Spacetime OPEs from the String Worldsheet

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Introduction

- The operator product expansion (OPE) is a basic feature of any local QFT

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\mathcal{O}_n(x_n)\rangle \sim \sum_k C_{12}^k(x_1-x_2)\,\langle \mathcal{O}_k(x_2)\ldots\mathcal{O}_n(x_n)\rangle$$

  as $|x_1 - x_2| \to 0$.
- For CFT's which admit holographic duals, the bulk theory should reproduce this. Indeed this has been analysed in detail providing important checks of AdS/CFT. [D'Hoker, Freedman, Skiba, (1998); Arutynov, Frolov, Petkou, (2000); Mathur, Rastelli, Freedman, D'Hoker, (2000)]
- Here we will study the OPE limit of holographic CFT correlators from the worldsheet description of the dual string theory. [Aharony, Komargodski, 2007]

Setup

- Consider (large N) CFTs that admit weakly coupled string duals. For e.g., those dual to strings in AdS_3 with NS-NS flux. [Maldacena, Ooguri; also see Rajesh's talk] or weakly curved $AdS_{d+1} \times W$ with R-R flux.
- Let $\mathcal{O}_{\Delta,q} \to$ scalar *single-trace* primary operators in the CFT with $\Delta \geq d/2$; $q \to$ additional (internal) global symmetry labels.
- These map to dual vertex operators in the worldsheet CFT which create single-particle states in the bulk

$$\mathcal{O}_{\Delta,q}(x) = \int d^2z \; \mathcal{V}_{\Delta,q}(x; z, \bar{z})$$

- $\mathcal{V}_{\Delta,q}(x; z, \bar{z}) \to$ level matched Virasoro primary with worldsheet scaling dimension $h = 2$.

Setup

- AdS/CFT relates n-point functions in the boundary CFT to integrated correlators of worldsheet vertex operators

$$\mathcal{A}_n = \left\langle \mathcal{O}_1(x_1)\mathcal{O}_2(0)\prod_{i=3}^{n-2}\mathcal{O}_i(x_i)\mathcal{O}_{n-1}(x_{n-1})\mathcal{O}_n(x_n)\right\rangle$$

$$= \int d^2z \left\langle \mathcal{V}_1(x_1;z)\hat{\mathcal{V}}_2(0;0)\prod_{i=3}^{n-2}\int d^2w_i\mathcal{V}_i(x_i;w_i)\hat{\mathcal{V}}_{n-1}(x_{n-1};1)\hat{\mathcal{V}}_n(x_n;\infty)\right\rangle$$

$\hat{\mathcal{V}}(x,z) \equiv \bar{c}c\mathcal{V}(x,z)$.

- We will analyse this in the limit $|x_1| \to 0$. Can't compute the full integral but will evaluate it in the small $|z|$ or worldsheet OPE regime. This will reproduce the *single-trace* terms in the boundary OPE.

Outline

- Review of Aharony-Komargodski Analysis
- Generalisation to spin ℓ exchange and external vectors
- Relating spacetime and worldsheet OPE coefficients
- Summary/ Future directions

Review of Aharony Komargodski Analysis

- The boundary CFT operators with $\Delta \geq d/2$ map to non-normalizable modes in the bulk AdS. The physical Hilbert space of the worldsheet CFT however consists of only normalizable vertex operators labelled by $\Delta = d/2 + 2is$, $s \in \mathbb{R}$.
- This requires performing an analytic continuation to analyse the boundary OPE limit.
- For normalizable vertex operators

$$\mathcal{V}_{\Delta_1,q_1}(x,z)\mathcal{V}_{\Delta_2,q_2}(0,0) = \sum_q \int_C d\Delta \int d^d x' F(z;x,x';\Delta_i,\Delta;q_i,q)\mathcal{V}_{\Delta,q}(x';0)$$

(ignoring contribution of worldsheet descendants)

- $F(z;x,x';\Delta_i,\Delta;q_i,q)$ is related to worldsheet 2 and 3-point functions.
- The contour C is along $\Delta = \frac{d}{2} + 2is$.

Remarks on Analytic Continuation

- We need to analytically continue to real values of Δ_1, Δ_2.
- Assume F to be an analytic function of the $\Delta's$ except at the location of some poles. The pole structure of F is known in specific cases [Maldacena, Ooguri, 2002].
- During analytic continuation some of these poles will be crossed giving rise to additional contributions. For now let's ignore these and assume that the structure of the OPE is preserved even after analytic continuation.

Worldsheet OPE

- Using worldsheet and boundary conformal symmetries

$$\mathcal{V}_1(x,z)\mathcal{V}_2(0,0) = \sum_q \int_C d\Delta \int d^d x' \frac{|z|^{-\tilde{h}}}{|x|^{\alpha}|x'|^{\beta}|x'-x|^{\gamma}} F^{(0,0,0)}_{12\Delta} \mathcal{V}_{\Delta,q}(x';0)$$

- $\tilde{h} = h(\Delta_1, q_1) + h(\Delta_2, q_2) - h(\Delta, q) = 4 - h(\Delta, q)$
- $F^{(0,0,0)}_{12\Delta} \rightarrow$ worldsheet OPE coefficient.
- Keeping only the leading terms in the $|x| \rightarrow 0$ limit,

$$\mathcal{V}_1(x,z)\mathcal{V}_2(0,0) = \sum_q \int_C d\Delta \frac{|z|^{-\tilde{h}}}{|x|^{\alpha+\beta+\gamma-d}} F^{(0,0,0)}_{12\Delta}\, \mathcal{V}_{\Delta,q}(0;0) \int \frac{d^d y}{|y|^{\beta}|y-\hat{x}|^{\gamma}}$$

- $\alpha + \beta + \gamma - d = \Delta_1 + \Delta_2 - \Delta.$

Saddle point calculation

- Plugging this in the spacetime CFT n-point function

$$\mathcal{A}_n = |x_1|^{-\Delta_1-\Delta_2} \sum_q \int d\ln|z| \int_C d\Delta |z|^{h(\Delta,q)-4} |x_1|^{\Delta} B(\Delta_i, q_i; \Delta, q)$$

where

$$B(\Delta_i, q_i; \Delta, q) \equiv 2\pi F^{(0,0,0)}_{12\Delta} \langle \bar{c} c \mathcal{V}_{\Delta,q}(0;0) X \rangle \int \frac{d^d y}{|y|^{\beta} |y - \hat{x}_1|^{\gamma}}$$

and X denotes the remaining $n-2$ vertex operator insertions.

Saddle point analysis

- The Δ integral admits a saddle point at $\Delta = \Delta_0$ such that

$$\partial_\Delta h(\Delta, q)|_{\Delta=\Delta_0} = -2\,\frac{\ln|x_1|}{\ln|z|}$$

- For the dominant values of z in the relevant approximation regime here, $\Delta_0 \geq d/2$. The contour C then needs to shifted. Again this can result in additional contributions from the poles of $B(\Delta_i, q_i; \Delta, q)$.
- The z integral can also be done via saddle approximation. The location of the saddle is determined by the equation

$$h(\Delta_0, q) = 2$$

- This precisely is the requirement for the intermediate vertex operator to be non-normalizable.

Saddle point analysis

- The saddle point calculation turns out to be exact in the small $|x_1|$ limit, all perturbative corrections around the saddle cancel.
- Thus apart from possible additional discrete contributions

$$\mathcal{A}_n = -2i\pi \sum_q \frac{|x_1|^{\Delta_0-\Delta_1-\Delta_2}}{\partial_\Delta h(\Delta, q)|_{\Delta=\Delta_0}} B(\Delta_i, q_i; \Delta_0, q)$$

- The $|x_1|$ dependence shows that this is indeed the expected form of the boundary correlator in the OPE limit for the exchange of a scalar *single-trace* operator of dimension Δ_0.
- Some of these additional discrete contributions arising during analytic continuation can be attributed to *multi-trace* operators in the boundary OPE. But this is not generically true. [Aharony, Komargodski, 2007]

Generalisation to spin ℓ exchange

- Now consider the contribution to the worldsheet OPE from operators of the form $\mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_\ell}(x,z)$ dual to a spin-ℓ operator in the boundary CFT

$$\mathcal{V}_1(x_1,z_1)\mathcal{V}_2(x_2,z_2) = \sum_q \int_C d\Delta\,|z_{12}|^{-\tilde{h}} F_{12\Delta}^{(0,0,\ell)} L_{\mu_1\cdots\mu_\ell}(x_1,x_2,x)\mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_\ell}(x,z_1)$$

- Under inversion of the boundary coordinates the form of the OPE is preserved if,

$$L_{\mu_1\cdots\mu_\ell}(x_1,x_2,x) \to |x_1|^{2\Delta_1}|x_2|^{2\Delta_2}|x|^{2(d-\Delta)}\left(J_{\nu_1}^{\mu_1}\cdots J_{\nu_\ell}^{\mu_\ell}\right)^{-1} L_{\nu_1\cdots\nu_\ell}(x_1,x_2,x)$$

with $J^{\mu\nu} \equiv J^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{x^\mu x^\nu}{x^2}$ and $J^{\mu\rho}(x)J_{\rho\nu}(x) = \delta^\mu_\nu$

Generalisation to spin ℓ exchange

- This implies

$$L_{\mu_1\cdots\mu_\ell}(x_1,x_2,x) \propto \left\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\tilde{\mathcal{O}}^{\mu_1\cdots\mu_\ell}_{d-\Delta}(x)\right\rangle$$

- $\tilde{\mathcal{O}}^{\mu_1\cdots\mu_\ell}_{d-\Delta}(x)$ is the shadow of the conformal primary $\mathcal{O}^{\mu_1\cdots\mu_\ell}_{\Delta}(x)$.

$$\tilde{\mathcal{O}}^{\mu_1\cdots\mu_\ell}_{d-\Delta}(x) \equiv \frac{k_{\Delta,\ell}}{\pi^{d/2}}\int d^dy\,\frac{1}{|x-y|^{2(d-\Delta)}}J^{\mu_1}_{(\nu_1}\cdots J^{\mu_\ell}_{\nu_\ell)}(x-y)\mathcal{O}^{\nu_1\cdots\nu_\ell}_{\Delta}(y)$$

- The x-dependence of $L_{\mu_1\cdots\mu_\ell}$ can then be fixed to give

$$\mathcal{V}_1(x_1,z_1)\mathcal{V}_2(x_2,z_2)$$

$$=\sum_q\int_C d\Delta\int d^dx\,\frac{|z_{12}|^{-\tilde{h}}\;\;Z_{\mu_1}\cdots Z_{\mu_\ell}}{|x_1-x_2|^{\alpha}|x_2-x|^{\beta}|x-x_1|^{\gamma}}F^{(0,0,\ell)}_{12\Delta}\mathcal{V}^{\mu_1\cdots\mu_\ell}_{\Delta,q}(x,z_1)$$

where, $Z^\mu(x_1,x_2,x_3)\equiv\frac{(x_1-x_3)^\mu}{(x_1-x_3)^2}-\frac{(x_2-x_3)^\mu}{(x_2-x_3)^2}$

Generalisation to spin ℓ exchange

- With $x_2 = 0, z_2 = 0$ and Taylor expanding around $|x_1| = 0$

$$\mathcal{V}_1(x_1, z_1)\mathcal{V}_2(0,0)$$

$$= \sum_q \int_C d\Delta \frac{|z_1|^{-\tilde{h}}}{|x_1|^{\alpha+\beta+\gamma+\ell-d}} F_{12\Delta}^{(0,0,\ell)} \mathcal{V}_{\Delta,q}^{\mu_1\cdots\mu_\ell}(0,0)\, G_{\mu_1\cdots\mu_\ell}(x_1)$$

$$G^{\mu_1\cdots\mu_\ell}(x_1) \equiv \int d^d y \; \frac{Z_{\mu_1}(\hat{x}_1, 0, y)\cdots Z_{\mu_\ell}(\hat{x}_1, 0, y)}{|y|^\beta \, |y - \hat{x}_1|^\gamma}$$

$$= \pi^{d/2} \lambda_\ell(\beta,\gamma) \frac{\Gamma\left(\frac{d-\beta}{2}\right)\Gamma\left(\frac{d-\gamma}{2}\right)\Gamma\left(\frac{\beta+\gamma-d}{2}+\ell\right)}{\Gamma\left(\frac{\beta}{2}+\ell\right)\Gamma\left(\frac{\gamma}{2}+\ell\right)\Gamma\left(\frac{2d-\beta-\gamma}{2}\right)} \left[\hat{x}_1^{\mu_1}\cdots\hat{x}^{\mu_\ell}\right]$$

- $\alpha = \Delta_1 + \Delta_2 + \Delta - d + \ell$, $\beta = \Delta_2 - \Delta_1 - \Delta + d - \ell$, $\gamma = \Delta_1 - \Delta_2 - \Delta + d - \ell$.

Generalisation to spin ℓ exchange

- Using this form of the worldsheet OPE and the previous saddle point analysis in a spacetime CFT 4-point function,

$$\left\langle \mathcal{O}_1(x_1)\mathcal{O}_2(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\right\rangle$$

$$= (-2i\pi)\sum_q \frac{|x_1|^{\Delta_0-\Delta_1-\Delta_2}}{\partial_\Delta h(\Delta,q)|_{\Delta=\Delta_0}}\bar{B}(\Delta_i,q_i;\Delta_0,q)$$

with

$$\bar{B} \equiv 2\pi F^{(0,0,\ell)}_{12\Delta_0}\left\langle \bar{c}c\mathcal{V}^{\mu_1\cdots\mu_\ell}_{\Delta_0,q}(0;0)\bar{c}c\mathcal{V}_3(x_3;1)\bar{c}c\mathcal{V}_4(x_4;\infty)\right\rangle G_{\mu_1\cdots\mu_\ell}(x_1)$$

- Again we recover the expected result in the OPE limit along with the correct tensor structure furnished by the factor $G_{\mu_1\ldots\mu_\ell}(x_1)$.

External vectors

- Now consider the 4-point function in the boundary CFT

$$\mathcal{A}_4 = \langle \mathcal{O}^\mu(x_1)\mathcal{O}^\nu(0)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle$$

  $\mathcal{O}^\mu(x) \rightarrow$ spin 1 primary with conformal dimension $d-1$ (conserved current).

- Let's consider the exchange of a scalar primary in the OPE between the two spin 1 operators.
- Extending the previous worldsheet analysis now to the case of external (spacetime) vectors now yields

$$\mathcal{A}_4 = -2i\pi \sum_q \frac{|x_1|^{\Delta-2(d-1)}}{\partial_\Delta h(\Delta,q)|_{\Delta=\Delta_0}} B^{\mu\nu}(\Delta_i, q_i; \Delta_0, q)$$

  where

$$B^{\mu\nu} \equiv 2\pi F^{(1,1,0)}_{12\Delta_0} \langle \bar{c}c\mathcal{V}_{\Delta,q}(0;0)\bar{c}c\mathcal{V}_3(x_3;1)\bar{c}c\mathcal{V}_4(x_4;\infty)\rangle \, H^{\mu\nu}(x_1)$$

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External vectors

$$H^{\mu\nu}(x) = \frac{\pi^{d/2}\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\frac{\Delta}{2}\right)\Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma\left(\frac{d-\Delta}{2}\right)\Gamma\left(\frac{d-\Delta}{2}\right)\Gamma\left(\Delta\right)}\left[\frac{-2\Delta(\Delta+2-2d)}{(d-\Delta)^2(\Delta+1-d)}\right]$$
$$\left[\hat{x}^\mu\hat{x}^\nu - \frac{(\Delta+1-d)}{(\Delta+2-2d)}\delta^{\mu\nu}\right]$$

- The x-dependence of $H^{\mu\nu}$ correctly reproduces structure of the OPE as expected in the boundary CFT.

Relating spacetime and worldsheet OPE coefficients

- Matching the worldsheet results with the expected form of the corresponding boundary CFT 4-point functions in the OPE regime we can also relate the worldsheet and spacetime OPE coefficients as

$$C^{(0,0,\ell)}_{12\Delta_0} = \left(\sqrt{\frac{4\pi^{d+2} C^g_{S_2} D_\ell(\Delta_0)}{\partial_\Delta h(\Delta, q)|_{\Delta=\Delta_0} K_\ell(\Delta_0)}} \; I_{12} \right) F^{(0,0,\ell)}_{12\Delta_0}$$

$$I_{12} = \frac{\lambda_\ell \, \Gamma\left(\frac{\Delta_0+\Delta_1-\Delta_2+\ell}{2}\right) \Gamma\left(\frac{\Delta_0+\Delta_2-\Delta_1+\ell}{2}\right) \Gamma\left(\frac{d-2\Delta_0}{2}\right)}{\Gamma\left(\frac{\Delta_1-\Delta_2-\Delta_0+d+\ell}{2}\right) \Gamma\left(\frac{\Delta_2-\Delta_1-\Delta_0+d+\ell}{2}\right) \Gamma\left(\ell+\Delta_0\right)}$$

- $D_\ell(\Delta_0) \rightarrow$ normalization of worldsheet 2-point function, $K_\ell(\Delta_0) \rightarrow$ normalization for boundary 2-point function of spin ℓ operators.

Relating spacetime and worldsheet OPE coefficients

- For the case external vectors with scalar exchange

$$C^{(1,1,0)}_{12\Delta_0} = \left(\sqrt{\frac{4\pi^{d+2} C^{g}_{S_2} D_0(\Delta_0)}{\partial_\Delta h(\Delta, q)|_{\Delta=\Delta_0}}}\, \tilde{I}_{12} \right) F^{(1,1,0)}_{12\Delta_0}$$

$$\tilde{I}_{12} = \left[\frac{2\Delta_0(2d-\Delta_0-2)}{(d-\Delta_0)^2(\Delta_0+1-d)} \right] \left(\frac{\Gamma\left(\frac{\Delta_0}{2}\right)}{\Gamma\left(\frac{d-\Delta_0}{2}\right)} \right)^2 \frac{\Gamma\left(\frac{d-2\Delta_0}{2}\right)}{\Gamma\left(\Delta_0\right)}$$

Summary and Future Directions

- We revisited the Aharony, Komargodski analysis of worldsheet description of spacetime OPEs and generalised their result for arbitary spin ℓ exchange and external spin 1 operators.
- We obtained a general and explicit relation between worldsheet and spacetime CFT OPE coefficients. Our expressions are consistent with known results in AdS_3/CFT_2. [Gabardiel, Kirsch (2007)]
- Understand in general the origin of multi-trace contributions in the spacetime OPE from the worldsheet.
- $1/N$ corrections.
- Relate the bootstrap conditions on the worldsheet and boundary theory as a step towards the program of carving out the landscape of large N theories with weakly coupled string duals.

THANK YOU