equations are not closed. Another equation for vortex line density $\mathcal{L}$ is required!

Phenomenological equation for $\mathcal{L}$ (Vinen 1957)

$$\frac{d\mathcal{L}}{dt} = \mathcal{P}(t) - \mathcal{D}(t).$$

- Production term $\mathcal{P}(t) \propto \alpha$-The growth of $\mathcal{L}$ due to the extension of the vortex rings by mutual friction, which is caused by the relative velocity between the normal and super component (V_{ns}).
- Decay term, $\mathcal{D}(t) \propto \alpha$ is again caused by the mutual friction.
- Dimensional reasoning:
  $\mathcal{P}_{cl} = \alpha\kappa\mathcal{L}^2 F(x)$, ($[\kappa]=m^2/\text{sec}$, $[\mathcal{L}]=m^{-2}$)
  $D_{cl} = \alpha\kappa\mathcal{L}^2 G(x)$.
  $F(x)$ and $G(x)$: dimensionless Functions with $x = V_{ns}^2/\kappa^2\mathcal{L}$
- Earlier suggestions (Vinen 1956, 1957):
  $\mathcal{P}_1 = \alpha C_1 \mathcal{L}^{3/2}$ for $F(x) = x^{1/2}$
  $\mathcal{P}_2 = \alpha C_2 \mathcal{L} V_{ns}^2/\kappa$ for $F(x) = x$;
  $D_{cl} = \alpha C_{dec}\kappa\mathcal{L}^2$ for $G(x) = 1$

Dynamics of vortex line density ($\mathcal{L}$) in inhomogeneous channel



- Consider the condition in which normal component is laminar and superfluid is turbulent.
- There is inhomoegeneity in the normal flow.
- It imposes the inhomogeneous distribution of the tangle in the box.

Dynamics of vortex line density in inhomogeneous channel

The vortex line density equation ($\mathcal{L}$:)

$$\frac{\partial \mathcal{L}(y,t)}{\partial t} + \frac{\partial J_{\text{cl}}(y,t)}{\partial y} = \mathcal{P}_3(y,t) - \mathcal{D}_{\text{cl}}(y,t) \ ,$$

The extra term vortex-line density flux J is added which will be

suggested from our model to have the form as:

$$J_{\text{cl}}(y,t) = -C_{flux}(\alpha/2\kappa) V_{ns} \cdot \nabla V_s$$

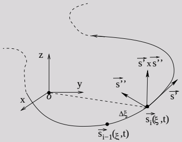
We propose the third form of $\mathcal{P}$ [corresponding to non-dimensional $F(x) \propto x^{3/2}$]:

$$\mathcal{P}_3 = \alpha C_{prod} \sqrt{L} V_{ns}^3 / \kappa^2 .$$

The above dynamical equation may serve the future studies for wall bounded superfluid turbulence.

Starting from the first principles: Analytical results

Dynamics of Vortex segment



$$d\delta\xi/(\delta\xi dt) \approx \alpha V_{\rm ns} \cdot (s' \times s'') \ .$$

$s(\xi, t)$: Coordinate of the quantized vortex lines parameterized by the arc length ξ; $s' = ds/d\xi$, $s'' = d^2s/d\xi^2$.
The counterflow V_{ns} is

$$V_{ns} = V^n - V^s, V^s = V_0^s + V_{BS}.$$

V_0^s: Macroscopic potential part of superfluid.

$$V_{\rm BS}(s) = \frac{\kappa}{4\pi} \int_{\mathcal{C}} \frac{(s - s_1) \times ds_1}{|s - s_1|^3} \Rightarrow V^{\rm s}_{\rm LIA} + V^{\rm s}_{\rm nl}(s) \ .$$

The logarithmically divergent ($s_1 \Rightarrow s$) can be regularized using vortex core radius a_0 and mean vortex line curvature $R = 1/\bar{S}$.

For $a_0 \leq |s_1 - s| \leq R$:

$$V^s_{LIA} = \beta s' \times s'', \beta \equiv (\kappa/4\pi) ln(R/a_0).$$

Microscopic theory of vortex line density ($\mathcal{L}$)

Dynamics of Vortex segment

The nonlocal term V_{nl}^{s} produced by rest of the configuration $\mathcal{C}'$, with $|s_1 - s| > R$

$$V_{\rm nl}^{\rm s}(s) = \frac{\kappa}{4\pi} \int_{\mathcal{C}'} \frac{(s - s_1) \times ds_1}{|s - s_1|^3} .$$

On Integrating over the slice Ω between y and $y + \delta y$ for all x and z:

$$d\delta\xi/(\delta\xi dt) \approx \alpha V_{\rm ns} \cdot (s' \times s'')$$

$\mathcal{L}(y,t) = \int_{C_\Omega} d\xi/\Omega$.

$$\frac{\partial \mathcal{L}(y,t)}{\partial t} + \frac{\partial J_{\rm num}(y,t)}{\partial y} = \mathcal{P}_{\rm num}(y,t) - \mathcal{D}_{\rm num}(y,t) ,$$

$$\text{Flux Term:} J_{\rm num}(y,t) = \frac{\alpha}{\Omega} \int d\xi \, V_{\rm ns,x} s'_z ,$$

$$\text{Production Term:} \mathcal{P}_{\rm num}(y,t) = \frac{\alpha}{\Omega} \int d\xi \, (V^{\rm n} - V_0^{\rm s} - V_{\rm nl}^{\rm s}) \cdot (s' \times s'') ,$$

$$\text{Decay Term:} \mathcal{D}_{\rm num}(y,t) = \frac{\alpha\beta}{\Omega} \int d\xi \, |s''|^2 = \alpha\beta\mathcal{L}\widetilde{S}^2 .$$

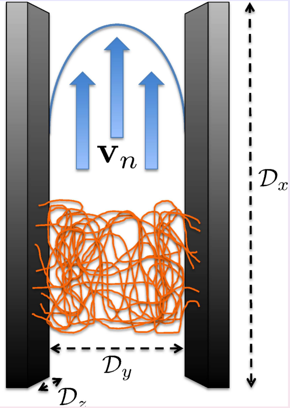
Numerical simulation using Vortex filament method

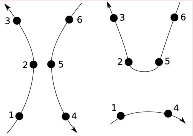
- Dynamical equation for vortex tangle:

$$\frac{d\mathbf{s}}{dt} = \mathbf{V}_s^{tot} + \alpha \mathbf{s}' \times (\mathbf{V}_n^{ext} - \mathbf{V}_s^{tot}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{V}_n - \mathbf{V}_s^{tot})]$$

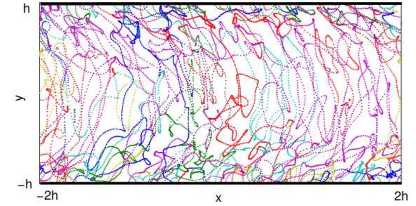
  $V_s(\mathbf{s})$: superfluid velocity due to the vortex tangle; α and α' : the mutual frictions parameters; V_n applied external normal velocity.

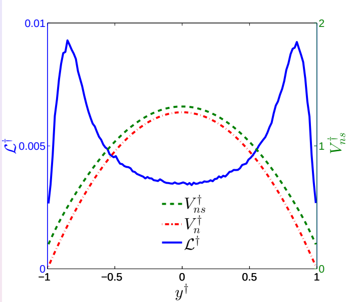
- Box-size: 0.1cm $\times$0.05 cm $\times$0.05 cm.
- Boundary condition: no-slip for V_n on the wall (y); periodic along x and z.
- $< V_n > = 1.0$ cm/sec., 1.2 cm/sec. and 1.5 cm/sec.
- Runge-Kutta Fourth order (CFL condition).
- T=1.0K, 1.6K, and 1.9K.
- Initial condition: 20 random oriented rings with resolution $\delta\xi = 1.6 \times 10^{-3}$.
- Dynamical criteria for the reconnection.





Numerical results





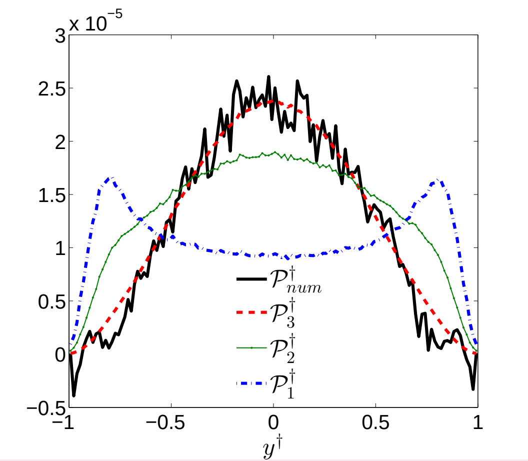
Prescribed normal velocity profile- red dash-dotted line.

Resulting normalized counter flow velocity profile- green dash-line.

vortex line density profile ($\mathcal{L}^\dagger(y)$)- blue solid line.

$$y^\dagger = y/h,\ V^\dagger = V/\sqrt{< V_{ns}^2 >},\ \mathcal{L}^\dagger = \kappa^2 \mathcal{L}/ < V_{ns}^2 > .$$

Comparison between the production terms of $\mathcal{L}$

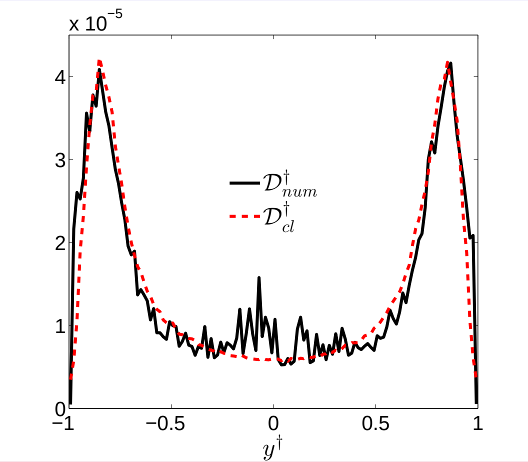


$$\mathcal{P}_{\text{num}}(y,t) = \frac{\alpha}{\Omega}\int d\xi\,(V^{\text{n}} - V^{\text{s}}_0 - V^{\text{s}}_{\text{nl}})\cdot(s' \times s'')$$

$$\mathcal{P}_1 = \alpha C_1 \mathcal{L}^{3/2}, \text{and} \mathcal{P}_2 = \alpha C_2 \mathcal{L} V_{ns}^2/\kappa$$

$$\mathcal{P}_3 = \alpha C_{prod}\sqrt{L} V_{ns}^3/\kappa^2.$$

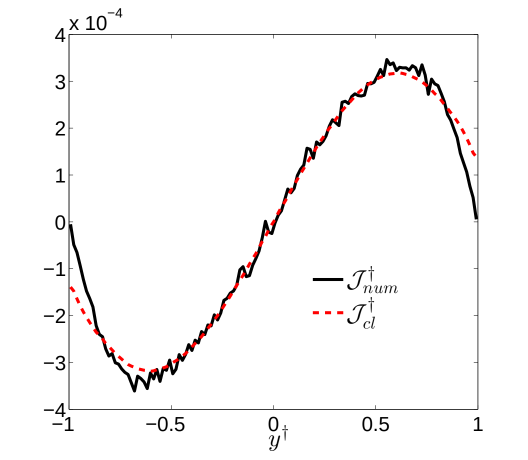
Comparison for the Decay of $\mathcal{L}$



$$D_{cl} = \alpha C_{dec} \kappa \mathcal{L}^2$$

$$\mathcal{D}_{\text{num}} = \frac{\alpha\beta}{\Omega} \int d\xi \, |s''|^2$$

Comparison for the flux of $\mathcal{L}$



$$J_{\text{num}}(y,t) = \frac{\alpha}{\Omega}\int d\xi\; V_{\text{ns,x}} s'_z$$

$$J_{cl} = -C_{flux}\frac{\alpha}{2\kappa}V_{ns}(y)\frac{dV_s}{dy}$$

Superfluid component

$$\frac{d\mathbf{s}}{dt} = \mathbf{V}_s^{tot} + \alpha \mathbf{s}' \times (\mathbf{V}_n^{ext} - \mathbf{V}_s^{tot}) - \alpha' \mathbf{s}' \times [\mathbf{s}' \times (\mathbf{V}_n - \mathbf{V}_s^{tot}]$$

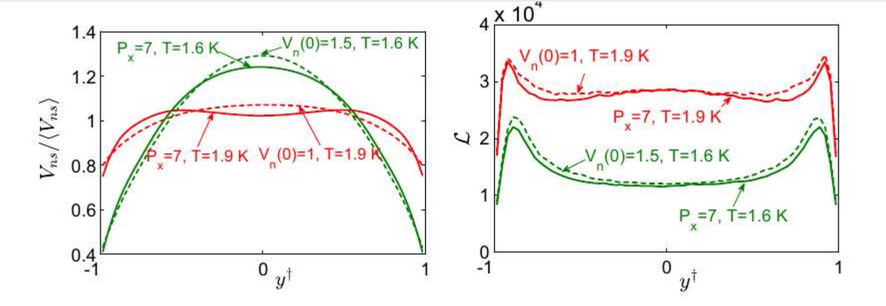
$V_s(\mathbf{s})$: superfluid velocity due to the vortex tangle; α and α' : the mutual frictions parameters; V_n normal velocity.

Normal component

$$\frac{\partial V_n(y,t)}{\partial t} = \frac{dP}{dx} + \frac{F_{ns}(y,t)}{\rho_n} + \nu_n \frac{\partial^2 V_n(y,t)}{\partial y^2}$$

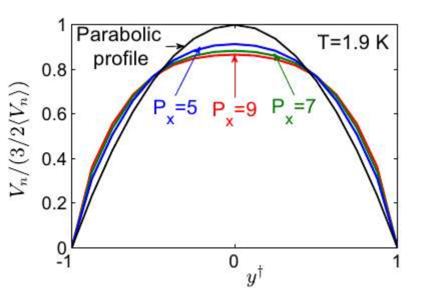
$$F_{ns} = \frac{\rho_s}{\Omega} \int_C^{\prime} (\alpha s' \times [s' \times v_{ns}] + \alpha' s' \times V_{ns}) d\xi$$

Profile in presence of frozen and dynamic normal component



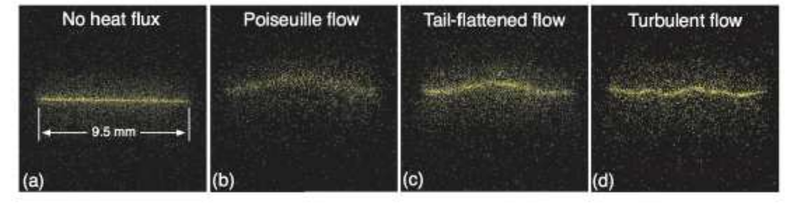
- Dashed lines: with frozen normal component.
- Solid lines: with dynamic normal component.

Coupled dynamics of superfluid turbulence



- Flattening of normal component profile in the middle of channel.

Recent experimental observation



Marakov *et al.*, Phys. Rev. B **91**, 094503 (2015).

Summary and perspectives

- We propose inhomogeneous equation for the vortex line density

$$\frac{\partial \mathcal{L}}{\alpha \partial t} - \frac{C_{flux}}{2\kappa} V_{ns} \frac{\partial V_s}{\partial y} = \frac{C_{prod}}{\kappa^2} \sqrt{\mathcal{L}} V_{ns}^3 - C_{dec} \kappa \mathcal{L}^2.$$

- The vortex line density attains maximum near boundary where counter flow is minimum.
- Profiles for vortex lines density and counter flow velocity remains same for both frozen and dynamic normal component.
- With coupled dynamics the parabolic normal component profile becomes flat near middle of the box on increase of external applied pressure (not consistent with the experimental observation).
- Need to include convective term in the normal component dynamical equation.

Thank you for your kind attention!