Out of Time Order correlators in systems interacting with a thermal bath

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Outline of the talk

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Motivation

- Out of time order correlators : little-studied objects in quantum mechanical systems.
- Nevertheless encode information about systems which may turn out to be interesting.
- For instance, 2-OTO correlators are being used as a measure of chaos.
- As yet, not much study of even higher OTO correlators.
- Interesting to explore such OTO correlators and see whether they tell us anything new about the physics of the system.

- One possible area of study : a system in contact with a thermal bath.
- Assume bath made of a bunch of harmonic oscillators.
- The system-bath interaction is r^{th} power in the bath degrees of freedom where $r \geq 1$.
- Action of the system+ thermal bath is

$$S_{total}[\vec{q}, \vec{X}] = S_{sys}[\vec{q}] + S_{SHO}[\vec{X}] + \lambda \int dt \vec{J}(t).\vec{X}^r(t)$$

where $\vec{J}(t)$ is a function of $\vec{q}(t)$. [Linear case $\rightarrow$ Feynman, Vernon, 1963]

- Integrate out the bath degrees of freedom to get an effective theory for the system. Study the behaviour of the OTO correlators as the system thermalizes.
- For this one needs an efficient formalism which allows one to calculate such OTO correlators. $\longrightarrow$ Generalized Schwinger-Keldysh formalism

- But as I'll explain later, there is a redundancy in this formalism.
- For thermal states,the column vector representation of correlation functions in this formalism makes this redundancy manifest
- It allows one to express correlation functions calculated in this formalism concisely in terms of the truly independent quantities.
- Then one can use this representation to integrate out the degrees of freedom of the thermal bath to construct an effective theory for the system.

Brief Review of Feynman-Vernon

- Consider a Brownian particle interacting with a harmonic bath.
- The interaction is linear in the SHO degrees of freedom
- Lagrangian is given by

$$L = \frac{1}{2}m\dot{q}^2 + \sum_{i=1}^{N}(\frac{1}{2}\dot{X}_i^2 - \frac{1}{2}\omega_0^{(i)}X_i^2) + \sum_{i=1}^{N}\lambda_i Q X_i$$

  where Q is a function of q.
- One can consider the Schwinger-Keldysh generating functional defined as

$$\mathcal{Z}_{\mathcal{SK}}[J_R, J_L] = Tr[U(J_R)\rho U(J_L)^\dagger]$$

- This can expressed as a path integral on a contour with 1 time-fold where the original degrees of freedom are doubled (to be discussed later in the more general setting).
- Integrating out the bath degrees of freedom in this path integral we get the effective dynamics of the particle which is encoded in the influence functional.

- The influence functional is exponential of a quantity called the influence phase which contains terms of the form

$$\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 Q_{\alpha_1}(t_1) Q_{\alpha_2}(t_2) \langle [X(\tilde{t_1}), X(\tilde{t_2})] \rangle$$

$$\text{and} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 Q_{\alpha_1}(t_1) Q_{\alpha_2}(t_2) \langle \{X(\tilde{t_1}), X(\tilde{t_2})\} \rangle$$

- The terms with the commutator contribute to the dissipative term in the Schwinger-Dyson equations of the particle.
- The terms with the anti-commutator contribute to the noise term in the Schwinger-Dyson equations of the particle.
- The KMS relations connect these two quantities $\longrightarrow$ Fluctuation-Dissipation relation.
- For a coupling linear in the SHO d.o.f, the influence functional is completely determined by 2 point functions of the variables X_i.
- Hence only 1-OTO correlators of the thermal bath can contribute to the equation of motion of the particle.

Questions

- When the interaction is nonlinear in the X_i's (say, X_i^r, where $r > 1$), what are the imprints of the higher OTO correlators of the bath on the equations satisfied by the correlators of the system?
- Are there analogous dissipative and noise terms?
- If so, can one find a Fluctuation-Dissipation relations for such quantities?
- To analyze these questions, we need to find the influence functional for such interactions in the Generalized Schwinger-Keldysh formalism.

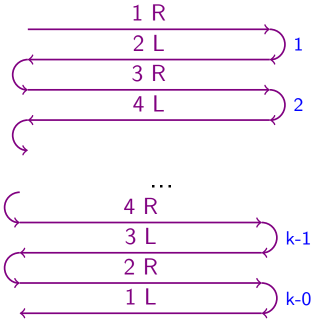
Generalized Schwinger-Keldysh formalism

- Consider a system which is described by a unitary theory defined on a set of fields collectively denoted by ϕ.
- Let $O_1, O_2, ...O_n$ be n functions of ϕ.
- The Generalized Schwinger-Keldysh formalism allows one to calculate k-OTO correlators of these operators where k can be any positive integer.
- In this formalism, one considers $2k$ copies of the field ϕ $\rightarrow \phi_{1R}$,$\phi_{1L}, ..., \phi_{kR}$,ϕ_{kL} .
- The path integral is given by

$$\mathcal{Z}_{k-fold} = \int_{\substack{\phi_{1R}(t_0)=\phi_{1R}^{(0)},\\ \phi_{1L}(t_0)=\phi_{1L}^{(0)}}} (\prod_{j=1}^{k}[D\phi_{jR}][D\phi_{jL}])\rho(\phi_{1R}^{(0)}, \phi_{1L}^{(0)})$$

$$e^{i\sum_{j=1}^{k}(S[\phi_{jR}]-S[\phi_{jL}])}$$

where the integral is over a k-fold contour of the form given below.



- There are $2k$ legs in this contour, k of which are right moving and k are left moving.
- The fields ϕ_{jR} and ϕ_{jL} are defined on the legs jR and jL respectively.
- The fields defined on 2 successive legs are identified at the turning point in the juncture of the 2 legs.

- Correlation functions in the generalized SK formalism are given by

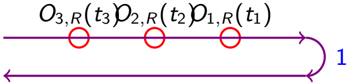
$$\langle \mathcal{T}_{\mathcal{C}}[O_{1,\alpha_1}(t_1)...O_{n,\alpha_n}(t_n)]\rangle_{k-fold}$$
$$= \int_{\substack{\phi_{1R}(t_0)=\phi_{1R}^{(0)},\\ \phi_{1L}(t_0)=\phi_{1L}^{(0)}}} (\prod_{j=1}^{k}[D\phi_{jR}][D\phi_{jL}])\rho(\phi_R^{(0)},\phi_L^{(0)})O_{1,\alpha_1}(t_1)...O_{n,\alpha_n}(t_n)$$
$$e^{i\sum_{j=1}^{k}(S[\phi_{jR}]-S[\phi_{jL}])}$$

where $\alpha_1, ..., \alpha_n$ take values from the set $\{1R, 1L, ...kR, kL\}$.

- Here $\mathcal{T}_{\mathcal{C}}$ indicates contour ordering which means that if one goes along the arrow shown in the diagram above the insertions are arranged from right to left.
- Such a contour-ordered correlator is equal to a Wightman correlator of the original theory with the insertions being in the same order, but the operators no longer having any index 'jR' or 'jL'.

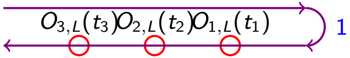
Examples:k=1 case, 3 point functions($t_1 > t_2 > t_3$)

- $\langle \mathcal{T}_{\mathcal{C}}[O_{1,R}(t_1)O_{2,R}(t_2)O_{3,R}(t_3)]\rangle_{SK} = \langle O_1(t_1)O_2(t_2)O_3(t_3)\rangle$



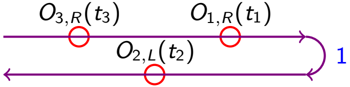
$\Longrightarrow$ Time-ordered

- $\langle \mathcal{T}_{\mathcal{C}}[O_{1,L}(t_1)O_{2,L}(t_2)O_{3,L}(t_3)]\rangle_{SK} = \langle O_3(t_3)O_2(t_2)O_1(t_1)\rangle$



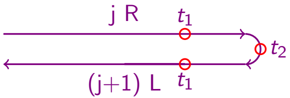
$\Longrightarrow$ Anti-time-ordered

- $\langle \mathcal{T}_{\mathcal{C}}[O_{1,R}(t_1)O_{2,L}(t_2)O_{3,R}(t_3)]\rangle_{SK} = \langle O_2(t_2)O_1(t_1)O_3(t_3)\rangle$



$\Longrightarrow$ 1-OTO

Sliding rule



- When one moves an insertion from t_1 to t_2 along a right-going leg, one evolves the corresponding operator in the correlator by the time-evolution operator $U(t_1, t_2)$, i.e.

$$O(t_1) \to U(t_1, t_2)^\dagger O(t_1) U(t_1, t_2) = O(t_2)$$

- Then if one moves the insertion from t_2 to t_1 along a left-going leg, one evolves the operator by $U(t_1, t_2)^\dagger$. i.e.

$$O(t_2) \to U(t_1, t_2) O(t_2) U(t_1, t_2)^\dagger = O(t_1)$$

- Because of unitarity, sliding an operator from a point on one leg to a turning point and then sliding it back to the same time along the next leg does not change the correlation function as long as this sliding is not obstructed by another insertion.

Thermal correlators

- In a thermal state the density matrix is $\rho = \frac{e^{-\beta H}}{Tr[e^{-\beta H}]}$ where $\beta = \frac{1}{k_B T}$.
- Wightman correlators satisfy the KMS relations

$$\begin{aligned}&\langle \tilde{O}_1(\omega_1)...\tilde{O}_n(\omega_n)\rangle \\ =&e^{-\beta\omega_n}\langle \tilde{O}_n(\omega_n)\tilde{O}_1(\omega_1)...\tilde{O}_{n-1}(\omega_{n-1})\rangle\end{aligned}$$

- So all Wightman correlators which are obtained from each other by a cyclic permutation of the insertions are equal upto a factor.
- There are $(n-1)!$ independent thermal Wightman correlators.
- But,in the generalized SK formalism on a k-fold contour there are $(2k)^n$ n-point contour correlators. Hence, for large enough k, there is a redundancy in this description.
- The column vector representation of the array of $(2k)^n$ contour correlators expresses the array of contour correlators in terms of the $(n-1)!$ independent Wightman correlators and makes the redundancy manifest.

Column vector representation of array of contour correlators

- For an n-point correlator on a k-fold contour, each point can lie on any one of the 2k legs of the contour.
- For the following discussion let us assume a particular ordering for the time instants of the n insertions: $t_1 > t_2 > ... > t_n$
- So, we have an n-dimensional array of correlators where there are 2k elements along each axis.
- One can choose a basis for the 2k-dimensional column vector along each axis of the n-dimensional array.
- The space of all n-dimensional arrays is then spanned by the tensor products of these column vectors.

- When $\omega \neq 0$,a suitable choice of basis of column vectors:

$$\bar{e}_P^{(j)}(\omega) \equiv \{ \underbrace{\mathfrak{f}, \mathfrak{f}, \ldots, \mathfrak{f}}_{2j-2 \text{ times}}, \underbrace{1+\mathfrak{f}, 1+\mathfrak{f}, \ldots, 1+\mathfrak{f}}_{2k-2j+2 \text{ times}} \}^T$$

$$\bar{e}_F^{(j)}(\omega) \equiv \{ \underbrace{\mathfrak{f}, \mathfrak{f}, \ldots, \mathfrak{f}}_{2j-1 \text{ times}}, \underbrace{1+\mathfrak{f}, 1+\mathfrak{f}, \ldots, 1+\mathfrak{f}}_{2k-2j+1 \text{ times}} \}^T$$

where $\mathfrak{f} \equiv \frac{1}{e^{\beta\omega}-1}$ and $j = 1, 2, \ldots, k$.

- The dual basis of these column vectors is given by

$$e_P^{(1)} \equiv (1, 0, 0, 0..., 0, -e^{-\beta\omega}), e_P^{(i)} \equiv (0, 0, 0, \ldots, -1_{2i-2}, 1_{2i-1}, 0, ..., 0),$$
$$e_F^{(j)} \equiv (0, 0, 0, \ldots, -1_{2j-1}, 1_{2j}, 0, ..., 0),$$

where $i = 2, \ldots, k$ and $j = 1, 2, \ldots, k$.

- The labels ‘P’ and ‘F’ labels indicate whether the 2 nonzero components are on legs which are connected by a past turning point or a future turning point.

- A basis for n-dimensional arrays is given

$$\bar{e}^{i_1 i_2 \ldots i_n}_{\alpha_1 \alpha_2 \ldots \alpha_n}(\omega_1, \omega_2, \ldots, \omega_n) \equiv \bar{e}^{i_1}_{\alpha_1}(\omega_1) \otimes \bar{e}^{i_2}_{\alpha_2}(\omega_2) \otimes \ldots \otimes \bar{e}^{i_n}_{\alpha_n}(\omega_n)$$

  where $\alpha_1, \ldots \alpha_n$ take values in the set $\{P, F\}$ and $i_1, i_2, \ldots, i_n$ take values in the set $\{1, \ldots k\}$.
- The dual basis is given by

$$e^{i_1 i_2 \ldots i_n}_{\alpha_1 \alpha_2 \ldots \alpha_n}(\omega_1, \omega_2, \ldots, \omega_n) \equiv e^{i_1}_{\alpha_1}(\omega_1) \otimes e^{i_2}_{\alpha_2}(\omega_2) \otimes \ldots \otimes e^{i_n}_{\alpha_n}(\omega_n)$$

- The component of the array of contour correlators along $\bar{e}^{i_1 i_2 \ldots i_n}_{\alpha_1 \alpha_2 \ldots \alpha_n}$ is obtained from a contraction of the array with $e^{i_1 i_2 \ldots i_n}_{\alpha_1 \alpha_2 \ldots \alpha_n}$.
- The reason for choosing this particular basis is that many of the components vanish as a consequence of the sliding rules mentioned earlier.
- Using the sliding rule , one can show that in many cases contractions with one of the dual basis vectors is zero because the 2 nonzero terms in one of the column vectors cancel each other's contribution.

Column vector representation:2 point and 3 point functions

- For $t_1 > t_2$, the array of 2 point contour correlators in the frequency space is given by

$$\langle T_{\mathcal{C}}\ 1 \otimes 2 \rangle = \langle [1,2] \rangle \sum_{j=1}^{k} \bar{e}_P^{(j)}(\omega_1) \otimes (\bar{e}_F^{(j-1)}(\omega_2) - \bar{e}_F^{(j)}(\omega_2))$$

- For $t_1 > t_2 > t_3$,array of 3 point contour correlators is given by

$$\begin{aligned} &\langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \rangle \\ &= \{-\langle [123] \rangle \sum_{j=1}^{k} (\bar{e}_P^{(j+1)}(\omega_1) - \bar{e}_P^{(j)}(\omega_1)) \otimes \bar{e}_F^{(j)}(\omega_2) \otimes \bar{e}_F^{(j)}(\omega_3) \\ &\quad + \langle [321] \rangle \sum_{j=1}^{k} \bar{e}_P^{(j)}(\omega_1) \otimes \bar{e}_P^{(j)}(\omega_2) \otimes (\bar{e}_F^{(j-1)}(\omega_3) - \bar{e}_F^{(j)}(\omega_3))\} \end{aligned}$$

where $[123] = [[1,2],3]$ & $[321] = [[3,2],1]$

Column vector representation:4 point functions

- For $t_1 > t_2 > t_3 > t_4$,array of 4 point contour correlators is given by

$$
\begin{aligned}
&\langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle \\
&= \langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PPPF} + \langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PFFF} \\
&+ \langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PFPF} + \langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PPFF}
\end{aligned}
$$

Here

$$\langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PPPF}$$

$$= -\langle [4321] \rangle \sum_{j=1}^{k} \bar{e}_P^{(j)}(\omega_1) \otimes \bar{e}_P^{(j)}(\omega_2) \otimes \bar{e}_P^{(j)}(\omega_3) \otimes \left(\bar{e}_F^{(j-1)}(\omega_4) - \bar{e}_F^{(j)}(\omega_4)\right)$$

$$\langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PFFF}$$

$$= \langle [1234] \rangle \sum_{j=1}^{k} \left(\bar{e}_P^{(j+1)}(\omega_1) - \bar{e}_P^{(j)}(\omega_1)\right) \otimes \bar{e}_F^{(j)}(\omega_2) \otimes \bar{e}_F^{(j)}(\omega_3) \otimes \bar{e}_F^{(j)}(\omega_4)$$

where $[1234] = [[[1,2],3],4]$ & $[4321] = [[[4,3],2],1]$

$$\langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PFPF}$$

$$= \{\langle [3,4][1,2] \rangle \sum_{i<j} (\bar{e}_P^{(i+1(\omega_1))} - \bar{e}_P^{(i)}(\omega_1)) \otimes \bar{e}_F^{(i)}(\omega_2) \otimes \bar{e}_P^{(j)}(\omega_3) \otimes (\bar{e}_F^{(j-1)}(\omega_4)$$

$$+ \langle [1,2][3,4] \rangle \sum_{i \geq j} (\bar{e}_P^{(i+1)}(\omega_1) - \bar{e}_P^{(i)}(\omega_1)) \otimes \bar{e}_F^{(i)}(\omega_2) \otimes \bar{e}_P^{(j)}(\omega_3) \otimes (\bar{e}_F^{(j-1)}(\omega_4)$$

$$
\begin{aligned}
&\langle T_{\mathcal{C}}\ 1 \otimes 2 \otimes 3 \otimes 4 \rangle_{PPFF} \\
&= \sum_{i,j} (\Theta_{i<j} \langle [2,4][1,3] \rangle + \Theta_{i>j} \langle [1,3][2,4] \rangle) \\
&\qquad \bar{e}_P^{(i)}(\omega_1) \otimes \bar{e}_P^{(j)}(\omega_2) \otimes (\bar{e}_F^{(i-1)}(\omega_3) - \bar{e}_F^{(i)}(\omega_3)) \otimes (\bar{e}_F^{(j-1)}(\omega_4) - \bar{e}_F^{(j)}(\omega_4)) \\
&+ \sum_{i,j} (\Theta_{i<j} \langle [2,3][1,4] \rangle + \Theta_{i>j} \langle [1,4][2,3] \rangle) \\
&\qquad \bar{e}_P^{(i)}(\omega_1) \otimes \bar{e}_P^{(j)}(\omega_2) \otimes (\bar{e}_F^{(j-1)}(\omega_3) - \bar{e}_F^{(j)}(\omega_3)) \otimes (\bar{e}_F^{(i-1)}(\omega_4) - \bar{e}_F^{(i)}(\omega_4)) \\
&- \langle ([1,3][2,4] + [2,3][1,4]) \rangle \\
&\qquad \sum_{j} \bar{e}_P^{(j)}(\omega_1) \otimes \bar{e}_P^{(j)}(\omega_2) \otimes \bar{e}_F^{(j)}(\omega_3) \otimes (\bar{e}_F^{(j-1)}(\omega_4) - \bar{e}_F^{(j)}(\omega_4)) \\
&+ \langle ([1,4][2,3] + [2,4][1,3]) \rangle \\
&\qquad \sum_{j} \bar{e}_P^{(j)}(\omega_1) \otimes \bar{e}_P^{(j)}(\omega_2) \otimes \bar{e}_F^{(j-1)}(\omega_3) \otimes (\bar{e}_F^{(j-1)}(\omega_4) - \bar{e}_F^{(j)}(\omega_4)) \\
&+ \langle [2314] \rangle \sum_{j} \bar{e}_P^{(j)}(\omega_1) \otimes \bar{e}_P^{(j)}(\omega_2) \otimes (\bar{e}_F^{(j-1)}(\omega_3) - \bar{e}_F^{(j)}(\omega_3) \otimes \bar{e}_F^{(j)}(\omega_4))
\end{aligned}
$$

Application:System interacting with a harmonic bath

- Now, let us go back to the case mentioned at the beginning of the talk i.e. a system brought in contact with a harmonic bath.
- Let the thermal bath (at temperature $\frac{1}{k_B \beta}$) consist of N simple harmonic oscillators with frequencies $\omega_0^{(1)}, \omega_0^{(2)}, ..., \omega_0^{(N)}$.
- Let the interaction be quadratic in the degrees of freedom of the SHO.
- The total action of the system and the thermal bath is

$$S_{total}[\vec{q}, \vec{X}] = S_{sys}[\vec{q}] + S_{SHO}[\vec{X}] + \lambda \int dt \vec{J}(t).\vec{X}^2(t)$$

where $\vec{J}(t)$ is a function of $\vec{q}(t)$.

- The path integral of this total system in the generalized SK formalism on a k-fold contour is given by

$$\mathcal{Z}_{k-fold} = \int (\prod_{j=1}^{k} [D\vec{q}_{jR}][D\vec{q}_{jL}][D\vec{X}_{jR}][D\vec{X}_{jL}])\rho(\vec{q}_{1R}(t_0), \vec{q}_{1L}(t_0))$$
$$e^{i\sum_{j=1}^{k}(S_{sys}[\vec{q}_{jR}]-S_{sys}[\vec{q}_{jL}]])} \times e^{i\sum_{j=1}^{k}(S_{SHO}[\vec{X}_{jR}]-S_{SHO}[\vec{q}_{jL}]])}$$
$$\times e^{i\lambda\sum_{j=1}^{k}(\int \vec{J}_{jR}.\vec{X^2}_{jR}-\int \vec{J}_{jL}.\vec{X^2}_{jL}]])} e^{-\beta H_{SHO}}(\vec{X}_{1R}(t_0), \vec{X}_{1L}(t_0))$$

- Integrating out the degrees of freedom of the bath would give us the influence functional for the system.
- But this influence functional is nothing but the generating functional of the thermal bath with $J_{i,R}$ and $J_{i,L}$ being the sources.
- Thus,

$$\mathcal{Z}_{k-fold} = \int (\prod_{j=1}^{k} [D\vec{q}_{jR}][D\vec{q}_{jL}])$$
$$e^{i\sum_{j=1}^{k}(S_{sys}[\vec{q}_{jR}]-S_{sys}[\vec{q}_{jL}]])} \mathcal{Z}_{SHO}[\{\vec{J}_{jR}\}, \{\vec{J}_{jL}\}]$$

- One can evaluate this generating functional for the thermal bath as a perturbation theory in the sources.
- The coefficients in this expansion are contour-ordered correlators of X_α^2 where $\alpha \in \{1R, 1L, ...kR, kL\}$.
- One can employ the column vector representation to express these contour correlators concisely in terms of the independent Wightman correlators.
- Caution: The results we got are incomplete. They don't include the terms in Wightman correlators which contain $\delta(\omega_i)$.
- Once, one has the column vector representation, one can just evaluate the independent n-point Wightman correlators for the SHOs and that would give the influence functional for the system upto the n-th order in the coupling.

Future directions

- Extend the column vector representation to the case when $\omega_i = 0$ for some i.
- Extend this representation to general n point functions.
- Address the questions posed earlier.
- Extend this analysis to more complicated thermal baths (Eg. Bath of anharmonic oscillators)