Dynamical phase transitions in SYK-like models and higher dimensional generalization

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Part 1:
Solvable model for a dynamical quantum phase transition from fast to slow scrambling

Work done with Ehud Altman (UC Berkeley)

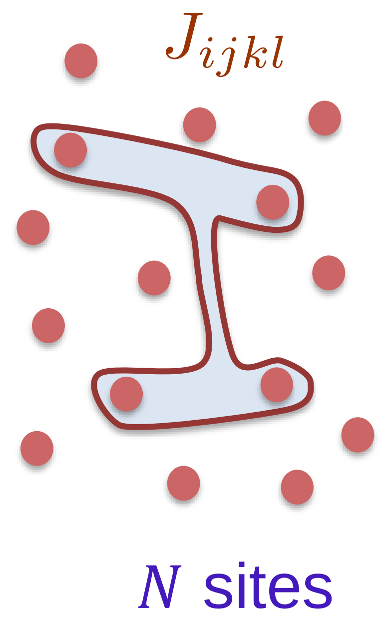
SB & E. Altman, Phys. Rev. B **95**, 134302 (2017)

Sachdev-Ye-Kitaev (SYK) model

Sachdev & Ye, PRL (1993)
Kitaev, KITP (2015)
Sachdev, PRX (2015)

$$H_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l - \mu \sum_{i}^{N} c_i^\dagger c_i$$

$$P(J_{ijkl}) \sim e^{-\frac{|J_{ijkl}|^2}{J^2}}$$



- Solvable in strong coupling for large N
- Emergent near-conformal symmetry at low energy.
- 'Maximally chaotic'
Quantum chaos or 'scrambling' with Lyapunov exponent, $\lambda_L = 2\pi T$

'Upper bound' to quantum chaos as in a black hole

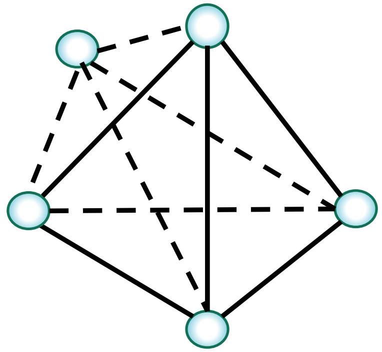
Maldacena, Shenker & Stanford (2016)

Solvable model for thermalization

Kitaev → Solvable model for holography (?)

Contrast with quadratic infinite range model (model for quantum dot)

$$H = \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} c_i^\dagger c_j$$



$$P(t_{ij}) \sim e^{-\frac{|t_{ij}|^2}{t^2}}$$

- Fermions occupying states of a $N \times N$ random matrix.
- No thermalization or chaos in the many-body sense.

Add weak interaction →

- Fermi liquid state at infinite N.

Quasi-particle lifetime → $\tau \sim 1/T^2$

Expectation, Lyapunov exponent $\lambda_L \sim \frac{1}{\tau} \sim T^2$

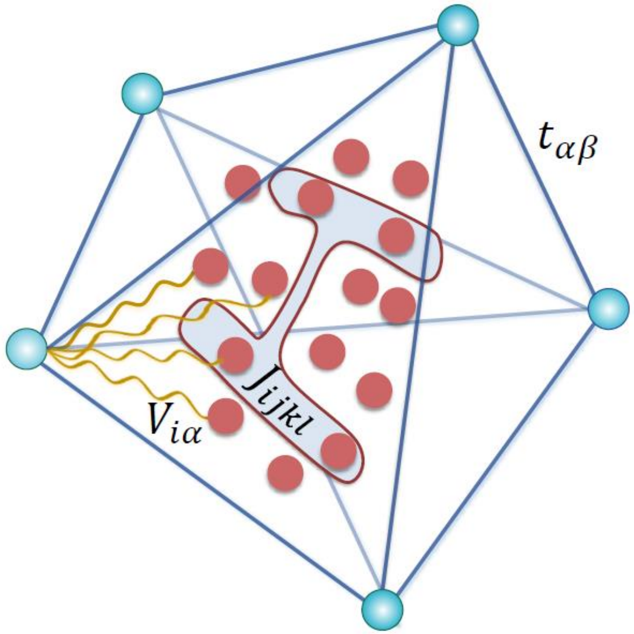
** Many body localization (MBL) at low T for finite N.

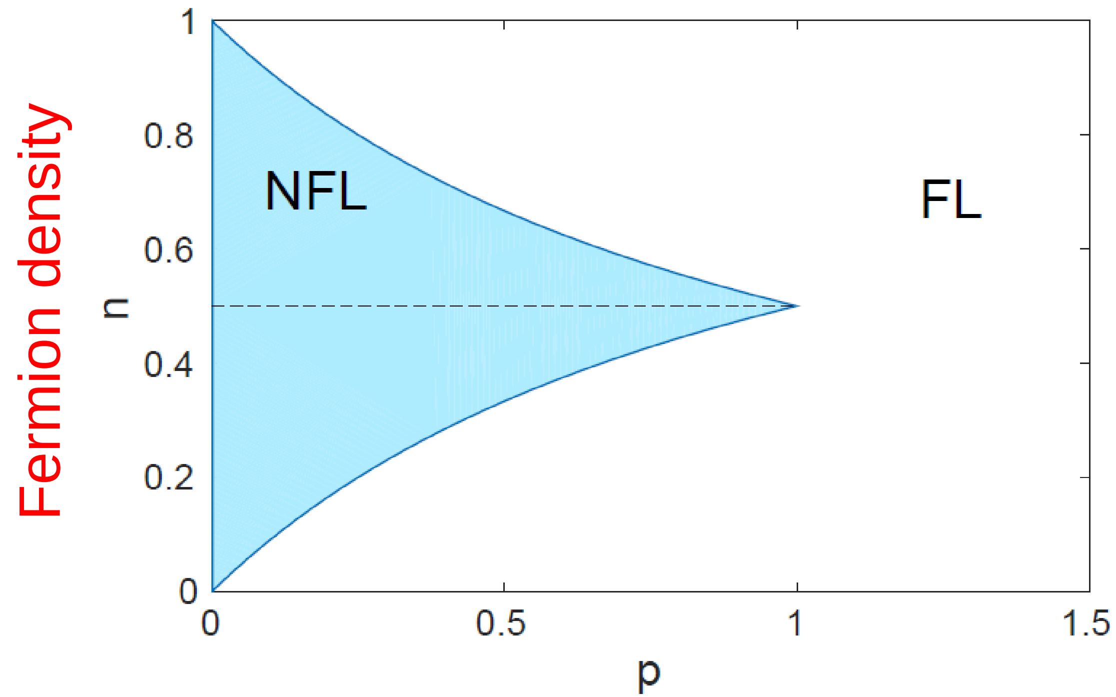
Altshuler, Gefen, Kamenev & Levitov, PRL (1997) $\lambda_L = 0$

Part 1: Solvable model with a quantum critical point two distinct quantum chaotic fixed points, SYK and Fermi-liquid.

Classifying phases and phase transitions in terms of quantum chaos?

How spectrum and quantum chaos evolve?





Two-species fermion model

Ratio of number of sites of two species

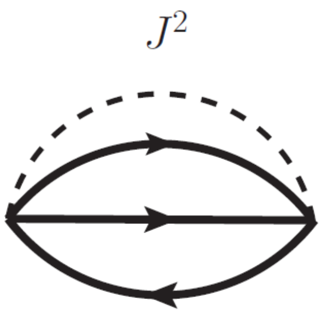
Saddle point and emergent reparameterization symmetry

- Disordered averaged saddle point for $N \to \infty$

$$G^{-1}(\omega) = \cancel{\omega} + \mu - \Sigma(\omega)$$

$$\Sigma(\tau) = -J^2 G^2(\tau) G(-\tau)$$

$$\hat{\Sigma}(\omega) = \Sigma(\omega) - \mu$$



- Reparameterization symmetry at low-energy $(\omega, T \ll J)$

$$\int d\tau \, G(\tau, \tau_1) \Sigma(\tau_1, \tau') = -\delta(\tau - \tau')$$

$$\tau = f(\sigma)$$

$$f'(\sigma) = \frac{\partial f}{\partial \sigma}$$

$$\tilde{G}(\sigma_1, \sigma_2) = [f'(\sigma_1) f'(\sigma_2)]^{1/4} \, G(f(\sigma_1), f(\sigma_2))$$

$$\tilde{\Sigma}(\sigma_1, \sigma_2) = [f'(\sigma_1) f'(\sigma_2)]^{3/4} \, \hat{\Sigma}(f(\sigma_1), f(\sigma_2))$$

→ Power-law solution for $G(\omega)$

Non Fermi liquid (SYK) fixed point

Sachdev, PRX (2015)

$$G_R(\omega) = \Lambda \frac{\mathrm{e}^{-i\left(\frac{\pi}{4}+\theta\right)}}{\sqrt{J\omega}} \rightarrow G(\tau) \sim -\frac{\mathrm{sgn}(\tau)}{\sqrt{J|\tau|}}$$

$$\Sigma_R(\omega) \sim -\Lambda^3 e^{i\left(\frac{\pi}{4}+\theta\right)} \cos 2\theta \sqrt{J\omega}$$

○ Spectral asymmetry, $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$

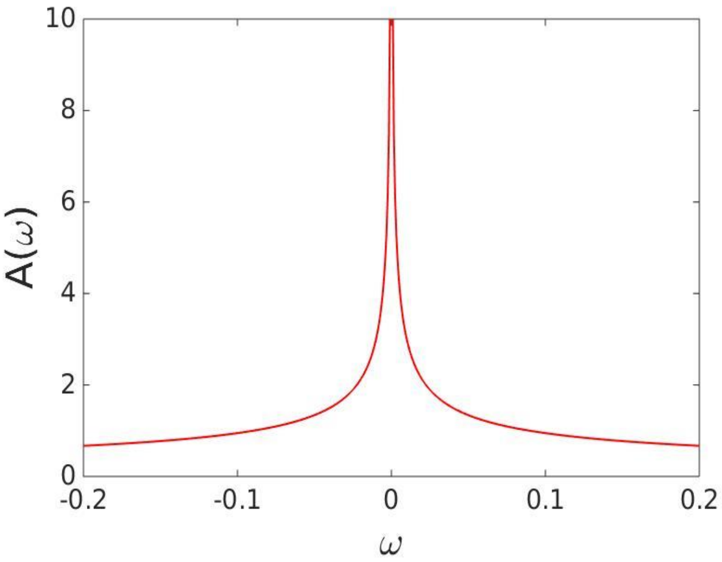
→ $\mathrm{Im} G_R(\omega) < 0$

$$\Lambda = \left(\frac{\pi}{\cos 2\theta}\right)^{1/4}$$

○ Diverging DOS for $\omega \rightarrow 0$ at $T = 0$

Λ, strength of the divergence

Half filling, $\theta = 0$



Quantum chaos in SYK model

Kitaev, KITP (2015)
Polchinski & Rosenhaus (2016)
Maldacena & Stanford (2016)

Out-of-time-order correlation

$$\langle\, c_i^\dagger(t) c_j(0) c_i^\dagger(t) c_j(0) \rangle \sim 1 - \left(\frac{\beta J}{N}\right) e^{\lambda_L t}$$

Srambling time

$$\lambda_L = 2\pi T \qquad t_* \sim \frac{1}{\lambda_L} \ln N \qquad N \to \ 1/\hbar$$

Fastest scrambler! Maximally chaotic.

Solvable model
of thermalization!

Upper bound to quantum chaos

Maldacena, Shenker & Stanford (2015)

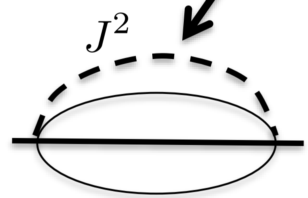
How to drive a phase transition out of this maximally chaotic non-Fermi liquid fixed point?

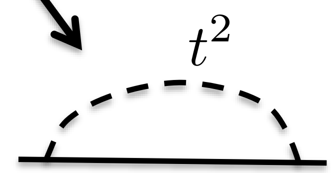
Naive attempt: add a quadratic term

$$\mathcal{H}_{SYK} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{N}} \sum_{ij} t_{ij} c_i^\dagger c_j$$

$$G^{-1}(\omega) = \omega - \Sigma_J(\omega) - t^2 G(\omega)$$

$$\Sigma_J(\tau) = -J^2 G^2(\tau) G(-\tau) =$$





The ansatz $G(\omega) \sim 1/\sqrt{\omega}$ is not self-consistent in the limit $\omega \to 0$

$\Sigma_J(\omega) \sim \sqrt{J\omega}$

$$G^{-1}(\omega) \sim \omega - \sqrt{J\,\omega} - \frac{t^2}{\sqrt{\omega}}$$

The free fermion ansatz is self consistent: $G(\omega) \sim -i/t$

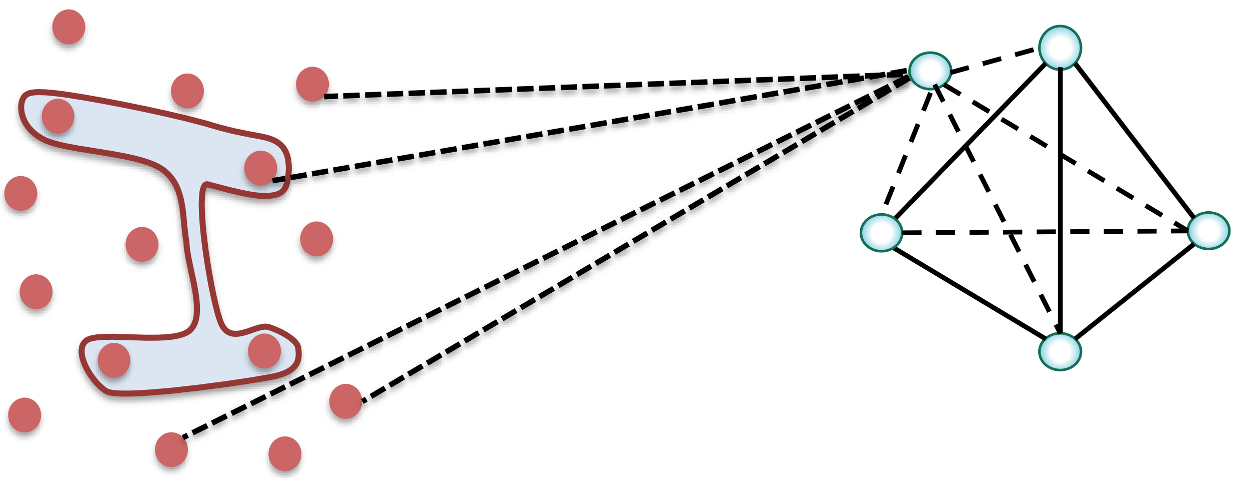
Quadratic interaction is relevant. Always a Fermi liquid. No transition!

Consider a model with two types of sites

N sites:
SYK coupling

M sites:
Random hopping

$$\overline{J_{ijkl}^2} = J^2 \qquad \overline{V_{i\alpha}^2} = V^2 \qquad \overline{t_{\alpha\beta}^2} = t^2$$



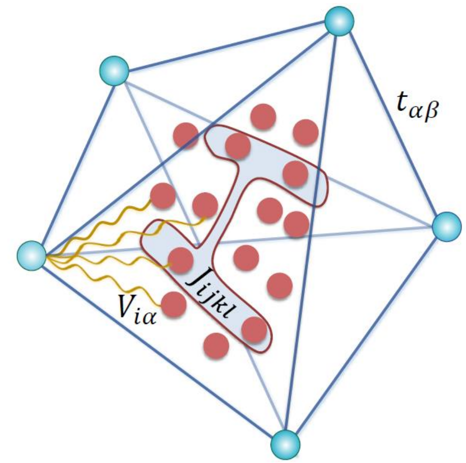
$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{M}} \sum_{\alpha\beta} t_{\alpha\beta} \psi_\alpha^\dagger \psi_\beta - \mu\left(\sum_i c_i^\dagger c_i + \sum_\alpha \psi_\alpha^\dagger \psi_\alpha\right)$$

$$+ \frac{1}{(MN)^{1/4}} \sum_{i\alpha} \left(V_{i\alpha} c_i^\dagger \psi_\alpha + h.c.\right)$$

Saddle point equations at large N

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{N}} \sum_{\alpha\beta} t_{\alpha\beta} \psi_\alpha^\dagger \psi_\beta + \frac{1}{(MN)^{1/4}} \sum_{i\alpha} (V_{i\alpha} c_i^\dagger \psi_\alpha + h.c.)$$

$$-\mu\Big(\sum_i c_i^\dagger c_i + \sum_\alpha \psi_\alpha^\dagger \psi_\alpha\Big)$$

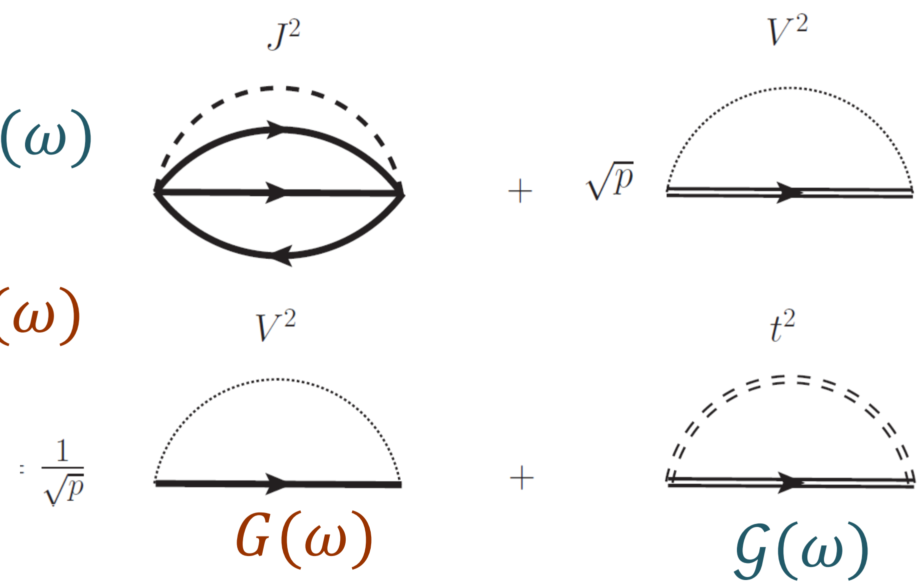


$p = M/N$ $\qquad\qquad$ $N, M \to \infty$

$$G^{-1}(\omega) = \omega + \mu - \Sigma_J(\omega) - V^2\sqrt{p}\,\mathcal{G}(\omega)$$

$$\mathcal{G}^{-1}(\omega) = \omega + \mu - t^2\mathcal{G}(\omega) - \frac{V^2}{\sqrt{p}}\,G(\omega)$$

$$\Sigma_J(\tau) = -J^2 G^2(\tau) G(-\tau)$$

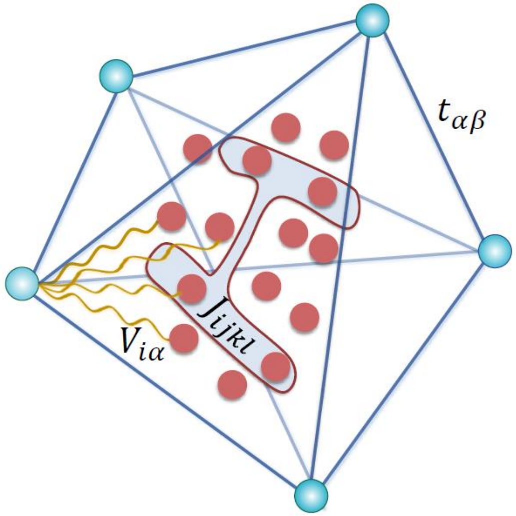


Non Fermi liquid fixed point

$$G^{-1}(\omega) = \cancel{\omega + \mu} - \Sigma_{\mathrm{J}}(\omega) - V^2\sqrt{p}\,\mathcal{G}(\omega)$$

$$\mathcal{G}^{-1}(\omega) = \cancel{\omega + \mu - t^2\mathcal{G}(\omega)} - \frac{V^2}{\sqrt{p}}\,G(\omega)$$

$$\Sigma_{\mathrm{J}}(\tau) = -J^2G^2(\tau)G(-\tau)$$



→ Emergent reparameterization symmetry

$$\tau = f(\sigma)$$

$$\tilde{G}(\sigma_1, \sigma_2) = [f'(\sigma_1)f'(\sigma_2)]^{\Delta_c}\,G(f(\sigma_1), f(\sigma_2))$$

$$\tilde{\mathcal{G}}(\sigma_1, \sigma_2) = [f'(\sigma_1)f'(\sigma_2)]^{\Delta_\psi}\,\mathcal{G}\big(f(\sigma_1), f(\sigma_2)\big)$$

$$\sim \tilde{\Sigma}(\sigma_1, \sigma_2)$$

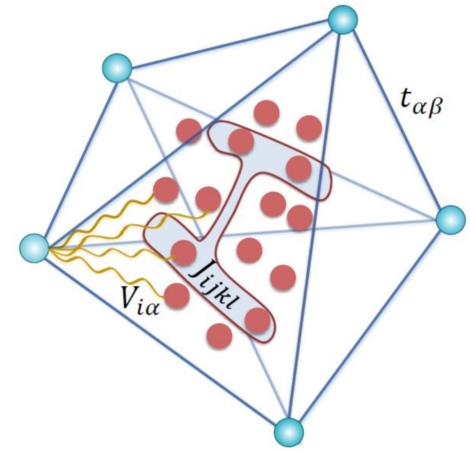
Scaling dimension

$$\Delta_c = \frac{1}{4}$$

$$\Delta_\psi = \frac{3}{4}$$

→ Power-law solution

Half filling: Non-Fermi liquid



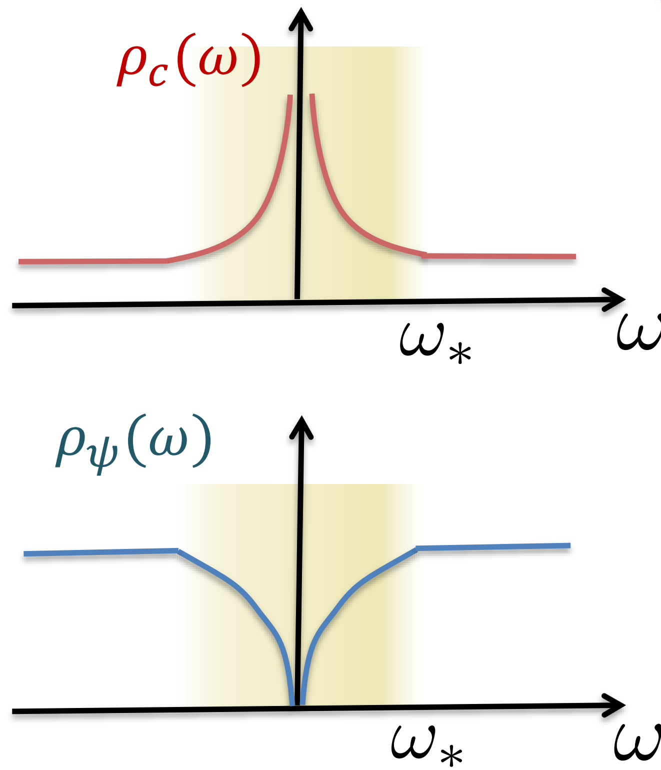
$\mu = 0$

→ Solution at $T = 0$, for $p = \frac{M}{N} < 1$

$$G_R(\omega) \sim \frac{(1-p)^{\frac{1}{4}}}{\sqrt{J\omega}} e^{-i\pi/4}$$

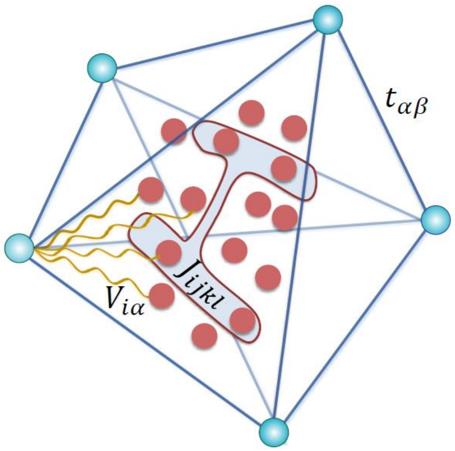
$$\mathcal{G}_R(\omega) \sim -\frac{\sqrt{p}}{(1-p)^{\frac{1}{4}}} \frac{\sqrt{J\omega}}{V^2} e^{i\pi/4}$$

$$\omega_* = \frac{V^4(1-p)^{1/2}}{t^2 J p}$$



Weight and bandwidth of the singularity in $G_R(\omega)$ vanishes continuously as $p \to p_c = 1$

Half filling: Fermi-liquid



- Solution for $p = M/N > 1$

$$G_R(\omega) \simeq -i \frac{1}{\sqrt{p(p-1)}} \frac{t}{V^2}$$

$$\mathcal{G}_R(\omega) \simeq -i \sqrt{\frac{p-1}{p}} \frac{1}{t}$$

→ Constant DOS for $\omega < \omega_* \approx \left(\frac{V^2}{t}\right) \sqrt{p(p-1)}$

Self-energy, $\mathrm{Im}\Sigma_{\mathrm{J}}(\omega) \sim -\left(\frac{J^2 t^3}{V^6}\right) \frac{1}{(p(p-1))^{3/2}}\ \omega^2$

- Free fermion fixed point, emergent reparameterization symmetry

$$\tilde{G}(\sigma_1, \sigma_2) = [f'(\sigma_1) f'(\sigma_2)]^{1/2}\, G\big(f(\sigma_1), f(\sigma_2)\big) \sim \tilde{\mathcal{G}}(\sigma_1, \sigma_2)$$

→Critical point at $p = M/N = 1$ separates NFL and FL fixed points

Quantum critical point (QCP) between NFL and FL

Half filling

$$\omega^*_{NFL} \sim (1-p)^{\frac{1}{2}}$$

$$\omega^*_{FL} \sim (p-1)$$

$$G_R(\omega) \sim (1-p)^{\frac{1}{4}}/\sqrt{\omega}$$

$$\mathcal{G}_R(\omega) \sim \sqrt{\omega}\,/(1-p)^{\frac{1}{4}}$$

$$G_R(\omega) \sim 1/(p-1)^{\frac{1}{2}}$$

$$\mathcal{G}_R(\omega) \sim (p-1)^{\frac{1}{2}}$$

$p = 1$

p

Away from half filling

$$G_R(\omega) = \Lambda \frac{e^{-i\left(\frac{\pi}{4}+\theta\right)}}{\sqrt{J\omega}}$$

$$\mathcal{G}_R(\omega) = -\frac{\sqrt{p}}{V^2\Lambda}\sqrt{J\omega}\, e^{i\left(\frac{\pi}{4}+\theta\right)}$$

$$\Lambda = \left(\frac{\pi(1-p)}{\cos 2\theta}\right)^{1/4}$$

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ ← Spectral asymmetry

What controls θ ?

$\theta = \theta(n) \leftrightarrow$ "k_F"

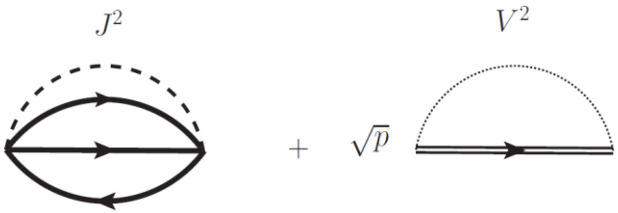
n, fermion density

Luttinger theorem for the NFL →

Luttinger theorem for the NFL

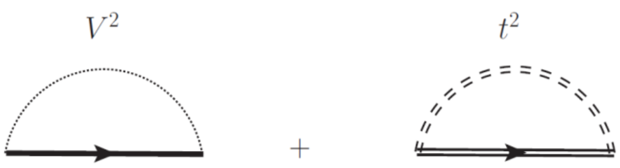
$$n = \frac{i}{1+p}\,[\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(\frac{\partial \ln G}{\partial\omega} + p\,\frac{\partial \ln \mathcal{G}}{\partial\omega}\right)e^{i\omega 0^+} - \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left(G\,\frac{\partial\Sigma_c}{\partial\omega}\cancel{+}\,p\,\frac{\partial\Sigma_\psi}{\partial\omega}\right)e^{i\omega 0^+}]$$

$\Sigma_c =$



$$\Lsh = (1-p)\,\frac{\sin 2\theta}{4}$$

$\Sigma_\psi = \frac{1}{\sqrt{p}}$



Georges et al. (2001)

$$n = \frac{1}{1+p}\left[\left(\frac{1}{2} - \frac{\theta}{4}\right) + p\left(\frac{1}{2} + \frac{\theta}{4}\right) - (1-p)\,\frac{\sin 2\theta}{4}\right]$$

Zero-temperature NFL-FL phase boundary

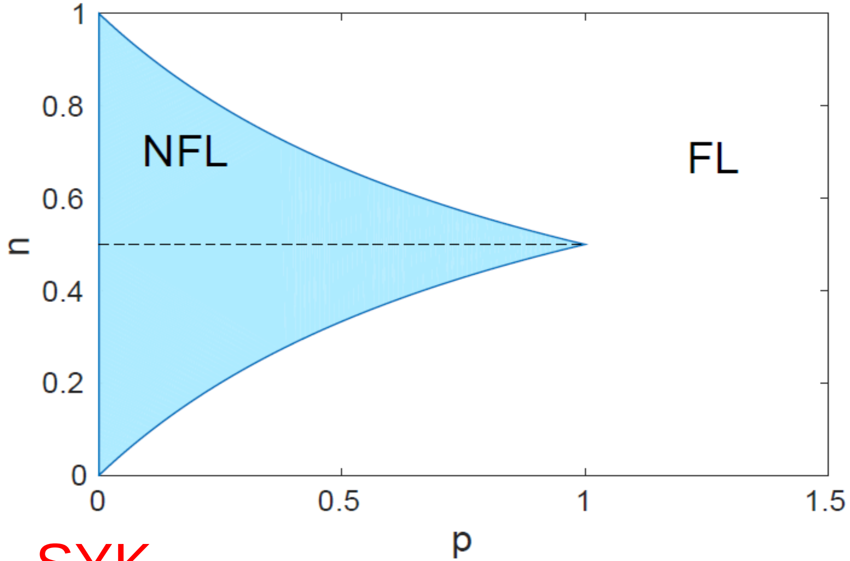
$$n = \frac{1}{1+p}\left[\left(\frac{1}{2} - \frac{\theta}{4}\right) + p\left(\frac{1}{2} + \frac{\theta}{4}\right) - (1-p)\frac{\sin 2\theta}{4}\right]$$

$-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$ →

Allowed density range

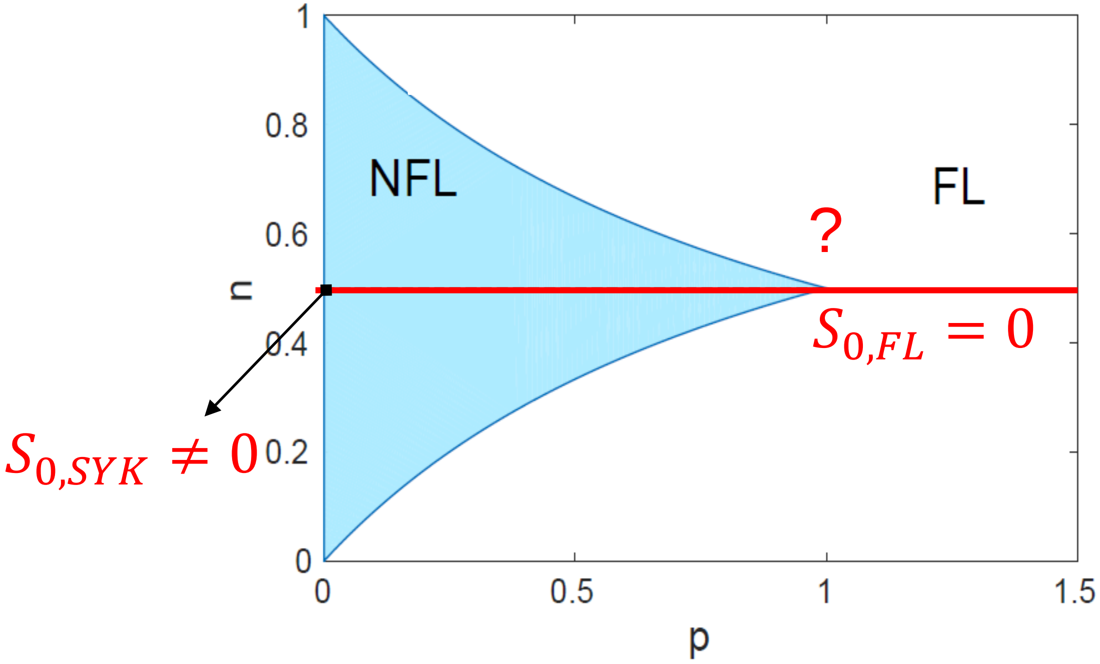
$$\frac{p}{1+p} \leq n \leq \frac{1}{1+p}$$

→ NFL-FL phase boundary



SYK

Zero-temperature entropy



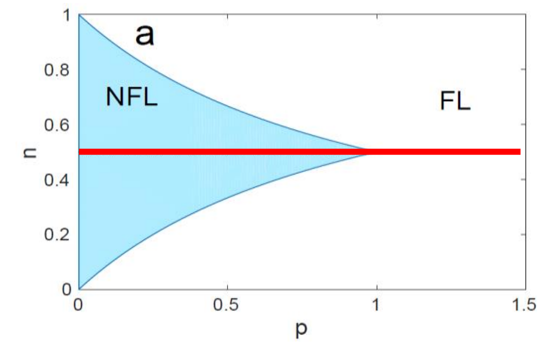
What happens to the residual entropy across QCP?

Low-temperature entropy

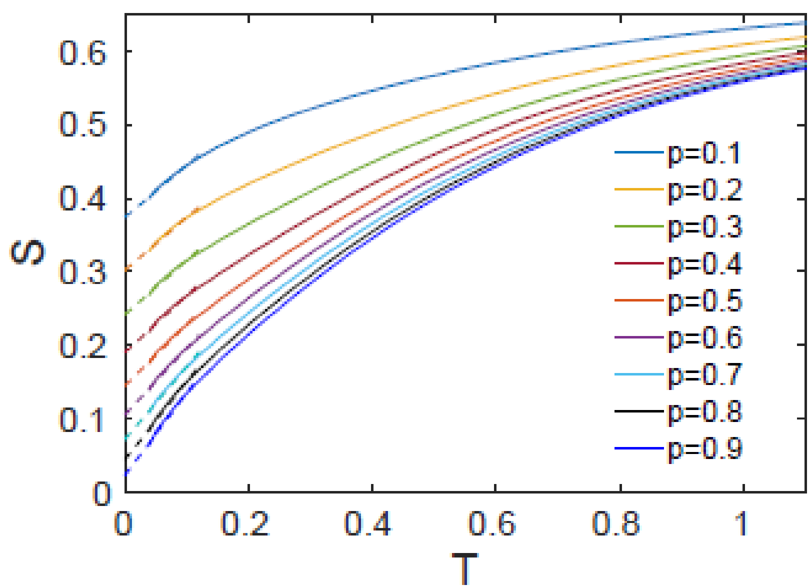
Saddle-point free-energy → $F = -T\,\overline{\ln Z} = -T \lim_{n\to 0} \frac{\overline{Z^n} - 1}{n}$

→ Entropy $S = -\frac{\partial F}{\partial T}$

Half filling

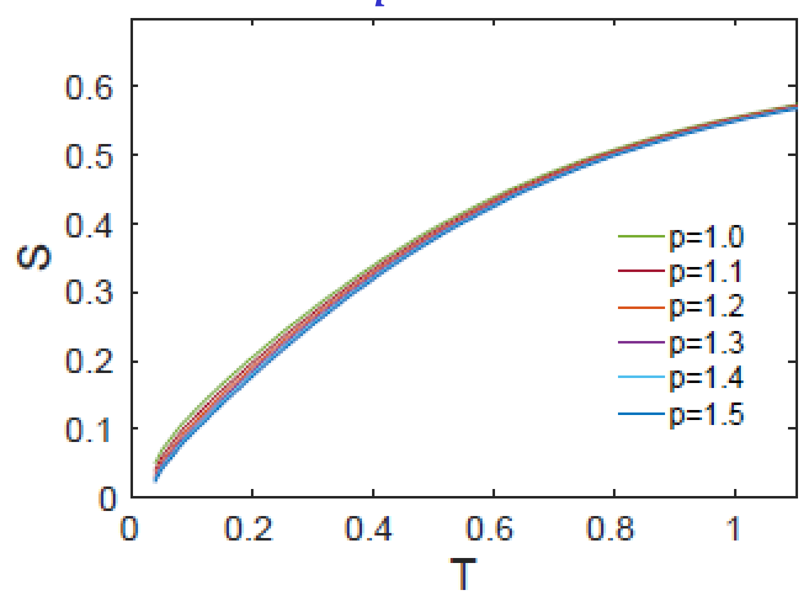


$p < 1$



NFL

$p \geq 1$

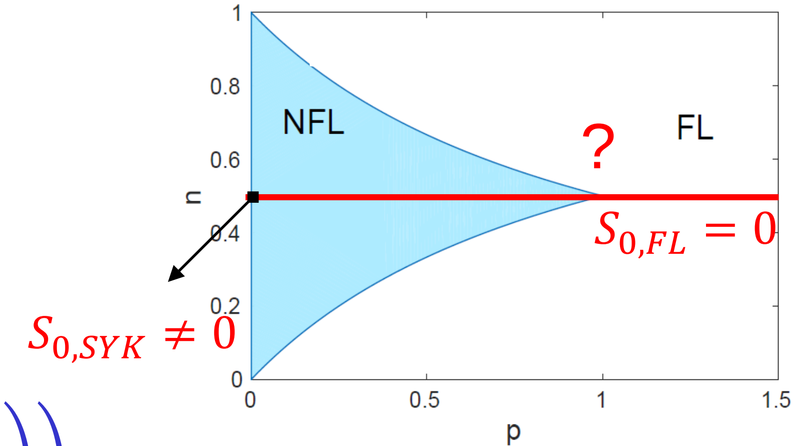


FL

Zero-temperature entropy

Thermodynamic integration

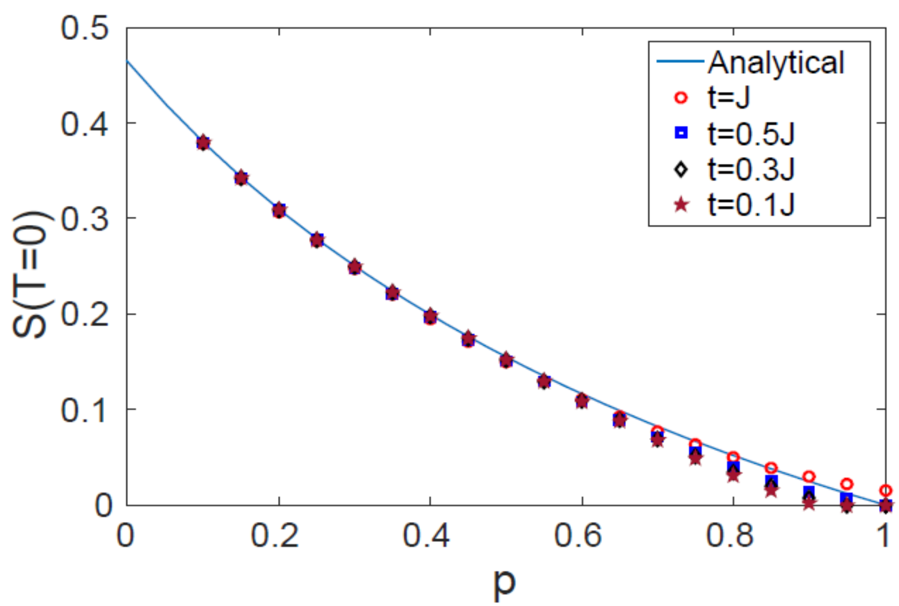
$$\left(\frac{\partial S}{\partial n}\right)_{T=0} = -\left(\frac{\partial \mu}{\partial T}\right)_n = -\ln\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$$



$$S_0(n) = \cancelto{0}{S(n_0)} + \int_{-\infty}^{\infty} dn \ln\left(\tan\left(\frac{\pi}{4}+\theta(n)\right)\right)$$

Luttinger theorem → $\theta(n)$

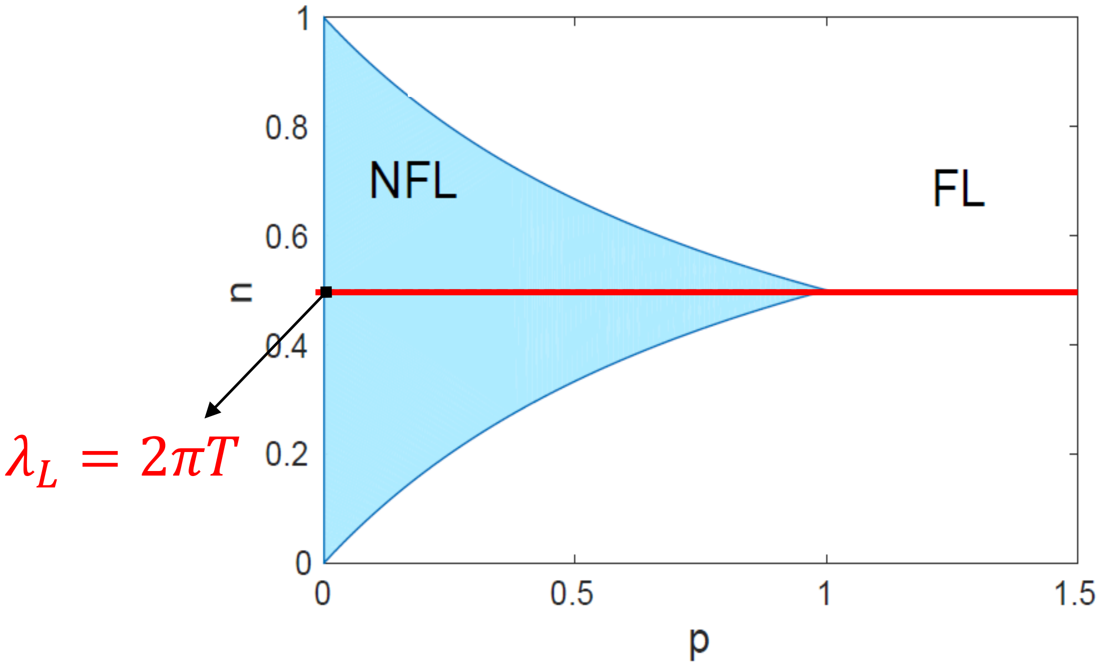
$$S_0(n=1/2) = \frac{1-p}{1+p}\, S_{0,SYK}$$



Zero-T entropy vanishes continuously at the transition

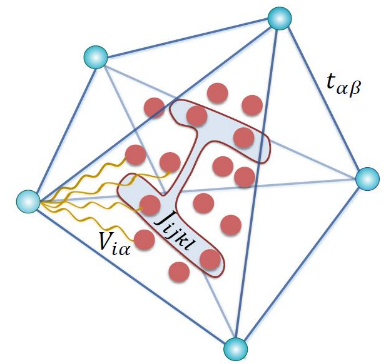
→ Change of geometry in dual gravity across QCP?

Quantum chaos across QCP?



Fast to slow scrambling

$$\mathcal{H} = \frac{1}{4!}\sum_{ijkl} J_{ijkl}\chi_i\chi_j\chi_k\chi_l + \frac{i}{2!}\sum_{\alpha\beta} t_{\alpha\beta}\eta_\alpha\eta_\beta + i\sum_{i\alpha} V_{i\alpha}\chi_i\eta_\alpha$$



$c_i \to \chi_i$

$\psi_\alpha \to \eta_\alpha$

$$p = M/N$$

Out-of-time-order (OTO) correlation

Kitaev, KITP (2015)

$p = 0$ $$\overline{\langle \chi_i(t)\chi_j(0)\chi_i(t)\chi_j(0)\rangle} \simeq f_0 - \frac{f_1}{N}e^{\lambda_L t} + \cdots$$

$$t \sim t^* = \left(\frac{1}{\lambda_L}\right)\ln N$$

$$\lambda_L = 2\pi T$$

Two OTO correlators

$$F_1(t_1, t_2) \sim \overline{\langle \chi_i(t)\chi_j(0)\chi_i(t)\chi_j(0)\rangle}$$

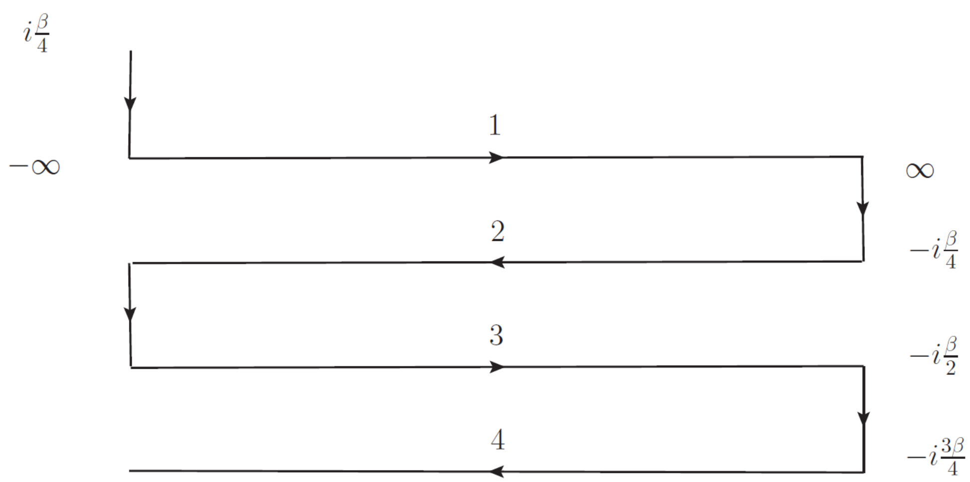
$$F_2(t_1, t_2) \sim \overline{\langle \eta_\alpha(t)\chi_i(0)\eta_\alpha(t)\chi_i(0)\rangle}$$

$$F_1(t_1,t_2) = \frac{1}{N^2}\sum_{ij} \overline{\mathrm{Tr}[y\chi_i(t_1)y\chi_j(0)y\chi_i(t_2)y\chi_j(0)]}$$

$$F_2(t_1,t_2) = \frac{1}{NM}\sum_{ij} \overline{\mathrm{Tr}[y\eta_\alpha(t_1)y\chi_i(0)y\eta_\alpha(t_2)y\chi_i(0)]}$$

$$y^4 = e^{-\beta\mathcal{H}}/Z$$

Keldysh contour



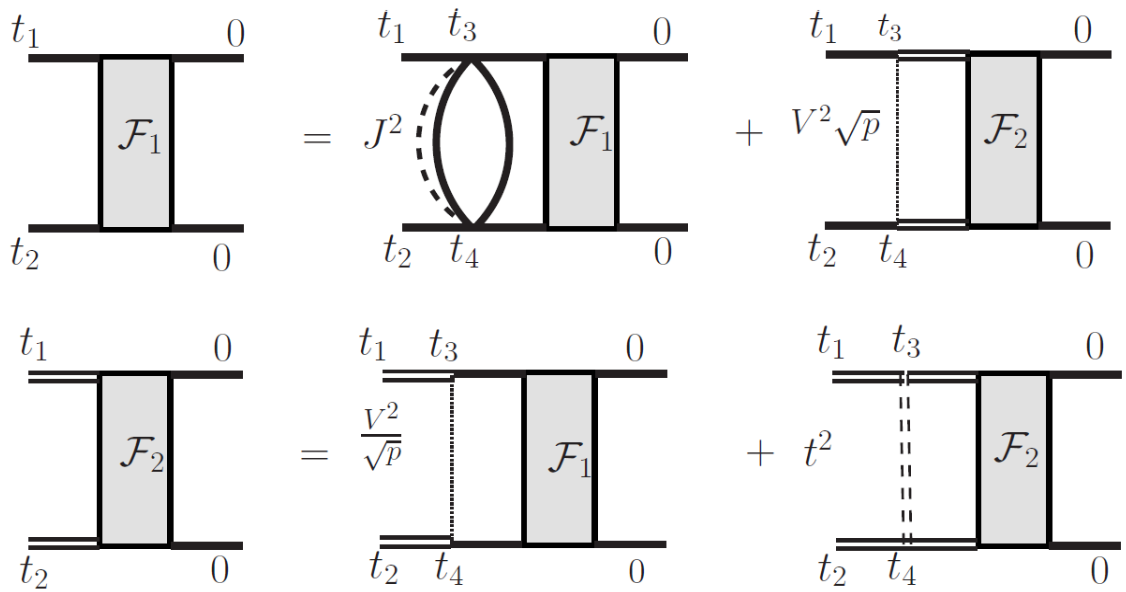
Kitaev (2015); Stanford (2016); Aleiner, Faoro & Ioffe (2016); Haehl, Loganayagam & Rangamani (2016)

$$F_1(t_1,t_2) = \frac{1}{N^2}\sum_{ij} \overline{\mathrm{Tr}[y\chi_i(t_1)y\chi_j(0)y\chi_i(t_2)y\chi_j(0)]}$$

$$F_2(t_1,t_2) = \frac{1}{NM}\sum_{ij} \overline{\mathrm{Tr}[y\eta_\alpha(t_1)y\chi_i(0)y\eta_\alpha(t_2)y\chi_i(0)]}$$

$$\simeq F^{(0)}(t_1,t_2) + \frac{1}{N}\mathcal{F}(t_1,t_2) + \mathcal{O}\left(\frac{1}{N^2}\right)$$

Ladder series →



$$\mathcal{F}_1(t_1,t_2) = \int dt_3 dt_4 [K_{11}(t_1,t_2,t_3,t_4)\mathcal{F}_1(t_3,t_4) + K_{12}(t_1,t_2,t_3,t_4)\mathcal{F}_2(t_3,t_4)]$$

$$\mathcal{F}_2(t_1,t_2) = \int dt_3 dt_4 [K_{21}(t_1,t_2,t_3,t_4)\mathcal{F}_1(t_3,t_4) + K_{22}(t_1,t_2,t_3,t_4)\mathcal{F}_2(t_3,t_4)]$$

→ Eigenvalue problem

$$\mathcal{K}|\mathcal{F}\rangle = k|\mathcal{F}\rangle$$

With eigenvalue $k=1$

→ Eigenvalue problem

$$\mathcal{K}|\mathcal{F}\rangle = k|\mathcal{F}\rangle$$

$$\mathcal{K} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \simeq \begin{pmatrix} 3J^2 G_R(t_{13}) G_R(t_{24}) G_{lr}^2(t_{34}) & -V^2\sqrt{p}\, G_R(t_{13}) G_R(t_{24}) \\ -\dfrac{V^2}{\sqrt{p}} \mathcal{G}_R(t_{13}) \mathcal{G}_R(t_{24}) & -t^2 \mathcal{G}_R(t_{13}) \mathcal{G}_R(t_{24}) \end{pmatrix}$$

$$t_{13} = t_1 - t_3$$

Wightmann correlator $G_{lr}(t) \equiv iG(it + \beta/2)$

Chaos ansatz

$$|\mathcal{F}\rangle = \begin{pmatrix} \mathcal{F}_1(t_1, t_2) \\ \mathcal{F}_2(t_1, t_2) \end{pmatrix} = e^{\lambda_L \frac{t_1 + t_2}{2}} \begin{pmatrix} f_1(t_1 - t_2) \\ f_2(t_1 - t_2) \end{pmatrix}$$

→ Lyapunov exponent $\quad k(\lambda_L) = 1 \;\Rightarrow\; \lambda_L$

→Solve numerically

→ Use conformal Green's functions in NFL to solve the eigenvalue problem for $T \to 0$

→ Integral equation

$$\frac{3(1-p)}{4\pi}\frac{\left|\Gamma\left(\frac{1}{4}+\frac{h}{2}+iu\right)\right|^2}{\left|\Gamma\left(\frac{3}{4}+\frac{h}{2}+iu\right)\right|^2}\int_{-\infty}^{\infty}du'\left|\Gamma\left(\frac{1}{2}+i(u-u')\right)\right|^2 f_1(u')=\left(k-\frac{p}{k}\right)f_1(u)$$

$$h=\frac{\lambda_L}{2\pi T}$$

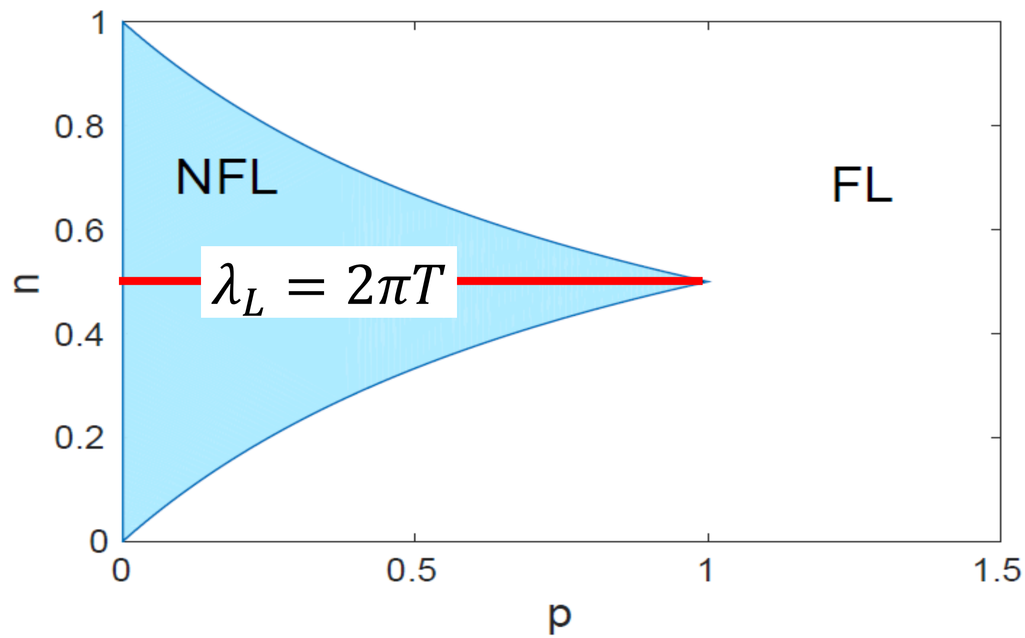
$h=1$, upper bound

→ Solution $f_1(u)=\left|\Gamma\left(\frac{1}{4}+\frac{h}{2}+iu\right)\right|^2$

$$\frac{3(1-p)}{1+2h}=\left(k-\frac{p}{k}\right)$$

$k=1 \quad \Rightarrow \quad h=1$

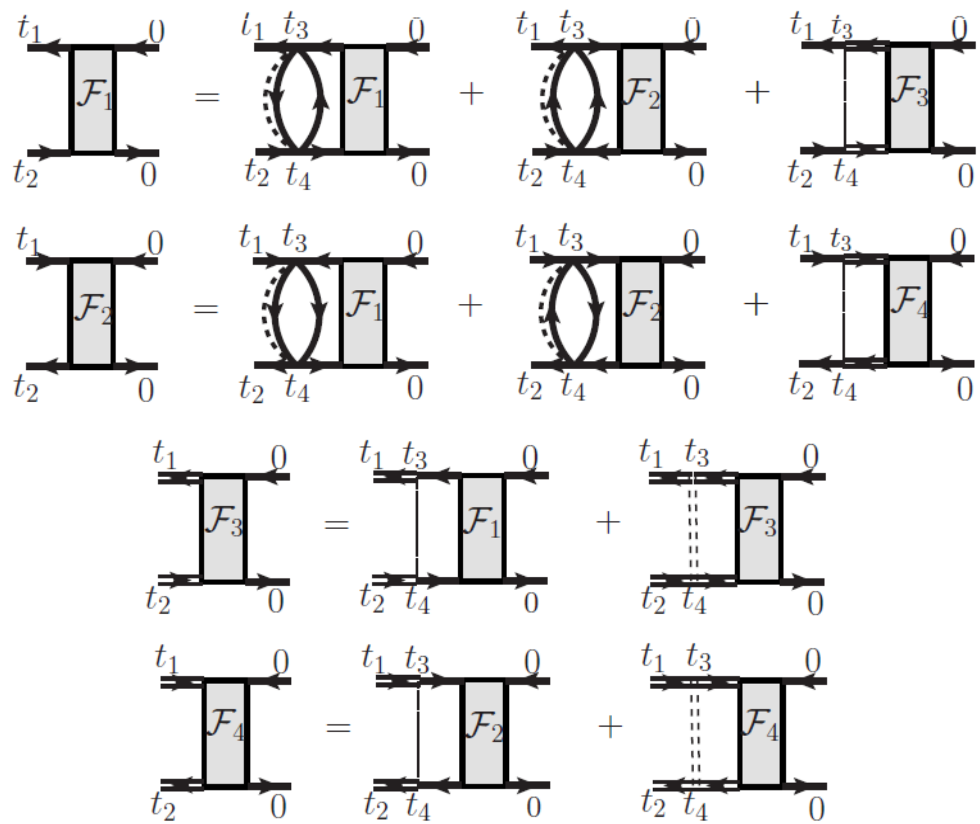
$\Rightarrow \quad \lambda_L=2\pi T$



Lyapunov exponent away from half filling

$$\mathcal{H} = \frac{1}{(2N)^{3/2}} \sum_{ijkl} J_{ijkl} c_i^\dagger c_j^\dagger c_k c_l + \frac{1}{\sqrt{N}} \sum_{\alpha\beta} t_{\alpha\beta} \psi_\alpha^\dagger \psi_\beta - \mu\Big(\sum_i c_i^\dagger c_i + \sum_\alpha \psi_\alpha^\dagger \psi_\alpha\Big)$$

$$+ \frac{1}{(MN)^{1/4}} \sum_{i\alpha} (V_{i\alpha} c_i^\dagger \psi_\alpha + h.c.)$$



$$\lambda_L = 2\pi T$$

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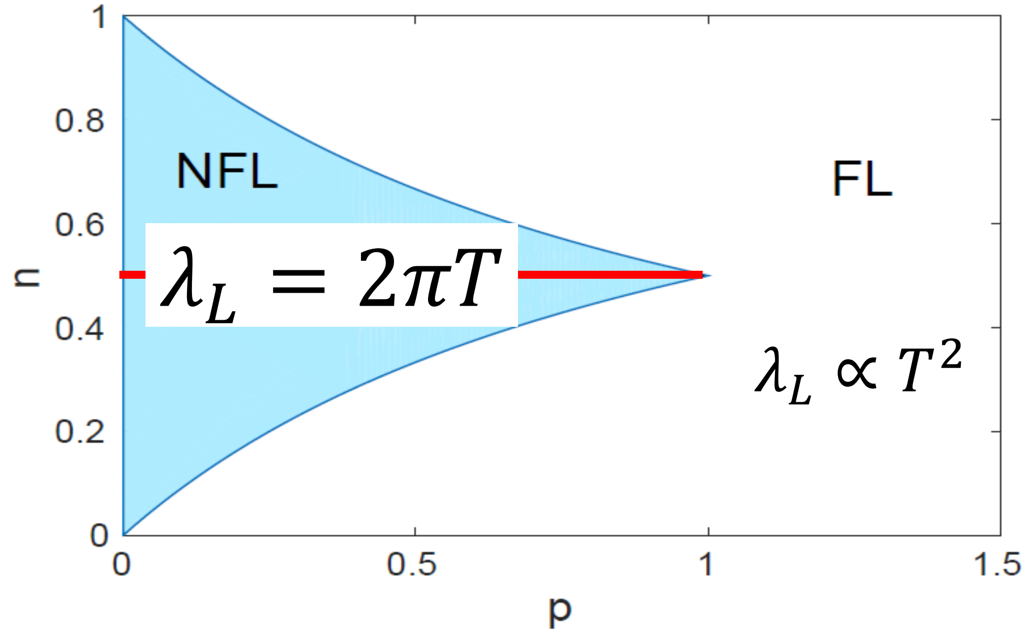
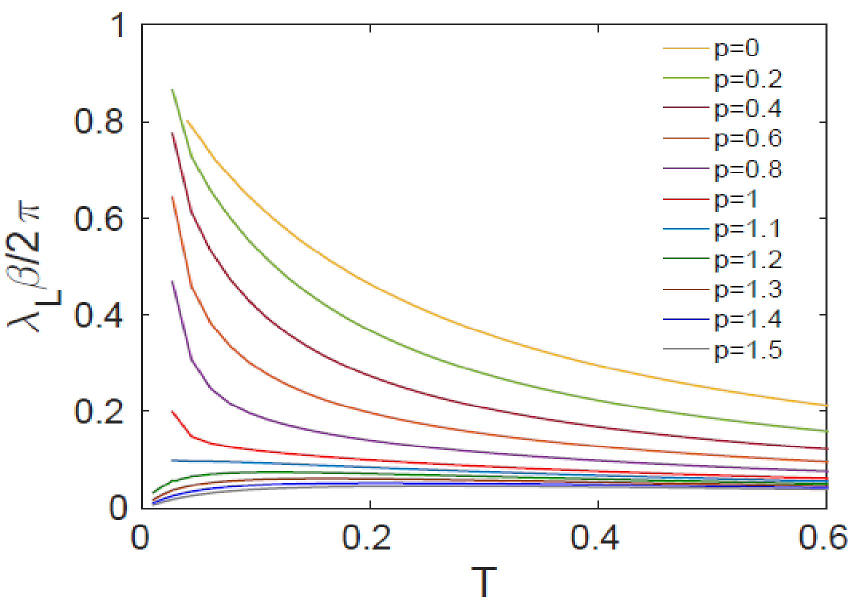
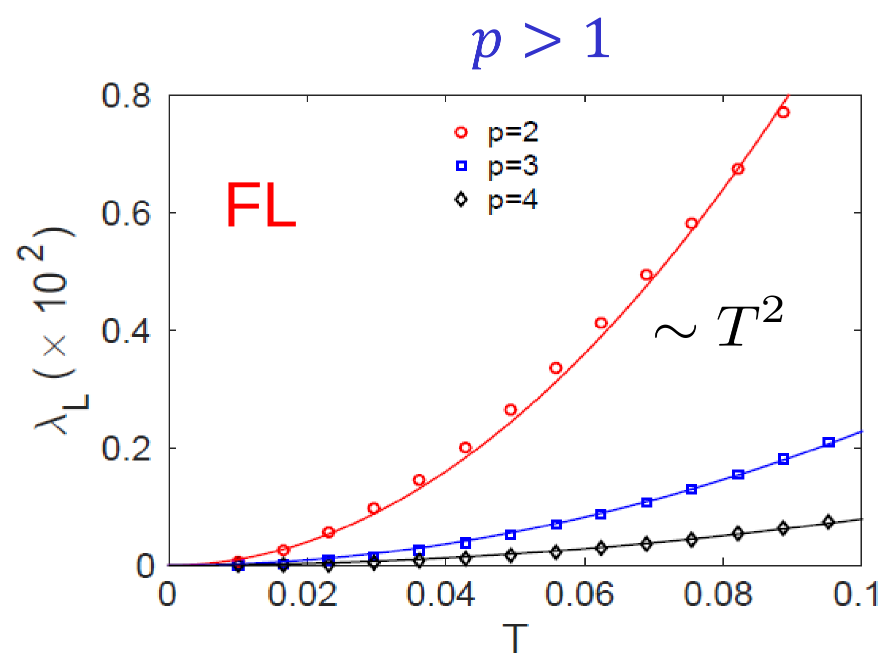




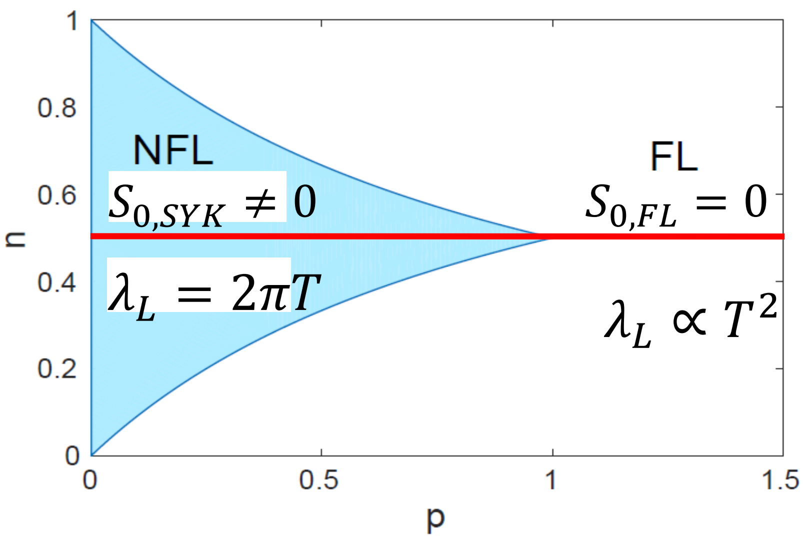


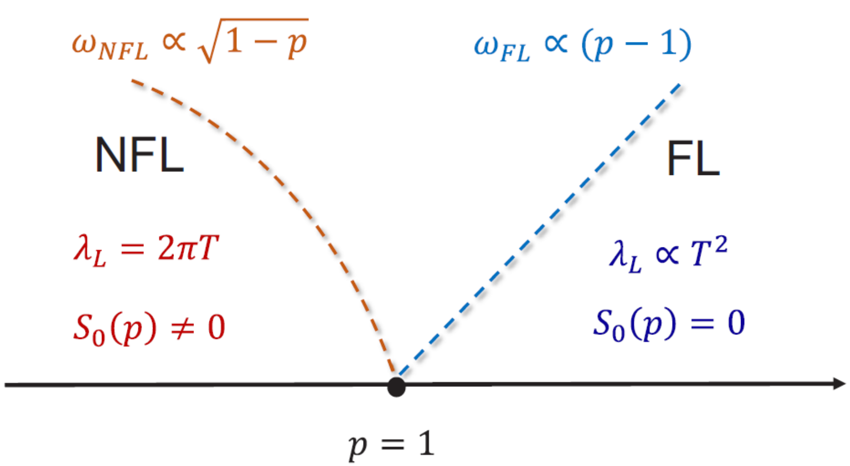
Transition from fast to slow scrambling

Part 1: Conclusions

Exactly Solvable model for a non-Fermi liquid to Fermi liquid transition.

- Spectral function
- Zero-temperature entropy
- Many-body quantum chaos,
  fast ($\lambda_L = 2\pi T$) to slow scrambling ($\lambda_L \propto T^2$).





- Theory for the critical point? New chaotic fixed point distinct from either SYK or FL?

- Extension to large-*N* description for MBL and MBL transition? No scrambling or power law scrambling.

- Holographic interpretation? Phase transition involving elimination of black hole?

→Quench across QCP.