Constraints on parity violating conformal field theories in $d = 3$

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References

- S. D. Chowdhury, J. R. David and S. Prakash, "Constraints on parity violating conformal field theories in $d = 3$", [arXiv:1707.03007[hep-th]].

Introduction and motivation

- Conformal field theories in $d = 3$ are of interest both in the context of holography as well as in condensed matter physics.
- A particularly important feature of conformal field theories in three dimensions is that physically relevant examples need not preserve parity, due to the presence of a Chern Simons term.
- There has been a systematic study of Chern Simons theories in the past several years.

- Consider a conformal field theory in $d=3$ with a $U(1)$ conserved current j and conserved stress tensor T and let the theory be parity violating.
- Conformal invariance restricts the structure of the three point functions of j and T, in $d=3$, to be of the following form,

$$\begin{aligned} \langle jjT \rangle &= n_s^j \langle jjT \rangle_{\text{free boson}} + n_f^j \langle jjT \rangle_{\text{free fermion}} + p_j \langle jjT \rangle_{\text{parity odd}}, \\ \langle TTT \rangle &= n_s^T \langle TTT \rangle_{\text{free boson}} + n_f^T \langle TTT \rangle_{\text{free fermion}} + p_T \langle TTT \rangle_{\text{parity odd}}, \end{aligned} \tag{1}$$

where $\langle .. \rangle_{\text{free boson}}$, $\langle .. \rangle_{\text{free fermion}}$ denote the correlator a real free boson and a real free fermion respectively, while $\langle .. \rangle_{\text{parity odd}}$ refers to the parity odd structure.

[Osborn, Petkou 1994]

[Giombi, Prakash, Yin 2011]

- The numerical coefficients $n_s^{j,T}, n_f^{j,T}$ correspond to the parity invariant sector, while $p_{j,T}$ is the parity violating coefficient. The parity violating structure is unique to $d=3$ and it appears only for interacting theories.
- For a general conformal field theory in $d=3$, can there be constraints on the parameter space of three point functions?

- For parity even conformal field theories in $d = 4$ and higher dimensions, conformal collider bounds impose constraints on the parity even coefficients that occur in the three point functions.

[Maldacena, Hofman 2008]
[De Boer et al , Sinha et al. 2009]

- This involves studying the effect of localized perturbations at the origin. The integrated energy flux per unit angle, over the states created by such perturbations, is measured at a large sphere of radius r.

$$\begin{aligned}
\langle E_{\hat{n}} \rangle &= \frac{\langle 0|\mathcal{O}^\dagger E_{\hat{n}} \mathcal{O}|0\rangle}{\langle 0|\mathcal{O}^\dagger \mathcal{O}|0\rangle}, \\
E_{\hat{n}} &= \lim_{r\to\infty} r^2 \int_{-\infty}^{\infty} dt n^i T_i^t(t, r\hat{n}), \\
\mathcal{O} &\sim \frac{\epsilon^{ij} T_{ij}}{\sqrt{\langle \epsilon^{ij} T_{ij} | T_{ij} \epsilon^{ij} \rangle}}, \qquad \frac{\epsilon^i j_i}{\sqrt{\langle \epsilon^j j_j | j_i \epsilon^i \rangle}},
\end{aligned} \tag{2}$$

where, $\hat{n}$ is a unit vector in R^3, which specifies the direction of the calorimeter and $\mathcal{O}$ is the operator creating the localised perturbation.

- By demanding the positivity of energy flux, one obtains various constraints on the parameters of the three point function of the CFT, depending on the operators used to create the states $\mathcal{O}$
- As an example let us consider the energy functional created by insertion of the stress tensor in $d = 4$. We look at the three point function of the stress tensor.

$$\langle TTT \rangle = n_s^T \langle TTT \rangle_{\text{free boson}} + n_f^T \langle TTT \rangle_{\text{free fermion}} + n_v^T \langle TTT \rangle_{\text{vector}}.$$

- The positivity of energy flux then translates to the following bounds,

$$\begin{aligned} -\frac{t_2}{3} - \frac{2t_4}{15} + 1 \geq 0, \\ 2\left(-\frac{t_2}{3} - \frac{2t_4}{15} + 1\right) + t_2 \geq 0, \\ \frac{3}{2}\left(-\frac{t_2}{3} - \frac{2t_4}{15} + 1\right) + t_2 + t_4 \geq 0, \end{aligned} \tag{3}$$

where,

$$t_2 = \frac{15(n_f^T - 4n_v^T)}{3n_f^T + n_s^T + 12n_v^T}, \qquad t_4 = \frac{15(-2n_f^T + n_s^T + 2n_v^T)}{2(3n_f^T + n_s^T + 12n_v^T)}. \tag{4}$$

- We investigate the bounds imposed by such a gedanken experiment on the parameter space of parity violating conformal field theories in $d = 3$.

Positivity of energy flux

- Let us consider localised perturbations of the CFT in minkowski space.

$$ds^2 = -dt^2 + dx^2 + dy^2. \quad (5)$$

- The perturbations evolve and spread out in time. In order to measure the flux, we consider concentric circles.
- The energy measured in a direction (denoted by $\hat{n}$) is defined by,

$$E_{\hat{n}} = \lim_{r\to\infty} r \int_{-\infty}^{\infty} dt\ n^i T^t_i(t, r\hat{n}), \quad (6)$$

  where r is the radius of the circle on which the detector is placed and $\hat{n}$ is a unit vector which determines the point on the circle where the detector is placed.
- The expectation value of the energy flux measured in such a way should be positive for any state.

[Maldacena, Hofman 2008]

$$\langle E_{\hat{n}} \rangle \geq 0. \quad (7)$$

- We can also consider the detector to be placed at null infinity from the very beginning and integrate over the null working time. The two definitions are equivalent.

[Zhiboedov 2013]

- In order to calculate energy flux at null infinity, we introduce the light cone coordinates, $x^{\pm} = t \pm y$. We place our detector along y direction. The energy functional becomes

$$E = \lim_{x^+ \to \infty} (\frac{x^+ - x^-}{2}) \int_{-\infty}^{\infty} \frac{dx^-}{2} T_{--}. \quad (8)$$

- We are interested in expectation value of the energy operator on states created by stress tensor or current with specific polarizations.

- The normalized states are defined as

$$\mathcal{O}_E|0\rangle = \frac{\int dtdxdy\ e^{iEt}\mathcal{O}(t,x,y)|0\rangle}{\sqrt{\langle\mathcal{O}_E|\mathcal{O}_E\rangle}}, \tag{9}$$

where $\mathcal{O}$ are operators constructed from the current or stress tensor with definite polarizations.

- We look at perturbations created by stress tensor and current insertions

$$\mathcal{O}(\epsilon;T) = \epsilon_{ij}T^{ij}, \qquad \text{or} \qquad \mathcal{O}(\epsilon;j) = \epsilon_i j^i. \tag{10}$$

- We demand that energy flux measured in such a way be positive,

$$\langle E\rangle \quad = \quad \frac{\langle 0|\mathcal{O}^\dagger E\mathcal{O}|0\rangle}{\langle 0|\mathcal{O}^\dagger\mathcal{O}|0\rangle} \geq 0. \tag{11}$$

Energy matrix for charge excitations

- Let us look at the states created by current excitations first. We choose two independent polarisations for the charge excitations,

$$\epsilon^x = 1, \qquad \epsilon'^y = 1. \tag{12}$$

- This results in the following energy matrix

$$\hat{E}(j) = \begin{pmatrix} \langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle \\ \langle 0|\mathcal{O}_E^\dagger(\epsilon'; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon'; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle \end{pmatrix}.$$

- As an example let us see one such matrix element written out explicitly

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle = \frac{1}{\langle \mathcal{O}_E(\epsilon; j)|\mathcal{O}_E(\epsilon, j)\rangle} \times \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+ - x_1^-}{4} \int dx_1^- \langle j_x(x) T_{--}(x_1) j_x(0)\rangle, \tag{13}$$

where

$$\langle \mathcal{O}_E(\epsilon; j)|\mathcal{O}_E(\epsilon, j)\rangle = \int d^3x e^{iEt} \langle j_x(t, x, y) j_x(0).\rangle \tag{14}$$

- The condition of positivity of energy functional measured by the calorimeter then translates to demanding that the eigenvalues of the energy matrix be positive.
- The basic ingredient in this computation is the three point function of the stress energy tensor including the parity odd term.

$$
\begin{aligned}
\langle j(x)T(x_1)j(0)\rangle &= \frac{1}{|x_1-x|^3|x_1|^3|x|}\epsilon_2^\sigma I_\sigma^\alpha(x-x_1)\epsilon_3^\rho I_\rho^\beta(-x_1)\epsilon_1^{\mu\nu}t_{\mu\nu\alpha\beta}(X) \\
&\quad + p_j\frac{Q_1^2 S_1 + 2P_2^2 S_3 + 2P_3^2 S_2}{|x_1-x||x||-x_1|},
\end{aligned} \tag{15}
$$

where p_j is the parity odd coefficient.

[Osborn, Petkou 1994]

[Giombi, Prakash, Yin 2011]

- The first line corresponds to the usual parity even contribution while the 2nd part is the parity odd contribution to the three point function.

- For normalising the excited states, we also need the two point function of currents which is given by

$$\langle j_\mu(x) j_\nu(0)\rangle = \frac{C_V}{x^4} I_{\mu\nu}(x), \tag{16}$$

with

$$C_V = \frac{8}{3}\pi(c+e), \tag{17}$$

where

$$c = \frac{3(2n_f^j + n_s^j)}{256\pi^3}, \qquad e = \frac{3n_s^j}{256\pi^3} \tag{18}$$

- The parity even part contributes to the diagonal elements of the energy matrix while the off-diagonal contribution is due to the parity violating part.

- We look at the contribution to the energy matrix corresponding to the following parity violating term in the three point function,

$$\frac{p_j Q_1^2 S_1}{|x_1 - x||x|| - x_1|}, \tag{19}$$

where,

$$\begin{aligned} Q_1^2 &= \epsilon_1^\mu \epsilon_1^\nu \left(\frac{x_{1\mu}}{x_1^2} - \frac{x_{1\mu} - x_\mu}{(x_1 - x)^2} \right) \left(\frac{x_{1\nu}}{x_1^2} - \frac{x_{1\nu} - x_\nu}{(x_1 - x)^2} \right), \\ S_1 &= \frac{1}{4|x_1 - x||x|^3| - x_1|} \left(\varepsilon^{\mu\nu}{}_{\rho} x_\mu (x_1 - x)_\nu \epsilon_2^\rho \epsilon_3^\alpha x_\alpha \right. \\ &\quad \left. - \frac{\varepsilon^{\mu}{}_{\nu\rho}}{2} \left(|x_1 - x|^2 x_\mu + |x|^2 (x_1 - x)_\mu \right) \epsilon_2^\nu \epsilon_3^\rho \right). \end{aligned} \tag{20}$$

- Recall that the parity odd terms contribute to the off diagonal elements of the energy functional. The appropriate choice of polarisations is therefore,

$$\epsilon_2^x = \epsilon_3^y = \epsilon_1^{--} = 1. \tag{21}$$

- We are looking at essentially the following quantity.

$$\langle j_x(x) T_{--}(x_1) j_y(0) \rangle. \tag{22}$$

- For such a choice of polarisations, the parity violating term can be explicitly written out to be,

$$\begin{aligned} \mathcal{I} &= p_j \frac{Q_1^2 S_1}{|x_1 - x||x|| - x_1|}\Big|_{\epsilon_1^{--}=\epsilon_2^x=\epsilon_3^y=1}, \\ &= \frac{16 p_j}{64|x_1 - x|^2|x|^4| - x_1|^2} \left(\frac{x_{1-}}{x_1^2} - \frac{(x_1 - x)_-}{(x_1 - x)^2} \right)^2 \\ &\quad \Big(\varepsilon^{\mu\nu}{}_x x_\mu (x_1 - x)_\nu (x_+ - x_-) \\ &\quad - \frac{\varepsilon^{\mu}{}_{x+} - \varepsilon^{\mu}{}_{x-}}{2} \left(|x_1 - x|^2 x_\mu + |x|^2 (x_1 - x)_\mu \right) \Big). \end{aligned} \tag{23}$$

- The contribution to energy functional by the parity violating term $\mathcal{I}$ is given by,

$$\hat{E}(j)_{\mathcal{I}} = \frac{g_j^1(E)}{g_j^2(E)}, \tag{24}$$

where,

$$\begin{aligned} g_j^1(E) &= \int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+ - x_1^-}{4} \int_{-\infty}^{\infty} dx_1^- \mathcal{I}, \\ g_j^2(E) &= \sqrt{\langle j_x(x)|j_x(0)\rangle\langle j_y(x)|j_y(0)\rangle}. \end{aligned} \tag{25}$$

- We note correlators are not time ordered. However, the energy functional must be inserted between the operator creating the state and the one annihilating it.
- This is achieved by assigning the operator to the left, a larger negative imaginary part in time than the operators to the right i.e, $t_1 \to t_1 - i\epsilon$, $t \to t - 2i\epsilon$. Hence the light-cone coordinates change $x_1^\pm \to x_1^\pm - i\epsilon$, $x^\pm \to x^\pm - 2i\epsilon$.

- The calculation proceeds by taking first taking the limit $x_1^+ \to \infty$.
- Integral over x_1^- is then performed by using a Schwinger parametrization.
- The integrals performed in this manner give the same result as the analytic continuation of even dimensional integrals.
- The rest of the integrals are then best performed by integrating along the spatial direction x followed by the light cone directions $x^\pm$.
- The final result is

$$\begin{aligned} g_j^1(E) &= -\frac{3p_j}{160} E^2 \pi^3, \\ g_j^2(E) &= -\frac{8}{3} E \pi^3 (c+e), \end{aligned} \tag{26}$$

where, c and e are defined in terms of n_s^j and n_f^j.

$$c = \frac{3(2n_f^j + n_s^j)}{256\pi^3}, \qquad e = \frac{3n_s^j}{256\pi^3}. \tag{27}$$

- The complete energy functional matrix corresponding to charge excitations is given by,

$$\hat{E}(j) = \begin{pmatrix} \frac{E}{4\pi}(1-\frac{a_2}{2}) & \frac{E}{8\pi}\alpha_j \\ \frac{E}{8\pi}\alpha_j & \frac{E}{4\pi}(1+\frac{a_2}{2}) \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned} a_2 &= \frac{2(3e-c)}{(e+c)} = -\frac{2(n_f^j - n_s^j)}{(n_f^j + n_s^j)}, \\ \alpha_j &= \frac{3p_j\pi}{32(c+e)} = \frac{4\pi^4 p_j}{(n_f^j + n_s^j)}. \end{aligned} \tag{29}$$

- The diagonal contributions are due to the usual parity even part of the three point function. Off diagonal elements are solely due to the parity violating part.
- Positivity of energy flux implies,

$$a_2^2 + \alpha_j^2 \leq 4. \tag{30}$$

Energy matrix for stress tensor excitations

- The stress tensor is traceless, symmetric and conserved. Hence the allowed polarizations are,

$$\epsilon^{xy} = \epsilon^{yx} = \frac{1}{2}, \\ \epsilon'^{xx} = -\epsilon'^{yy} = 1. \tag{31}$$

- We can then define the energy matrix between these states as

$$\hat{E}(T) = \begin{pmatrix} \langle 0|\mathcal{O}_E^\dagger(\epsilon;T)\mathcal{E}\mathcal{O}_E(\epsilon;T)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon;T)\mathcal{E}\mathcal{O}_E(\epsilon';T)|0\rangle \\ \langle 0|\mathcal{O}_E^\dagger(\epsilon';T)\mathcal{E}\mathcal{O}_E(\epsilon;T)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon';T)\mathcal{E}\mathcal{O}_E(\epsilon';T)|0\rangle \end{pmatrix}. \tag{32}$$

- Explicitly written out, the matrix elements have the following form,

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon;T)\mathcal{E}\mathcal{O}_E(\epsilon;T)|0\rangle = \frac{1}{\langle \mathcal{O}_E(\epsilon;T)|\mathcal{O}_E(\epsilon,T)\rangle} \times \tag{33}$$

$$\int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+ - x_1^-}{4} \int dx_1^- \langle T_{xy}(x) T_{--}(x_1) T_{xy}(0)\rangle,$$

where,

$$\langle \mathcal{O}_E(\epsilon;T)|\mathcal{O}_E(\epsilon,T)\rangle = \int d^3x e^{iEt} \langle T_{xy}(t,x,y) T_{xy}(0)\rangle. \tag{34}$$

- The three point function of the stress tensor is the starting point of this calculation.

$$\langle T(x)T(x_1)T(0)\rangle = \frac{\epsilon_1^{\mu\nu}\mathcal{I}^T_{\mu\nu,\mu'\nu'}(x)\epsilon_2^{\sigma\rho}\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(x_1)\epsilon_3^{\alpha\beta}t^{\mu'\nu'\sigma'\rho'}{}_{\alpha\beta}}{x^6x_1^6} + p_T\frac{(P_1^2Q_1^2+5P_2^2P_3^2)S_1+(P_2^2Q_2^2+5P_3^2P_1^2)S_2+(P_3^2Q_3^2+5P_3^2P_1^2)S_3}{|x-x_1||x_1||-x|}, \tag{35}$$

where p_T is the coefficient of the parity violating part.

- The tensor structures in the first line correspond to the parity even part while the parity violating contribution is due the structures in the second line.

- Proceeding similarly we have the energy matrix due to the stress tensor excitations

$$\hat{E}(T) = \begin{pmatrix} \frac{E}{4\pi}(1-\frac{t_4}{4}) & \frac{E}{16\pi}\alpha_T \\ \frac{E}{16\pi}\alpha_T & \frac{E}{4\pi}(1+\frac{t_4}{4}) \end{pmatrix}, \tag{36}$$

where,

$$\begin{aligned} t_4 &= -\frac{4(30\mathcal{A}+90\mathcal{B}-240\mathcal{C})}{3(10\mathcal{A}-2\mathcal{B}-16\mathcal{C})} = -\frac{4(n_f^T - n_s^T)}{n_f^T + n_s^T}, \\ \alpha_T &= \frac{p_T}{256}\frac{240\pi}{5\mathcal{A}-\mathcal{B}-8\mathcal{C}} = \frac{8\pi^4 p_T}{3(n_f^T+n_s^T)}. \end{aligned} \tag{37}$$

- The positivity of energy matrix then translates to

$$t_4^2 + \alpha_T^2 \leq 16. \tag{38}$$

Large N Chern Simons theories

- $U(N)$ Chern Simons theory at level κ coupled to either fermions or bosons in the fundamental representation are examples of conformal field theories which violate parity. In the large N limit, these can be solved to all orders in the 't Hooft coupling

[Giombi et al, 2011]

[Aharony et al, 2011]

- We consider $U(N)$ Chern Simons theory at level κ coupled to fundamental fermions.
- The coefficients for the three point functions at the large N limit (planar limit) are given by,

$$n_s^T(f) = n_s^j(f) = 2N\frac{\sin\theta}{\theta}\sin^2\frac{\theta}{2}, \qquad n_f^T(f) = n_f^j(f) = 2N\frac{\sin\theta}{\theta}\cos^2\frac{\theta}{2},$$

$$p_j(f) = \alpha' N\frac{\sin^2\theta}{\theta}, \qquad p_T(f) = \alpha N\frac{\sin^2\theta}{\theta},$$

where the t 'Hooft coupling is related to θ by

$$\theta = \frac{\pi N}{\kappa}. \tag{39}$$

[Maldacena, Zhiboedov 2012]

- The numerical coefficients α, α' can be determined by a one loop computation in the theory with fundamental fermions.

[Giombi et al, 2011]

- We repeat the analysis to precisely determine the factors α, α'.

$$\alpha = \frac{3}{\pi^4}, \qquad \alpha' = \frac{1}{\pi^4}. \tag{40}$$

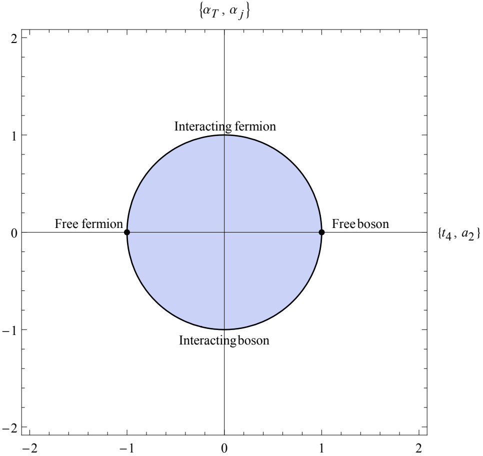
- The parameters of the three point function take the form

$$a_2 = -2\cos\theta, \qquad \alpha_j = 2\sin\theta, \qquad t_4 = -4\cos\theta, \qquad \alpha_T = 4\sin\theta. \tag{41}$$

- Thus the Chern-Simons theories with a single fundamental boson or fermion saturate the conformal collider bounds and they lie on the circles

$$a_2^2 + \alpha_j^2 = 4, \qquad t_4^2 + \alpha_T^2 = 16. \tag{42}$$

The location of the theory on the circle is given by θ the t 'Hooft coupling.


{α_T, α_j}
{t_4, a_2}
Interacting fermion
Free fermion
Free boson
Interacting boson
2
1
0
−1
−2
−2
−1
0
1
2

Conclusions

- We have obtained constraints on the three-point functions $\langle jjT \rangle, \langle TTT \rangle$ that apply to all (both parity-even and parity-odd) conformal field theories in $d = 3$ by imposing the condition that energy observed at the conformal collider be positive.
- In particular, if the parameters which determine the $\langle TTT \rangle$ correlation function be t_4 and α_T , where α_T is the parity-violating contribution and the parameters which determine $\langle jjT \rangle$ correlation function be a_2, and α_J , where α_J is the parity-violating contribution we have the following bounds

$$a_2^2 + \alpha_j^2 \leq 4, \qquad t_4^2 + \alpha_T^2 \leq 16 \tag{43}$$

- We have explicitly shown that for large N, $U(N)$ Chern-Simons theories with a single fundamental fermion or a fundamental boson, these parameters lie on the bounding circles of these disc.
- It will be interesting to generalise these observations to excitations created by higher spin currents.