

SYK/Tensor models

and

2-dim Quantum Gravity

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Sachdev-Ye-Kitaev Model

QM of N Majorana fermions

$$\Psi_i(t), i=1, \dots, N$$

$$H = \sum_{1 \leq i < j < k < l \leq N} J_{ijkl} \Psi_i \Psi_j \Psi_k \Psi_l$$

} Kitaev
Maldaena
Stanford

J_{ijkl} is a gaussian random coupling

$$\langle J_{ijkl} \rangle = 0 \quad \langle J_{ijkl}^2 \rangle = 6 \frac{J^2}{N^3}$$

Model has 2 important properties:

1. $N \rightarrow \infty, \beta J \gg 1$

Emergent reparametrization (almost)

2. $\langle \Psi_i(0) \Psi_j(t) \Psi_i(0) \Psi_j(t) \rangle \sim \frac{\beta J}{N} e^{\lambda_L t}$
 $\langle \Psi_i(0) \Psi_i(0) \rangle \langle \Psi_j(t) \Psi_j(t) \rangle$

holes $\lambda_L = 2\pi / \beta \hbar$ Chaos bound

$t_s \sim \frac{\hbar}{kT} (S - \ln \beta J)$ scrambling time
 $S \sim \ln N$

(large N + Strong coupling)

Soluble model to study holographic duality and black hole physics

Averaging over disorder, the large N

SD equations are expressed in terms of the bilocal fields:

$$G(z_1, z_2) + \Sigma(z_1, z_2) \quad (\text{euclidean time } z)$$

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega),$$

$$\Sigma(z_1, z_2) = J^2 G(z_1, z_2)$$

They are large N saddle points:

$$Z = \int \mathcal{D}G \mathcal{D}\Sigma \, e^{-NS} = Z_{\text{syk}} \text{ at large } N$$

$$S = -\frac{1}{2} \log \det (\partial_z - \Sigma)$$

$$+ \int dz_1 dz_2 \left(\Sigma(z_1, z_2) G(z_1, z_2) - \frac{J^2}{4} G(z_1, z_2)^4 \right)$$

Emergent reparametrization symmetry

$$\omega \ll J \quad (\text{or } \beta J \gg 1 \text{ at finite temp})$$

Action and SD eqns are invariant under:

$$G(z_1, z_2) \rightarrow [f'(z_1) f'(z_2)]^{\frac{1}{4}} G(f(z_1), f(z_2))$$

$$\Sigma(z_1, z_2) \rightarrow [f'(z_1) f'(z_2)]^{\frac{3}{4}} \Sigma(f(z_1), f(z_2))$$

Solution:

$$G_c(z) \sim \frac{\text{Sgn}(z)}{|Jz|^{\frac{1}{4}}}, \quad \Sigma_c(z) \sim J^2 G_c^3$$

finite temp. $\tau \rightarrow \tan \frac{\pi \tau}{\beta}$

$$G_c(z) \sim \text{Sgn}(z) \left(\frac{\pi}{\beta J \sin \frac{\pi z}{\beta}} \right)^{\frac{1}{2}}$$

Diff(1) spontaneously broken to $SL(2, \mathbb{R})$

Spontaneous Symmetry breaking

$$\text{Diff}(1) \rightarrow \text{SL}(2, \mathbb{R})$$

$$P^1 \quad S^1_\beta$$

$$G_c^{[f]}(z), \Sigma_c^{[f]}(z), \quad f(z) \in \text{Diff}(1)/\text{SL}(2, \mathbb{R})$$

$$S(G_c, \Sigma_c) = S(G_c^{[f]}, \Sigma_c^{[f]})$$

$$(G_c^{[f]}, \Sigma_c^{[f]})$$

$$(G_c, \Sigma_c)$$

$$Z \sim \int_{\frac{\text{Diff}(1)}{\text{SL}(2, \mathbb{R})}} d\mu(f) e^{-N S(G_c, \Sigma_c)} = \infty$$

[more careful analysis gives the same

$$\text{answer } Z \propto \text{Vol}\left(\frac{\text{Diff}(1)}{\text{SL}(2, \mathbb{R})}\right) = \infty$$

as $\beta J \rightarrow \infty$]

There is a J independent spectrum of evenly spaced dimensions $\{h_m\}$ $m=1, 2, \dots, \infty$

$$\text{corresponding to } \delta G(z_1, z_2) = G(z_1, z_2) - G_c(z_1, z_2)$$

$$\sim \mathcal{O}_m = \sum_i \psi_i \partial_z^{2m+1} \psi_i, \quad m=1, 2, \dots, \infty$$

Treatment of the Diff(1) / $SL(2, \mathbb{R})$ modes

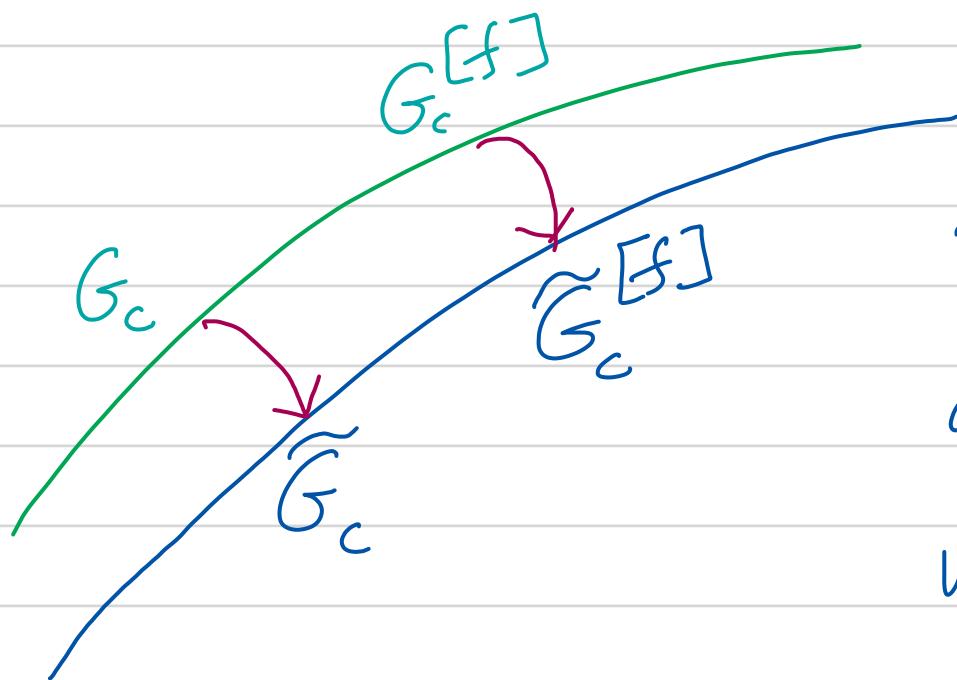
Solve SD eqs for large Jz / βJ :

$$\tilde{G}_c(z) = G_c(z) \left[1 - \frac{1}{2|Jz|} + \dots \right]$$

$$z \rightarrow \tan \frac{\pi z}{\beta}$$

$$\tilde{G}_c(z, \beta) = G_c(z, \beta) \left[1 - \frac{1}{2\beta J} \left(\frac{2 + \pi - 2\pi |z|/\beta}{\tan \frac{\pi |z|}{\beta}} \right) + \dots \right]$$

To explicitly break $SL(2, \mathbb{R})$ symmetry



Perturbation theory
around $\tilde{G}_c[f]$
well defined.

Effective action for low lying spectrum
is the Schwarzian

The Schwarzian

large N , large J

$$S_{\text{Sch}} \sim \frac{N}{J} \int_0^\beta dz \left[\left(\frac{f''}{f'} \right)^2 - \left(\frac{2\pi}{\beta} \right)^2 (f')^2 \right]$$

(Kitaev, Maldacena Stanford)

$$f(z) = z + \epsilon(z) \Rightarrow$$

$$\langle \epsilon(z) \epsilon(0) \rangle \sim \frac{\beta J}{N} \left(- \frac{(|z| - \pi)^2}{2} + (|z| - \pi) \sin |z| \right)$$

OTO $\Rightarrow$  $\sim \left(\frac{\beta J}{N} \right) e^{\frac{2\pi t}{k\beta}}$
 $\uparrow$
 real time

(dominant exchange in 4-point function).

Similar conclusions can be drawn for tensor models which do not have disorder.
(Witten, Gurau, Klebanov, Tarnopolsky)

To Summarize :

SYK/Tensor models at large N and strong coupling are soluble and exhibit chaotic behavior similar to that of a black hole.... also the appearance of $SL(2, \mathbb{R})$.

In contrast the $C=1$ matrix model does not exhibit chaotic behavior and hence could not describe the black hole of 2-dim. string theory. On the other hand it is exactly soluble for finite N . Hence it was possible to illustrate how an exact non-local description goes over into a 'effective local formulation' called 'collective field theory'. Also the Hilbert space of the exact gauge invariant non-local theory is far richer than the 'effective local theory'.

[Sengupta, Mandal, Dhar, SRW] $\longrightarrow$ LLM, Berenstein et al
[Sakita, Jenicki, Das,]

1. Holographic dual of SYK/TM

2. Finite N ($G_N \sim \frac{1}{N} \nrightarrow 0$) and the nature of the Hilbert space (Papadodimas, Raju, Jafferis)

$\rightarrow$ 1. Jackiw-Teitelboim dilaton gravity

$$S \sim \int d^2x \sqrt{g} \phi (R+2) + \text{boundary terms}$$

Allowed geometries are $R+2=0$

2. 2-dim. gravity with Polyakov action

(Mandal, Nayak, SRW 1702 04266 v2)

Motivation:

1 SYK model has emergent $\text{Diff}(1)$ symmetry
 $\xRightarrow{\text{broken}} SL(2, \mathbb{R})$ [large N , strong coupling]

2. Quantization of the coadjoint orbit of

$\text{Diff}(S^1)/SL(2, \mathbb{R})$ } loop group $F(z, \sigma)$

$\text{Diff}(\mathbb{R}^1)/SL(2, \mathbb{R})$ } (Witten, Alexeev Shatashvili,
Rai Rogers, Bakas)

$\Rightarrow$ Polyakov's 2-dim. gravity

$$\text{Action} \propto \frac{1}{G_N} \int d^2x \, R \frac{1}{\square} R \sqrt{g}$$

3. Requirement of asymptotic AdS_2

$$\text{geometry} \Rightarrow \mu \int d^2x \sqrt{g}$$

Also $R \left(\frac{1}{\square} R \right)^n$, $n > 1$, do not admit

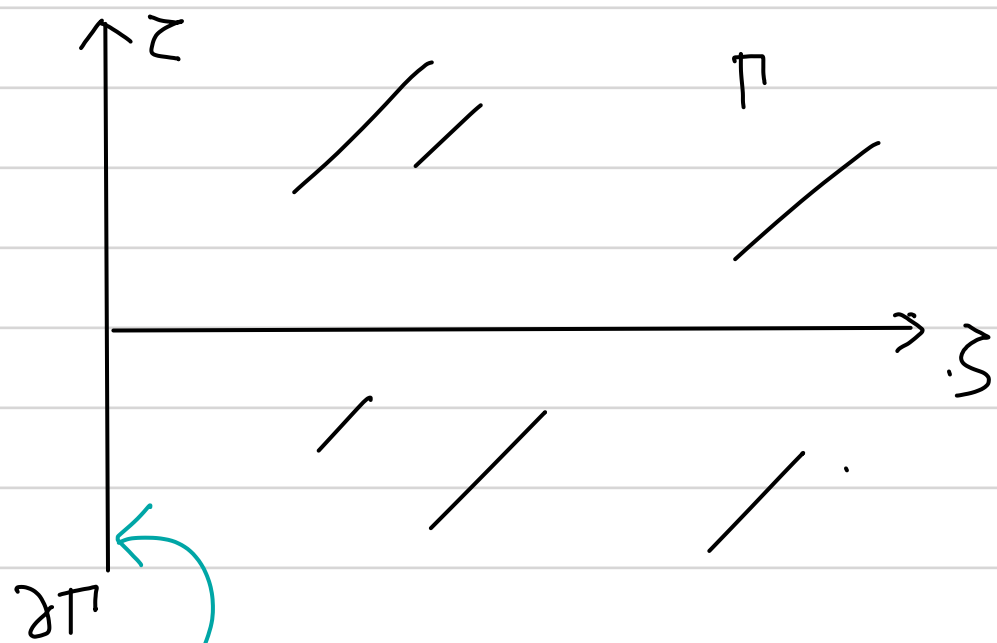
asymptotic AdS_2 geometries.

The Model:

$$S[g] = \frac{1}{16\pi b^2} \int_{\Gamma} \sqrt{g} \left[R \frac{1}{\square} R - 16\pi\mu \right]$$

$$\frac{1}{4\pi b^2} \int_{\partial\Gamma} \sqrt{\gamma} \left(K \frac{1}{\square} R + \overset{\substack{\text{extrinsic Curvature} \\ \text{on } \partial\Gamma}}{K \frac{1}{\square} K} \right)$$

Domain Γ is the right half plane



$$\frac{1}{\square} R(x) \equiv \int_{\Gamma} G(x,y) R(y)$$

(Euclidean signature)

SYK lives on $\partial\Gamma \simeq \mathbb{R}^1$

b^2 is the dimensionless Newton's constant,

$-\mu < 0$ is the cosmological constant.

Boundary terms are required so that eqns of motion follow from a variational principle.

Eqs of motion:

$$\textcircled{\text{I}} \quad R(x) = -8\pi\mu < 0$$

$$\begin{aligned} \textcircled{\text{II}} \quad & \int_{\Gamma} d^2x \sqrt{g} \left(\nabla_{\mu}^{(\omega)} \nabla_{\nu}^{(\omega)} G(\omega, x) - \frac{1}{2} g_{\mu\nu}(\omega) \square^{(\omega)} G(\omega, x) \right) R(x) \\ &= \frac{1}{2} \int_{\Gamma} d^2x \sqrt{g} \int_{\Gamma} d^2y \sqrt{g} \left[\left(\frac{\partial G(\omega, x)}{\partial \omega^M} \frac{\partial G(\omega, x)}{\partial \omega^M} \right. \right. \\ & \quad \left. \left. - g_{\mu\nu}(\omega) g^{\alpha\beta}(\omega) \frac{\partial G(\omega, x)}{\partial \omega^{\alpha}} \frac{\partial G(\omega, y)}{\partial \omega^{\beta}} \right) R(x) R(y) \right] \end{aligned}$$

If we use $g_{\alpha\beta} = \hat{g}_{\alpha\beta} e^{2\phi}$ $\left(\begin{array}{l} \hat{g} \rightarrow \hat{g} e^{\alpha} \\ \phi \rightarrow \phi - \frac{\alpha}{2} \end{array} \right)$

$$\textcircled{\text{I}} \quad -2\hat{\square}\phi + \hat{R} + 8\pi\mu e^{2\phi} = 0$$

$$\begin{aligned} \textcircled{\text{II}} \quad & \partial_{\mu}\phi \partial_{\nu}\phi - \nabla_{\mu}\nabla_{\nu}\phi - \frac{1}{2}\hat{g}_{\mu\nu}(\partial^{\alpha}\phi \partial_{\alpha}\phi - 2\hat{\square}\phi) \\ & - \frac{1}{2}\hat{g}_{\mu\nu}(4\pi\mu e^{2\phi}) = 0 \end{aligned}$$

I + II follow from a Liouville 'type' action by varying ϕ and $\hat{g}_{\alpha\beta}$

Liouville (type) action

$$S_L(\phi, \hat{g}) = -\frac{1}{4\pi b^2} \int_{\Gamma} \sqrt{\hat{g}} \left(\hat{g}^{\alpha\beta} \partial_{\alpha} \phi \partial_{\beta} \phi + \hat{R} \phi + 4\pi\mu e^{2\phi} \right) \\ + \frac{2}{4\pi b^2} \int_{\partial\Gamma} \sqrt{\hat{g}} \hat{K} \phi \\ + \frac{1}{4\pi b^2} \int_{\partial\Gamma} \sqrt{\hat{g}} \hat{n}^{\mu} \phi \partial_{\mu} \phi$$

Solutions

$$g_{\alpha\beta} = e^{2\phi} \hat{g}_{\alpha\beta}, \quad ds^2 = \hat{g}_{\alpha\beta} dx^{\alpha} dx^{\beta} \\ = \frac{d\zeta^2 + d\tau^2}{4\pi\mu \zeta^2}$$

Poincaré
right half plane: AdS_2
(a choice)

$$= \frac{dz d\bar{z}}{\pi\mu (z + \bar{z})^2}$$

$$\textcircled{\text{I}} \Rightarrow 2 \square \phi = \hat{R} + 8\pi\mu e^{2\phi}, \quad \hat{R} = -8\pi\mu$$

$$\phi = \frac{1}{2} \log \left[\frac{(z + \bar{z})^2 \partial g(z) \bar{\partial} \bar{g}(\bar{z})}{(g(z) + \bar{g}(\bar{z}))^2} \right]$$

$$\textcircled{\text{II}} \Rightarrow \partial^2 \phi - (\partial \phi(z, \bar{z}))^2 + 2 \frac{\partial \phi(z, \bar{z})}{(z + \bar{z})} = 0$$

$$\bar{\partial}^2 \phi - (\bar{\partial} \phi(z, \bar{z}))^2 + 2 \frac{\bar{\partial} \phi(z, \bar{z})}{(z + \bar{z})} = 0$$

constraints

$$\textcircled{\text{I}} + \textcircled{\text{II}} \Rightarrow \begin{cases} g(z), z \} = 0 \\ \{ \bar{g}(\bar{z}), \bar{z} \} = 0 \end{cases}$$

$\{, \}$ is the Schwarzian

$$g(z) = \frac{az + ib}{ic\bar{z} + d}, \quad \bar{g}(\bar{z}) = \overline{g(z)}$$

$$a, b, c, d \in \mathbb{C}, \quad ab + cd = 1$$

$$\text{If } g(z) + \bar{g}(\bar{z}) \Big|_{z+\bar{z}=0} = 0 \Rightarrow a, b, c, d \in \mathbb{R}^1$$

$$\text{and } g(z) \in SL(2, \mathbb{R}).$$

The rest of the 3 parameters lead to

$g(z)$ which do not preserve the boundary $(z + \bar{z} = 0)$ of AdS_2 .i.e. $g(z) + \bar{g}(\bar{z}) \Big|_{z+\bar{z}=0} \neq 0$

Small deformations of AdS_2 : $NAdS_2$

$$a = 1 + \delta a, \quad b = \delta b, \quad c = \delta c, \quad d = 1 - \delta a$$

$$\text{Re } \delta a = \text{Re } \delta b = \text{Re } \delta c = 0$$

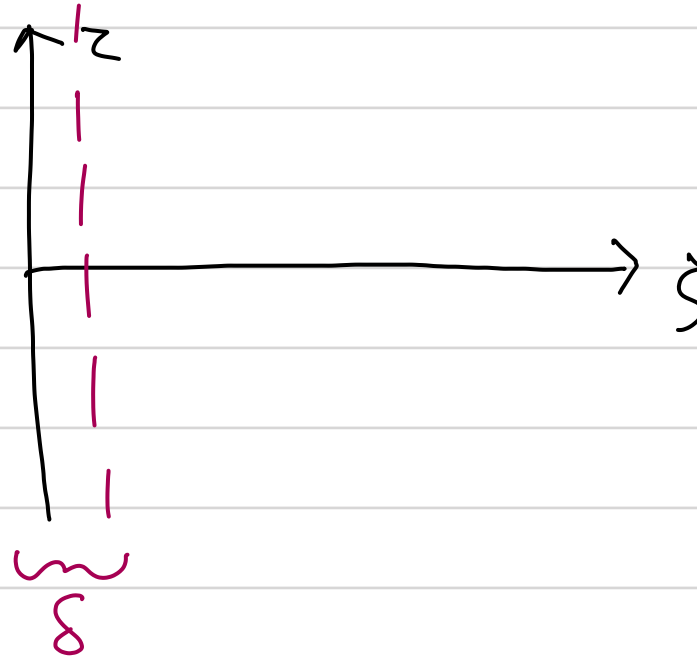
$$\phi = \frac{\delta g(z)}{\xi} + o(\delta a^2, \delta b^2, \delta c^2)$$

$$\delta g(z) = \text{Im}(\delta b) + 2 \underset{\text{O}}{\text{Im}(\delta a)} z + \underset{\text{O}}{\text{Im}(\delta c)} z^2$$

choose $\text{Im } \delta a = \text{Im } \delta c = 0$ ($g(z) = z - \text{Im } b$)

$$\phi = \frac{\text{Im}(\delta b)}{\xi} = \frac{\delta g}{\xi} \ll 1$$

If $(\delta g) \lesssim \delta = \text{cut off in } AdS_2$.



$$ds^2 = \frac{1}{4\pi\mu} \frac{d\xi^2 + dz^2}{\xi^2} \left(1 + \frac{2\delta g}{\xi} \dots \right)$$

$NAdS_2$

UV cut off

Asymptotically AdS_2 geometries ($AAIS_2$)

Fall off conditions!

1. $g_{\zeta\zeta} = \frac{1}{4\pi\mu\zeta^2}$, $g_{\zeta z} = o(\zeta^0)$, $g_{zz} = \frac{1}{4\pi\mu\zeta^2} + o(\zeta^0)$

2. Diff of AdS_2 that preserve these conditions:

$$\delta g_{\alpha\beta} = \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha$$

$$= \begin{bmatrix} -\frac{\epsilon^\zeta - \zeta \partial_\zeta \epsilon^\zeta}{2\pi\mu\zeta^3} & \frac{\partial_z \epsilon^\zeta + \partial_\zeta \epsilon^z}{4\pi\mu\zeta^2} \\ \frac{\partial_z \epsilon^\zeta + \partial_\zeta \epsilon^z}{4\pi\mu\zeta^2} & -\frac{\epsilon^z - \zeta \partial_z \epsilon^z}{2\pi\mu\zeta^3} \end{bmatrix}$$

3. Fix Fefferman-Graham gauge:

$$\delta g_{\zeta\zeta} = 0 = \delta g_{\zeta z}$$

4. Solution for Killing vectors:

$$\epsilon^\zeta = \zeta \delta f(z), \quad \epsilon^z = \delta f(z) - \frac{1}{2} \zeta^2 \delta f''(z)$$

$\delta f(z)$ is an arbitrary function $|\delta f| \ll 1$.

Diffs tangential to the boundary of AdS_2 .

Finite transformations:

$$\tilde{z} = f(z) - \frac{2\zeta^2 f''(z) f'(z)}{4f'(z)^2 + \zeta^2 f''(z)^2}$$

$$\tilde{\zeta} = \frac{4\zeta f'(z)^3}{4f'(z)^2 + \zeta^2 f''(z)^2}$$

can be obtained by restricting large diffs in AdS_3 to a plane $(\zeta, z, x=0)$.

$$\hat{ds}^2 = \frac{1}{4\pi\mu} \frac{(d\tilde{\zeta}^2 + d\tilde{z}^2)}{\tilde{\zeta}^2}$$

$\downarrow f(z)$

$\downarrow$

$\hat{g}[f]$

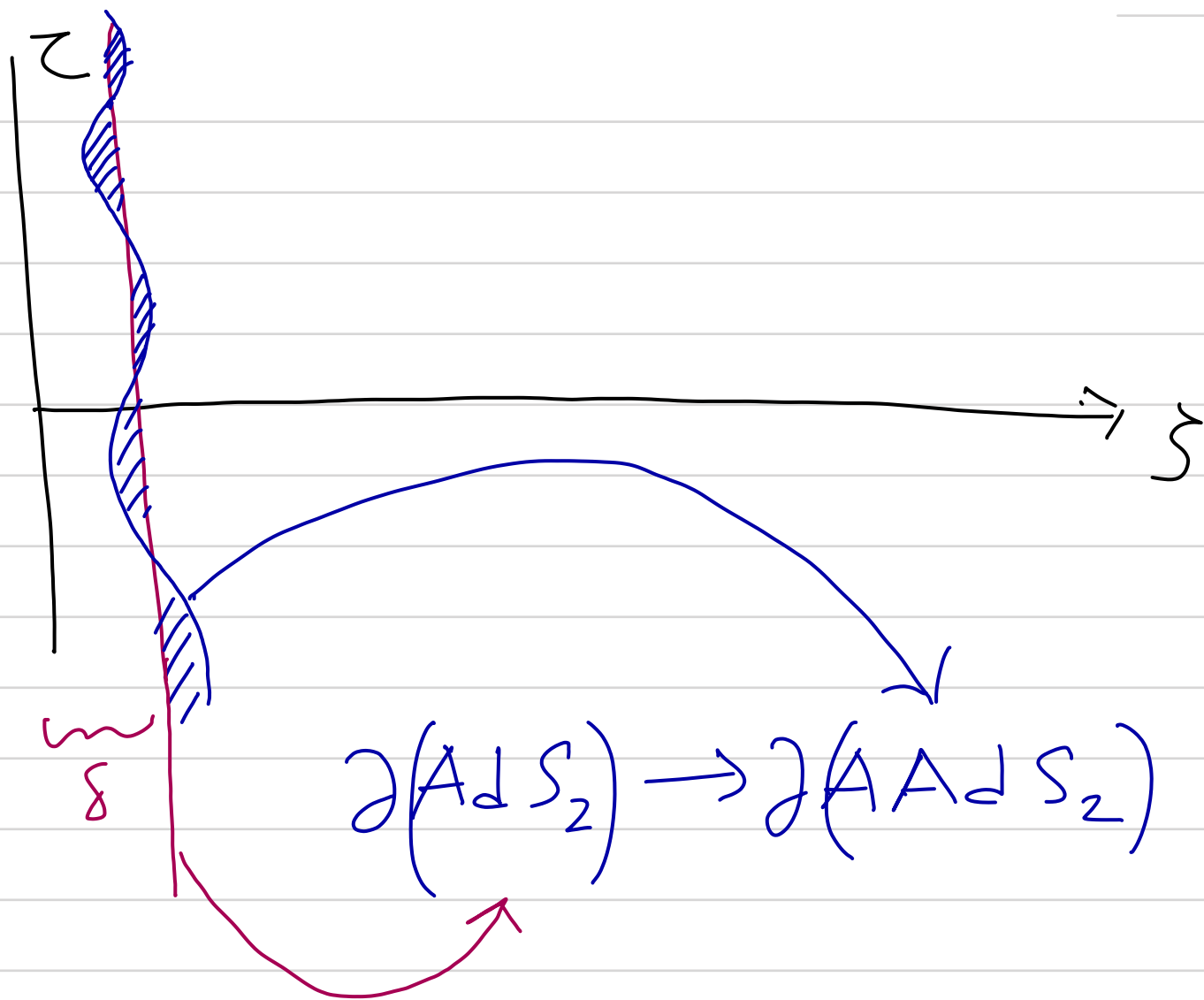
$$\hat{ds}_f^2 = \frac{1}{4\pi\mu} \zeta^2 \left(d\zeta^2 + dz^2 \left[1 - \zeta^2 \frac{\{f(z), z\}}{2} \right]^2 \right)$$

$$\{f, z\} = \frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2, \quad f' \neq 0$$

Cut-off: $\tilde{\zeta} = \delta$

$$\Rightarrow \zeta = \frac{2}{\delta} \frac{f'(z)}{f''(z)^2} \left[f'(z)^2 - \sqrt{f'^4 - \delta^2 f''(z)^2} \right]$$

$$\approx \delta (1 + \delta f'(z)) + \dots$$



In summary in the limit $b^2 \rightarrow 0$
the classical AAAdS_2 geometries are
entirely characterized by $\{f(z), f'(z) > 0\}$

$$ds^2 = e^{2\phi} \hat{ds}^2$$

Full Solution

$$\hat{ds}_f^2 = \frac{1}{4\pi\mu} \xi^2 \left(d\xi^2 + dz^2 \left[1 - \xi^2 \frac{\{f(z), z\}}{2} \right]^2 \right)$$

$$\phi = \underbrace{\delta g}_{\xi} + o(\delta a^2, \delta b^2, \delta c^2),$$

$$\delta g = \text{Im}(\delta b) + 2 \cancel{\text{Im}(\delta a)} z + \cancel{\text{Im}(\delta c)} z^2$$

The classical action:

Results:

$$1. \quad S_{\text{bulk}}(\hat{g}[f], \phi) - S_{\text{bulk}}(\hat{g}[f=z], \phi) \\ = \frac{1}{4\pi b^2} \frac{1}{\delta} [\epsilon(\infty) - \epsilon(-\infty)]$$

where $f(z) = z + \epsilon(z)$ (note $f(z) = z$ is the identity diff)

choosing $\epsilon(\infty) = \epsilon(-\infty) \Rightarrow$

$$\delta S_{\text{bulk}} = S_{\text{bulk}}(\hat{g}[f], \phi) - S_{\text{bulk}}(\hat{g}[f=z], \phi) \\ = 0$$

$$2. \quad \delta S_{\text{bd}} = S_{\text{bd}}^{\text{AAdS}_2} - S_{\text{bd}}^{\text{AdS}_2}$$

$$\delta S_{\text{bd}} = \frac{1}{2\pi b^2} \int d\tilde{z} \, \delta g(i\tilde{z}) \{ \tilde{f}(\tilde{z}), \tilde{z} \}$$

2-dim gravity path integral

$$Z \sim \int \underbrace{\begin{bmatrix} \partial f(z) \\ \frac{1}{f'(z)} \end{bmatrix}}_{\text{exclude integration over } SL(2, \mathbb{R})} e^{-\frac{\delta g}{2\pi b^2} \int dz \{f(z), z\}}$$

exclude integration over $SL(2, \mathbb{R})$.

At finite temperature using $\tau = \tan\left(\frac{\pi\theta}{\beta}\right)$

$$S_\beta = \frac{\delta g}{2\pi b^2} \int d\theta \left\{ \frac{\beta}{2} \tan\left(\pi \frac{f(\theta)}{\beta}\right), \theta \right\}$$

Correspondence with SYK.

$$\delta g \sim \frac{1}{J}, \quad b^2 \sim \frac{1}{N}$$

$$\ln Z = -\beta F = \frac{\delta g}{2b^2} \frac{1}{\beta} + (\text{constant})$$

matches with SYK

Summary

- The 2-dim gravity dual which is naturally motivated by the infrared Virasoro symmetry correctly accounts for the hydrodynamic Schwarzian action.
- Around smooth $AdS_2 + AAAdS_2$ geometries the only degree of freedom is the boundary diff. $\{f(z)\}$

Future

- It would be interesting to understand the operators
$$O_m = \sum_i \Psi_i \partial_z^{2m+1} \Psi_i, m \geq 1$$
 in this dual framework. Likely to source singular geometries in the bulk.
- $SL(2, \mathbb{R})$ current algebra may enable an exact solution of the proposed dual theory and an understanding of its Hilbert space beyond the semi-classical approximation.