BLACK HOLE DEGENERACIES FROM MOONSHINE

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• Consider the Elliptic genus of $K3$.

$$F(K3;\tau,z)=$$
$$\mathrm{Tr}_{RR}\left((-1)^{F^{K3}+\bar{F}^{K3}} e^{2\pi i z F^{K3}} e^{2\pi i\tau(L_0-c/24)} \bar{e}^{-2\pi i\bar{\tau}(\bar{L}_0-\hat{c}/24)}\right)$$
$$= \sum_{m\geq 0,l} c(4m-l^2) e^{2\pi i m\tau} e^{2\pi i l z}$$

The trace is taken over the Ramond sector.

The elliptic genus is holomorphic in τ, z.

Only the ground states of the left movers are counted.

Evaluating the index we obtain

$$F(K3;\tau,z) = 8\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right]$$

A simple model of $K3$: $T^4/\mathbb{Z}_2$

$$\mathbb{Z}_2 : (y_1, y_2, y_3, y_4) \to -(y_1, y_2, y_3, y_4)$$

The Hodge diamond of $K3$ is given by

$$\begin{aligned} h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} &= 1, \\ h_{(1,1)} &= 20 \end{aligned}$$

What is the connection with the Mathieu group M_{24}.

Rewrite the elliptic genus in terms of the characters of the $\mathcal{N} = 4$ super conformal algebra

$$Z_{K3}(\tau,z) = 8\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2} + \frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2} + \frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right]$$

Decompose the elliptic genus into the elliptic genera of the short and the long representations of the $\mathcal{N}=4$ super conformal algebra.

$$\begin{aligned} \mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,z) &= -i\frac{e^{\pi i z}\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n=-\infty}^{\infty}\frac{e^{\pi i\tau n(n+1)}e^{2\pi i(n+\frac{1}{2})}}{1-e^{2\pi i(n\tau+z)}}, \\ \mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau,z) &= e^{2\pi i\tau(n-\frac{1}{8})}\frac{\theta_1(\tau,z)^2}{\eta(\tau)^2}. \end{aligned}$$

Then it can be seen

$$Z_{K3}(\tau, z) = 24\mathrm{ch}_{h=\frac{1}{4}, l=0}(\tau, z) + \sum_{n=0}^{\infty} A_n^{(1A)} \mathrm{ch}_{h=n+\frac{1}{4}, l=\frac{1}{2}}(\tau, z).$$

where the first few values of $A_n^{(1A)}$ are given by

$$A_n^{(1A)} = -2,\ 90,\ 462,\ 1540,\ 4554,\ 11592, \ldots$$

These coefficients are either the dimensions or the sums of dimensions of the irreducible representations of the group M_{24}.
Eguchi, Ooguri, Tachikawa (2010)

$$A_n^{(1A)} = \mathrm{Tr}(\mathbf{1})_n, \qquad n > 0$$

There exists $\mathbb{Z}_N$ quotients of $K3$ for which the Hodge diamond of $K3/\mathbb{Z}_N$ becomes

$$h_{(0,0)} = h_{(2,2)} = h_{(0,2)} = h_{(2,0)} = 1,$$
$$h_{(1,1)} = 2\left(\frac{24}{N+1} - 2\right) = 2k$$

N	$h_{(1,1)}$	k
1	20	10
2	12	6
3	8	4
5	4	2
7	2	1

Let us refer to these $\mathbb{Z}_N$ action by g'.

• Let g' be action of this quotient, the twisted elliptic genus of $K3$ is defined as

$$
\begin{aligned}
& F^{(r,s)}(\tau, z) \\
= & \frac{1}{N}\mathrm{Tr}^{K3}_{RR;g'^r}\left((-1)^{F^{K3}+\bar{F}^{K3}} g'^s e^{2\pi i z F^{K3}} e^{2\pi i\tau(L_0 - c/24)} \bar{q}^{-2\pi i\bar{\tau}(\bar{L}_0 - c/24)}\right) \\
= & \sum_{b=0}^{1} \sum_{m\geq 0\in\mathbb{Z}/N, l\in 2\mathbb{Z}+b} c_b^{(r,s)}(4m - l^2) e^{2\pi i m\tau} e^{2\pi i l z} \\
& 0 \leq r, s, \leq (N-1).
\end{aligned}
$$

These twisted elliptic genera for the $\mathbb{Z}_N$ quotients of $K3$ by g' with $N = 2, 3, 5, 7$ have been written down in David, Jatkar, Sen (2006)

For the $N = 2$ orbifold the twisted indices are

$$\begin{aligned}
F^{(0,0)}(\tau,z) &= 4\left[\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}+\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}+\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2}\right],\\
F^{(0,1)}(\tau,z) &= 4\frac{\theta_2(\tau,z)^2}{\theta_2(\tau,0)^2}, \quad F^{(1,0)}(\tau,z) = 4\frac{\theta_4(\tau,z)^2}{\theta_4(\tau,0)^2},\\
F^{(1,1)}(\tau,z) &= 4\frac{\theta_3(\tau,z)^2}{\theta_3(\tau,0)^2}.
\end{aligned}$$

- The twining character $F^{(0,1)}$ admits the decomposition

$$2F^{(0,1)}(\tau,z) = 8\mathrm{ch}_{h=\frac{1}{4},l=0}(\tau,z) + \sum_{n=0}^{\infty} A_n^{(2A)} \mathrm{ch}_{h=n+\frac{1}{4},l=\frac{1}{2}}(\tau,z).$$

the first few values of $A_n^{(2A)}$ are given by

$$A_n^{(2A)} = -2,\ -6,\ 14,\ -28,\ 42,\ -56,\ 86,\ -138,\ \ldots$$

These coefficients can be identified with McKay-Thompson series constructed out of trace of the elements g'_{2A} in the $2A$ conjugacy class of the Mathieu group M_{24}.

$$A_n^{(2A)} = \mathrm{Tr}(g'_{2A})_n$$

Cheng (2010), Gaberdiel, Hohenegger, Volpato (2010)

A similar expansion of the twining character $F^{(0,1)}(\tau, z)$ for each of the $\mathbb{Z}_N$ quotients with $N = 2, 3, 5, 7$ can be made in terms of the McKay-Thompson series associated with the pA conjugacy class of the Mathieu group M_{24} with $p = 2, 3, 5, 7$.

Now it is known that the M_{24} admits 26 conjugacy classes.

Conjucay Class	Order	Cycle shape	Cycle
1A	1	1^{24}	()
2A	2	$1^8 \cdot 2^8$	(1, 8)(2, 12)(4, 15)(5, 7)(9, 22)(1
3A	3	$1^6 \cdot 3^6$	(3, 18, 20)(4, 22, 24)(5, 19, 17)(6, 1
5A	4	$1^4 \cdot 5^4$	(2, 21, 13, 16, 23)(3, 5, 15, 22, 14)(4, 12
7A	7	$1^3 \cdot 7^3$	(1, 17, 5, 21, 24, 10, 6)(2, 12, 13, 9, 4, 2
7A	7	$1^3 \cdot 7^3$	(1, 21, 6, 5, 10, 17, 24)(2, 9, 20, 13, 23,
11A	11	$1^2 \cdot 11^2$	(1, 3, 10, 4, 14, 15, 5, 24, 13, 17, 18)(2, 21,
23A	23	$1^1 \cdot 23^1$	(1, 7, 6, 24, 14, 4, 16, 12, 20, 9, 11, 5, 15, 1
23B	23	$1^1 \cdot 23^1$	(1, 4, 11, 18, 8, 6, 12, 15, 17, 21, 14, 9, 19,
4B	4	$1^4 \cdot 2^2 \cdot 4^4$	(1, 17, 21, 9)(2, 13, 24, 15)(3, 23)(4, 14
6A	6	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$	(1, 8)(2, 24, 11, 12, 23, 18)(3, 20, 10)(4, 15
8A	8	$1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$	(1, 13, 17, 24, 21, 15, 9, 2)(3, 16, 23, 6)(4,
14A	14	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 12, 17, 13, 5, 9, 21, 4, 24, 23, 10, 20, 6,
14B	14	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$	(1, 13, 21, 23, 6, 12, 5, 4, 10, 2, 17, 9, 24, 2
15A	15	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 13, 23, 21, 16)(3, 7, 9, 5, 4, 18, 15, 12, 1
15B	15	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$	(2, 23, 16, 13, 21)(3, 12, 24, 15, 17, 18, 14, 4

Table: Conjugacy classes of $M_{23} \subset M_{24}$ (Type 1)

Conjucay Class	Order	Cycle shape	
2B	4	2^{12}	(1, 8)(2, 10)(3, 20)(4, 22)(5, 17)(6, 11
3B	9	3^{8}	(1, 10, 3)(2, 24, 18)(4, 13, 22)(5, 19,
12B	144	12^{2}	(1, 12, 24, 23, 10, 8, 18, 6, 3, 21, 2,
6B	36	6^{4}	(1, 24, 10, 18, 3, 2)(4, 11, 13, 20, 22
4C	16	4^{6}	(1, 23, 18, 21)(2, 12, 10, 6)(3, 7, 24,
10A	20	$2^{2} \cdot 10^{2}$	(1, 8)(2, 18, 21, 19, 13, 10, 16, 24, 2
21A	63	$3^{1} \cdot 21^{1}$	(1, 3, 9, 15, 5, 12, 2, 13, 20, 23, 17,
21B	63	$3^{1} \cdot 21^{1}$	(1, 12, 17, 22, 16, 5, 23, 21, 11, 15,
4A	8	$2^{4} \cdot 4^{4}$	(1, 4, 8, 15)(2, 9, 12, 22)(3, 6)(5, 24,
12A	24	$2^{1} \cdot 4^{1} \cdot 6^{1} \cdot 12^{1}$	(1, 15, 8, 4)(2, 19, 24, 9, 11, 7, 12, 1

Table: Conjugacy classes of $M_{24} \not\in M_{23}$ (Type 2)

Using the McKay thompson series associated with each of these 26 conjugacy classes one can work backwards and write down the twining character $F^{(0,1)}$ for each of the 26 classes.

Closed form expressions for these were given by
Cheng (2010), Gaberdiel, Hohenegger, Volpato (2010)

Therefore M_{24} symmetry of the elliptic genus, points to the existence of 26 quotients of $K3$.

Given the twining character $F^{(0,1)}$ can we obtain all the elements $F^{(r,s)}$ of the twisted elliptic genus?

Modular transformations relate these elements by

$$F^{(r,s)}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=\exp\left(2\pi i\frac{cz^2}{c\tau+d}\right)F^{(cs+ar,ds+br)}(\tau,z)$$

with

$$a,b,c,d\in\mathbb{Z},\qquad ad-bc=1.$$

The indices $cs+ar$ and $ds+br$ belong to $\mathbb{Z}$ mod N.

For example the $(0,1)$ sector on the LHS is related to the $(1,0)$ sector its arguments is evaluated at $(-1/\tau, z/\tau)$.
Choose $(a=0, b=-1, c=1, d=0)$

There 2 technical obstacles in using modular transformations to finding all the elements of the twisted elliptic genus.

For practical purposes we require a Fourier ($q = e^{2\pi i\tau}$) expansion say of the of the $(1,0)$ sector.

The relation of the $(1,0)$ sector with the $(0,1)$ sector will result in a expansion in terms of $e^{-2\pi i/\tau}$.

We need to use identities involving modular forms to relate these expansions.

For example consider the conjugacy class $11A$, to relate the $F^{(1,5)}$ element to the $F^{(2,10)}$ element we need to demonstrate the identity

$$\mathcal{E}_{11}(-\frac{1}{11\tau}+\frac{2}{11}) \quad = \quad \tau^2\mathcal{E}_{11}\left(\frac{\tau+5}{11}\right)$$

where

$$\mathcal{E}_N(\tau)=\frac{12i}{\pi(N-1)}\partial_\tau[\ln\eta(\tau)-\ln\eta(N\tau)]$$

Here is another one

$$\mathcal{E}_6(\frac{\tau}{6}+\frac{1}{2})=\frac{\tau^2}{5}(18\mathcal{E}_2(3\tau)-12\mathcal{E}_3(\tau)-9\mathcal{E}_2(\frac{3\tau}{2}))$$

The second technical obstacle is when the order of the class is a composite number it is not possible to relate all the sectors to the twining character $(0, 1)$ by modular transformations.

The various sectors of the twisted elliptic genus break up into sub-orbits under the action of modular transformations.

For example for the class $4B$ with $N = 4$ in table 1, the sectors $F^{(0,2)}, F^{(2,0)}, F^{(0,2)}$ form a sub-orbit and cannot be related to $F^{(0,1)}$.

To determine the twisted elliptic genus in these sectors we use the cycle shape of the conjugacy class in M_{24} to fix the twisted elliptic genus in these sectors.

We can determine the twisted elliptic genus for all the conjugacy classes listed as type 1.

For the conjugacy classes listed as type 2 this method does not work. It was noticed that for quotients of $K3$ by type 2 classes, the sum of the twisted elliptic genus in all the sectors vanish Gaberdiel, Taormina, Volpato, Wendland

Using this property we can fix the twisted elliptic genus in $2B$ and the $3B$ classes.

Explicit closed form expressions for the twisted elliptic genus was obtain for [all the classes in type 1 and the first two classes in type 2.
Chattopadhyaya, David (2017)

Using the twisted elliptic genus to evaluate the Hodge numbers we can identify the classes $4B, 6A, 8A$ together with $2A, 3A, 5A, 7A$ to that which where known earlier as CHL orbifolds.

For the remaining classes q expansion of the twisted elliptic genus can be extracted using the methods of
Gaberdiel, Perssson, Ronellenfitsch, Volpato (2012)

Note that all these twisted elliptic genera were shown to exist and constructed using the symmetry M_{24}.

Many of these are not geometric quotients.

For eg. the $11A$ conjugacy class , using the twisted elliptic genus we can obtain what would have correspond to the hodge number $h^{(1,1)}$.
It turns out this vanishes. Thus even the Kähler form of $K3$ is projected out.

The non-geometric ones are $11A, 23A/B, 2B, 3B$.

The class $3B$ has an explicit CFT construction in terms of 6 $SU(2)$ WZW theories at level 1.

What is the use of all these twisted elliptic genera?

Consider type II B/A theory on $K3 \times T^2/\mathbb{Z}_N$ where the $\mathbb{Z}_N$ action is g' on $K3$ and a shift of $1/N$ on one of the circles of S^1.

These compactifications of type IIB/A preserve $\mathcal{N} = 4$ supersymmetry in $d = 4$.

Thus we have a class of new $\mathcal{N} = 4$ string vacua.

Each of these vacua admit $1/4$ BPS states.
These are dyons with both electric and magnetic charges. For large charges they can be identified with supersymmetric black hole solutions.

For the cases of $pA, p = 1, 2, 3, 5, 7$, the generating function for the degeneracy (index) of dyons in these $\mathcal{N} = 4$ theories is given by

$$-B_6 = -(-1)^{Q\cdot P}\int_{\mathcal{C}} \mathrm{d}\rho\mathrm{d}\sigma\mathrm{d}v\; e^{-\pi i(N\rho Q^2 + \sigma/NP^2 + 2vQ\cdot P)}\frac{1}{\tilde{\Phi}(\rho, \sigma, v)},$$

where $\mathcal{C}$ is a contour in the complex 3-plane. Q, P refer to the electric and magnetic charge of the dyons.

Dijkgraaf, Verlinde, Verlinde (1996), Jatkar Sen (2005), David, Jatkar, Sen (2006), David, Sen (2006), Dabholkar Nampuri (2006)

The contour $\mathcal{C}$ is defined over a 3 dimensional subspace of the 3 complex dimensional space
$(\rho = \rho_1 + i\rho_2, \sigma = \sigma_1 + i\sigma_2, v = v_1 + iv_2)$.

$$\rho_2 = M_1, \qquad \sigma_2 = M_2, \qquad v_2 = -M_3,$$
$$0 \leq \rho_1 \leq 1, \qquad 0 \leq \sigma_1 \leq N, \qquad 0 \leq v_1 \leq 1.$$
$$M_1, M_2 >> 0, \quad M_3 << 0, \qquad |M_3| << M_1, M_2$$

$\tilde{\Phi}(\rho, \sigma, v)$ is the Siegel modular form associated with the twisted elliptic genus is given by

$$\tilde{\Phi}(\rho, \sigma, v) = e^{2\pi i(\tilde{\alpha}\rho + \tilde{\beta}\sigma + v)}$$

$$\prod_{b=0,1} \prod_{r=0}^{N-1} \prod_{\substack{k' \in \mathcal{Z} + \frac{r}{N}, l \in \mathcal{Z}, \\ j \in 2\mathcal{Z} + b \\ k', l \geq 0, j < 0 k' = l = 0}} (1 - e^{2\pi i(k'\sigma + l\rho + jv)})^{\sum_{s=0}^{N-1} e^{2\pi i s l/N} c_b^{r,s}(4k'l - j^2)}.$$

where

$$\tilde{\beta} = \frac{1}{N}, \qquad \tilde{\alpha} = 1$$

Here N is the order of the orbifold action.
This Siegel modular form transforms as a weight k form under appropriate sub-groups of $Sp(2, \mathbb{Z})$.

The modular property is defined as follows. Let

$$\Omega = \begin{pmatrix} \rho & v \\ v & \sigma \end{pmatrix}$$

Then

$$\tilde{\Phi}_k((C\Omega + D)^{-1}(A\Omega + B)) = [\det(C\Omega + D)]^k \tilde{\Phi}_k(\Omega)$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_{4\times 4} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

A, B, C, D are 2×2 matrices with integer elements.

The weight k is related to the low lying coefficients of the twisted elliptic genus and is given by

$$k = \frac{1}{2}\sum_{0}^{N-1} c_0^{(0,s)}(0).$$

There has been several detailed tests for this degeneracy formula.

We will discuss 2 of the them

Comparison of the statistical entropy with the Wald entropy.

Using a saddle point analysis of the contour determining the degeneracy one can find out what is the degeneracy for large charges and evaluate the entropy

$$
\begin{aligned}
S(Q,P) &= \pi\sqrt{Q^2P^2-(Q\cdot P)^2} \\
&\quad + \ln(h^{(k+2)}(\tau)) + \ln(h^{(k+2)}(-\bar{\tau})) - (k+2)\ln(2\tau_2)
\end{aligned}
$$

with

$$
\tau_1 = \frac{Q\cdot P}{P^2}, \qquad \tau_2 = \frac{1}{P^2}\sqrt{Q^2P^2-(Q\cdot P)^2}
$$

The leading term in the asymptotic formula for the entropy agrees with Hawking Bekenstein entropy of the corresponding black hole.

The subleading term agrees with the contribution of entropy from the 4 derivative term (Gauss Bonnet term) in the effective action of these $\mathcal{N} = 4$ theories using the Wald formula.

Now that we have new $\mathcal{N} = 4$ vacuua, corresponding to the conjugacy classes of M_{24},
we can check if the degeneracies given by the Siegel modular form constructed from the corresponding twisted elliptic genus agrees with the entropy given by the Wald formula.

Yes, it does.

The weights of the Siegel modular forms are given by

Type 1	pA	4B	6A	8A	14A	15A
Weight	$\frac{24}{p+1} - 2$	3	2	1	0	0

Table: Weight of Siegel modular forms corresponding to classes in M_{23}

Type 2	2B	3B
Weight	0	-1

Table: Weight of Siegel modular forms corresponding to the classes $\notin M_{23}$

The modular functions which determine the sub-leading corrections are given by

Conjugacy Class	$h^{(k+2)}(\rho)$
pA	$\eta^{k+2}(\rho)\eta^{k+2}(p\rho)$
4B	$\eta^4(4\rho)\eta^2(2\rho)\eta^4(\rho)$
6A	$\eta^2(\rho)\eta^2(2\rho)\eta^2(3\rho)\eta^2(6\rho)$
8A	$\eta^2(\rho)\eta(2\rho)\eta(4\rho)\eta^2(8\rho)$
14A	$\eta(\rho)\eta(2\rho)\eta(7\rho)\eta(14\rho)$
15A	$\eta(\rho)\eta(3\rho)\eta(5\rho)\eta(15\rho)$
2B	$\frac{\eta^8(4\rho)}{\eta^4(2\rho)}$
3B	$\frac{\eta^3(9\rho)}{\eta(3\rho)}$

Table: $p \in \{1, 2, 3, 5, 7, 11, 23\}$

Secondly and perhaps a more stringent test:

The coefficients $-B_6(Q, P)$ certainly must be integers.
It was conjectured that:
from the fact that for single centered black holes,
due to spherical symmetry and the regularity of the horizon,
the only angular momentum it caries is from the fermionic zero modes.

then $-B_6(Q, P)$ for single centered black holes must be positive.
Sen (2010)

The sufficient condition which ensures this property is that for charges which satisfy

$$Q \cdot P \geq 0, \qquad (Q \cdot P)^2 < Q^2 P^2, \qquad Q^2, P^2 > 0.$$

the coefficient $-B_6(Q, P)$ evaluated from the Fourier expansion of the Siegel modular form should be positive.

This is quite a non-trivial condition on the Fourier expansion of the inverse of Siegel modular forms which are generating functions for the index $-B_6(Q, P)$.

For the case of $1A$, (compactification of type II on $K3 \times T^2$) for a specific class of charges, this conjecture has been proved by Bringmann, Murthy (2013)

For the orbifolds corresponding to classes pA, $p = 2, 3, 5, 7,$ it has been verified by explicit computation of the Fourier coefficients of $-B_6(Q, P)$ for low lying charges.
Sen (2010)

We have constructed the twisted elliptic genus for orbifolds corresponding to all the conjugacy classes in type 1 and the first two classes in type 2.

Using this we can explicitly evaluate the Fourier coefficients which evaluate $-B_6$ of the dyons for low lying charges.

$(Q^2,\ P^2)\quad \backslash Q\cdot P$	-2	0	1	2	3
(1/2, 2)	-512	176	8	0	0
(1/2, 4)	-1536	896	80	0	0
(1/2, 6)	-4544	3616	480	0	0
(1/2, 8)	11752	12848	2176	24	0
(1,4)	-4592	5024	832	16	0
(1,6)	-13408	22464	36786	224	0
(1,8)	-33568	88320	26176	1760	0
(3/2, 6)	-37330	112316	36786	2998	38
(3/2, 8)	-80896	491920	196960	23616	592

Table: Some results for the index $-B_6$ for the $4B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q \cdot P$	-2	0	1	2	3
(1/3, 2)	-98	40	1	0	0
(1/3, 4)	-224	148	12	0	0
(1/3, 6)	-546	478	49	0	0
(1/3,8)	-1120	1352	186	0	0
(2/3, 4)	-512	592	92	0	0
(2/3, 6)	-1240	2080	436	8	0
(2/3, 8)	-2504	6416	1676	0	0
(1, 6)	-2926	7880	2172	116	0
(1, 10)	-2450	81380	32300	3494	49
(1, 12)	-4696	234900	104176	13856	316

Table: Some results for the index $-B_6$ for the $6A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2) \quad \backslash Q \cdot P$	-2	0	1	2	3
(1/4, 2)	-60	20	0	0	0
(1/4, 4)	-120	68	2	0	0
(1/4, 6)	-280	196	10	0	0
(1/4, 8)	-520	504	40	0	0
(1/2, 6)	-560	724	96	0	0
(1/2,8)	-1038	1998	352	2	0
(3/4, 6)	-1114	2280	450	6	0
(3/4, 8)	-2024	6704	1728	56	0
(3/4, 10)	-3860	18256	5564	300	0
(3/4, 12)	-6168	46456	16296	1192	4

Table: Some results for the index $-B_6$ for the $8A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q\cdot P$	-2	0	1	2	3
(2/11, 2)	-50	10	0	0	0
(2/11, 4)	-100	30	0	0	0
(2/11, 6)	-200	82	1	0	0
(4/11, 6)	-400	276	18	0	0
(6/11, 6)	-800	806	83	0	0
(6/11, 8)	-1438	2064	314	2	0
(6/11, 10)	-2584	4962	937	16	0
(6/11, 12)	-4328	11132	2558	72	0
(6/11, 22)	-34000	366378	139955	12760	114

Table: Some results for the index $-B_6$ for the $11A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q \cdot P$	-2	0	1	2	3
(1/7, 2)	-18	4	0	0	0
(1/7, 4)	-24	10	0	0	0
(1/7, 6)	-54	24	0	0	0
(2/7, 6)	-72	70	5	0	0
(2/7, 8)	-96	156	16	0	0
(3/7, 8)	-216	406	65	0	0
(3/7, 10)	-412	890	165	2	0
(4/7, 12)	-710	4682	1443	58	0
(5/7, 12)	-1180	11512	4156	292	0
(5/7, 14)	-1622	24744	9816	908	5

Table: Some results the index $-B_6$ for the $14A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad\backslash Q\cdot P$	-2	0	1	2	3
(2/15, 2)	-8	4	0	0	0
(2/15, 4)	-16	8	0	0	0
(2/15, 6)	-24	20	0	0	0
(2/5, 8)	-120	274	45	0	0
(2/5, 10)	-203	578	113	1	0
(4/15, 6)	-48	50	4	0	0
(4/15, 8)	-80	102	13	0	0
(8/15, 12)	-440	2844	898	40	0
(2/3, 12)	-638	6818	2498	178	0
(4/5, 18)	8236	141252	73651	12124	419

Table: Some results for the index $-B_6$ for the 15A orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q \cdot P$	-2	0	1	2	3
(2/23, 2)	-8	1	0	0	0
(2/23, 4)	-12	3	0	0	0
(2/23, 6)	-20	7	1	0	0
(4/23, 6)	-30	53	6	0	0
(4/23, 8)	-42	91	11	0	0
(6/23, 6)	-48	103	23	2	0
(6/23, 8)	-66	190	47	4	0
(6/23, 10)	-104	312	74	6	0

Table: Some results for the index $-B_6$ for the $23A$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q \cdot P$	-2	0	1	2	3
(1/2, 2)	320	288	24	0	0
(1/2, 4)	0	512	256	0	0
(1/2, 6)	-752	1120	888	48	0
(1/2, 8)	384	3328	2048	384	0
(1,4)	32	4416	2240	32	0
(1,6)	-2304	22464	13248	224	0
(1,8)	5920	42944	27328	5920	64
(3/2, 6)	-2008	102380	66172	9032	28
(3/2, 8)	59392	372736	243712	59392	2048

Table: Some results for the index B_6 for the $2B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

$(Q^2,\ P^2)\quad \backslash Q \cdot P$	-2	0	1	2	3
(2/9, 2)	0	18	0	0	0
(2/9, 4)	18	27	0	0	0
(2/9, 6)	0	78	21	0	0
(4/9, 4)	42	150	33	0	0
(4/9, 6)	0	270	81	0	0
(4/9, 8)	0	378	162	0	0
(2/3, 6)	0	918	297	0	0
(2/3, 8)	0	2460	1239	93	0

Table: Some results for the index $-B_6$ for the $3B$ orbifold of $K3$ for different values of Q^2, P^2 and $Q \cdot P$

As a check of the mathematica program used to evaluate these Fourier coefficients, we verified the coefficients listed by Sen(2010) for the orbifolds $pA, p = 2, 3, 5, 7$.

It is interesting to note that the non-geometric orbifolds $11A, 23A, 23B, 2B, 3B$ also satisfy the positivity constraints.

Other applications.

Compactifications of heterotic string theory on $K3 \times T^2/\mathbb{Z}_N$ where g' is an automorphism corresponding to the conjugacy class in M_{24} generalize the compactification of heterotic on $K3 \times T^2$.

These examples provide a class of $\mathcal{N} = 2$ string vaccua dual to type II compactification on Calabi-Yau manifolds.

The spectrum and the one loop effective action in the gauge sector of these compactifications were explored in
Datta, David, Lust (2015), Chattopadhyaya, David (2016)

At present we are evaluating the one loop effective action in the gravitational sector.
These provide information of the couplings

$$\mathcal{F}_{2g-2}(T,U)R^2F^{2g-2}$$

in the effective action. $g \geq 1$.
T, U are the Kähler and complex structure moduli of the torus T^2.
F is the self-dual graviphoton field strength.
R^2 is the self-dual Riemann tensor squared.

$\mathcal{F}_{2g-2}$ are known as topological string amplitudes and capture important toplogical data of the Calabi-Yau. They contain information of Gopakumar-Vafa invariants.

Working out these amplitudes in these orbifold models will help us understand a generalization (twisted) of these invariants.