Verifying the predictions of Conformal Bootstrap through lattice calculations

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Introduction

- A salient feature of second order transitions is *universality*, because of which very different systems show similar behaviour near the phase transition.
- At the critical point, fluctuations show up at all length scales making the system scale-invariant.
- It has been conjectured that systems are also conformal-invariant at the critical point. This has been rigorously demonstrated in the case of the two-dimensional Ising model.

CFT predictions for n-point correlators

- Conformal invariance imposes stringent conditions on n-point correlation functions of operators. In particular, the two-point function for a scalar field in $d \geq 2$ dimensions is fixed to be ($\mathbf{r_1} = (x_1, \ldots, x_d)$)

$$\langle \phi(\mathbf{r_1}) \phi(\mathbf{r_2}) \rangle \sim \frac{1}{|\mathbf{r_1} - \mathbf{r_2}|^{2\Delta_\phi}}, \tag{1}$$

  where Δ_ϕ is known as the anomalous dimension of the field $\phi(x)$.

- For four and higher point functions, the expressions are more difficult to calculate. One exact result is known for the four-point function for the two-dimensional Ising model, namely ($z_{12} = |\mathbf{r_1} - \mathbf{r_2}|$, etc.)

$$\frac{1}{\sqrt{2}} \left| \frac{z_{13} z_{24}}{z_{12} z_{34} z_{14} z_{23}} \right|^{\frac{1}{4}} \left[1 + \left| \frac{z_{12} z_{34}}{z_{13} z_{24}} \right| + \left| \frac{z_{14} z_{23}}{z_{13} z_{24}} \right| \right]^{\frac{1}{2}}. \tag{2}$$

CFT predictions for n-point correlators

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$$\langle \phi(\mathbf{r_1}) \phi(\mathbf{r_2}) \rangle \sim \frac{1}{|\mathbf{r_1} - \mathbf{r_2}|^{2\Delta_\phi}}, \tag{3}$$

  where Δ_ϕ is known as the anomalous dimension of the field $\phi(x)$.

- If we define $u = \left|\frac{z_{12}z_{34}}{z_{13}z_{24}}\right|^2$, $v = \left|\frac{z_{23}z_{14}}{z_{13}z_{24}}\right|^2$, then the expression becomes

$$\langle \phi(\mathbf{r_1}) \ldots \phi(\mathbf{r_4}) \rangle = \left| \frac{1}{4 z_{12} z_{34}} \right|^{\frac{1}{4}} \left[\frac{1 + u^{\frac{1}{2}} + v^{\frac{1}{2}}}{v^{\frac{1}{4}}} \right]^{\frac{1}{2}}. \tag{4}$$

CFT predictions for n-point correlators

- The above expression was derived by mapping the Ising model to a two-dimensional fermionic theory in which the fermion mass m plays the role of the temperature. As $T \to T_c$, the fermion mass $m \to 0$.
- In the limit $\mathbf{r_1} \to \mathbf{r_2}$, $\mathbf{r_3} \to \mathbf{r_4}$, the four point function becomes the two-point function of the field $\phi^2(x)$. Thus,

$$\lim_{\substack{\mathbf{r_1}\to\mathbf{r_2}\\ \mathbf{r_3}\to\mathbf{r_4}}} \Big\langle \phi(\mathbf{r_1})\phi(\mathbf{r_2})\phi(\mathbf{r_3})\phi(\mathbf{r_4}) \Big\rangle \sim \frac{1}{|\mathbf{r_2}-\mathbf{r_4}|^{2\Delta_{\phi^2}}}, \quad (5)$$

  where Δ_{ϕ^2} is the anomalous dimension of the field ϕ^2. Thus from the previous equation, $\Delta_{\phi^2} = 1/4$.
- We will be verifying these predictions through our Monte Carlo simulations. This is also my inititiation to calculating OPE coefficients on the lattice, which is a vast and ongoing field of research currently.

The Ising model

- The Ising model in d dimensions is a lattice of N spins, each of which can take a value of $+1$ or -1. The Hamiltonian of the system is given by

$$\mathcal{H} = -J \sum_{<ij>} s_i s_j. \tag{6}$$

The sum $\sum_{<ij>}$ is taken over all nearest neighbour pairs of spins.

- The thermodynamic properties of the system are derived from the partition function $\mathcal{Z}$ viz.

$$\mathcal{Z} = \sum_{s_i=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_N=\pm 1} e^{-\beta\mathcal{H}}. \tag{7}$$

Issues with a Direct Determination

- To observe scaling behaviour directly, one needs to be very close to the critical temperature $T = T_c$. In fact, CFT predictions are valid exactly at $T = T_c$.
- As we approach T_c however, the correlation length starts to diverge, $\xi \sim t^{-\nu}$. Our studies indicate that it is already $\gtrsim 500$ by $k_B T \sim 2.30J$.
- This means that we need to go to very large lattices to observe scaling behaviour. On small lattices, the two-point function would saturate instead of falling to zero.
- Currently however, we cannot go beyond lattices of size $L \sim 1500$.

Issues with a Direct Determination (contd.)

- Actually, this problem is not specific to just two-point correlators. It had already been noticed while trying to determine the critical exponents associated with standard thermodynamic observables such as the magnetic susceptibility χ.
- Direct observations of scaling behaviour require measurements very close to T_c on very large lattices. Even if this were possible, one should try to do better.
- *Finite-Size Scaling* (FSS) is an extension of the predictions of RG to finite lattices. We illustrate the method by first applying it to χ.

Finite-Size Scaling

- On an infinite lattice $\chi \sim t^{-\gamma}$. Since t itself is given by $t \sim \xi^{-1/\nu}$, we may write

$$\chi(t, L) = \xi^{\gamma/\nu} \tilde{f}(L/\xi). \tag{8}$$

- The function $\tilde{f}$ represents the finite-lattice corrections. On a finite lattice, instead of diverging, χ must saturate at a finite value. Thus we must have

$$\tilde{f}(x') \begin{cases} \rightarrow \text{constant}, & x' \rightarrow \infty, \\ \sim (x')^{\gamma/\nu}, & |x'| \ll 1. \end{cases} \tag{9}$$

Finite-Size Scaling (contd.)

- The previous equation is exact, however it contains the correlation length ξ which is difficult to determine. We can trade ξ for t and define a new variable x and a function f as follows

$$x' = x^{\nu}, \qquad f(x) = x^{\gamma}\,\tilde{f}(x^{\nu}). \tag{10}$$

- Thus, $\chi = L^{-\gamma/\nu} f\left(L^{1/\nu} t\right)$. In other words, a plot of $\chi L^{\gamma/\nu}$ versus $L^{1/\nu} t$ should yield the same curve for all lattice sizes L!

Finite-Size Scaling (contd.)

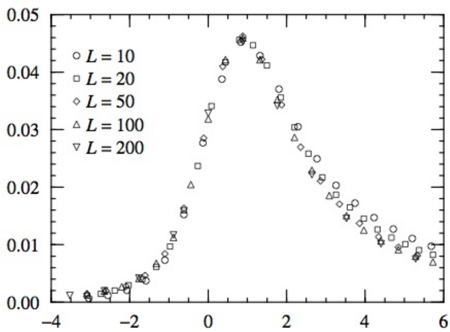


Fig. taken from "Monte Carlo Methods in Statistical Physics," M. Newman and G. Barkema (Clarendon Press).

Note however that for curve collapse to occur, it is necessary to use the right values of γ, ν as well as T_c. Thus FSS can be used to determine/verify the critical parameters of a theory.

Finite-Size Scaling for Correlators

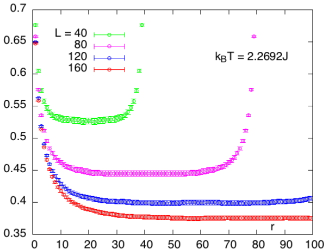
- With FSS, it is neither necessary to work with very large lattices nor be too close to T_c.
- We have extended the derivation above to make it applicable to correlators. Unlike χ, correlators are functions of r. Thus the scaling function will now be a function of *two* variables L/ξ and r/ξ. We may write

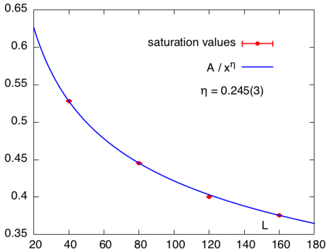
$$G_2(r,t,L) = \frac{1}{r^\eta}\,\tilde{\mathcal{G}}_2\left(\frac{r}{\xi},\frac{L}{\xi}\right). \tag{11}$$

- Studying limits as before, we have

$$\tilde{\mathcal{G}}_2\left(\frac{r}{\xi},\frac{L}{\xi}\to\infty\right)\to e^{-r/\xi},\quad \tilde{\mathcal{G}}_2\left(\frac{r}{\xi},\frac{L}{\xi}\lesssim 1\right)\xrightarrow[r\to L]{}\left(\frac{r}{L}\right)^\eta. \tag{12}$$

FSS for correlators (contd.)





- The second limit is because when $L < \xi$, the correlator must saturate instead of falling to zero. In fact, it predicts that the saturation must decrease as $L^{-\eta}$.
- We ran some quick tests on small lattices and found that this was indeed approximately the case (Right-hand fig. above).

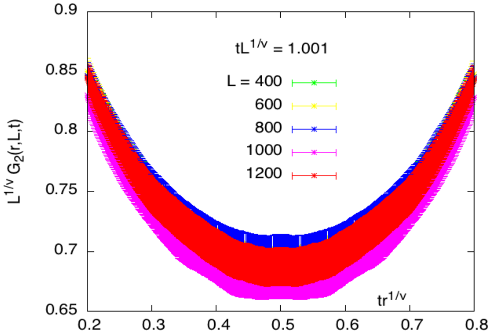
FSS for correlators (contd.)

- Trading ξ for t as before, we end up with

$$G_2(r,t,L) = \frac{1}{r^\eta}\,\mathcal{G}_2\left(r^{1/\nu}t, L^{1/\nu}t\right) \equiv \frac{1}{r^\eta}\,\mathcal{G}_2\left(x,y\right). \quad (13)$$

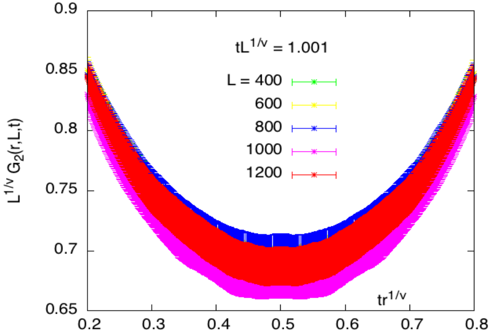
- Thus, $L^\eta G_2(r,t,L)$ (or even $r^\eta G_2(r,t,L)$) for different lattices should fall on top of each other provided we use the correct values of T_c, ν and most importantly, η.
- Note that since $\mathcal{G}_2$ is two-dimensional, this is actually a "surface collapse" rather than a curve collapse.
- We bypassed this by generating lattices for a fixed value of $L^{1/\nu}t$. We could do this because T_c and ν are well-known for the two-dimensional Ising model (And because we were only interested in η).

Curve Collapse for Correlators: First Results



- $\nu = 1$ for the 2d Ising model, while $k_B T_c = 2.26918\ldots J$. We calculated the two-point function for various values of L and t keeping $tL^{1/\nu}$ fixed to slightly above 1.0. We found good curve collapse within our statistical errors.

Curve Collapse for Correlators: First Results



- The above figure was produced by setting $\eta = 0.25$. Within our current errors, we found that we could vary η by a few percent to achieve similar results. Better statistics should lead to smaller errors, which in turn could lead to even better bounds on η.

Curve Collapse: Some Remarks

- A more straightforward, but vastly more tedious procedure to calculate η would have been:
  1. Calculate the two-point function (2PF) for various lattices $L_1 < L_2 < \ldots$ at fixed t.
  2. Beyond $L \gtrsim \xi$, the correlators are volume-independent. Fit the correlator to $f(x)$ and extract $\eta(t)$.
  3. Repeat for smaller t values.
  4. Finally, extrapolate to $t = 0$.

  In other words, calculate $\lim_{t\to 0} \lim_{L\to\infty}$.
- FSS is clearly more economical. One might still need to extrapolate to $L^{1/\nu}t = \infty$. We checked with somewhat larger values of $L^{1/\nu}t$ that the volume dependence was minimal.
- One could also try and fit the data directly to perhaps an RG-inspired fit ansatz and use it to try and extract η.

Curve Collapse for Four-Point Functions

The scaling function for four-point functions (4PFs) is more complicated. If the four points are denoted by $\mathbf{r_1}, \ldots, \mathbf{r_4}$, then

- Translation invariance implies that the 4PFs should actually depend only on the three difference vectors $\mathbf{r_{12}} = \mathbf{r_1} - \mathbf{r_2}$, etc.
- Furthermore, rotation invariance now reduces these to six scalars (in all dimensions), namely the three lengths $|\mathbf{r_{12}}|$, etc. and the three angles $\mathbf{r_{12}} \cdot \mathbf{r_{13}}$, etc. (In two dimensions, this number comes down to five).
- Lastly, conformal invariance predicts that in fact the only dependence is through the two ratios u and v defined earlier. This prediction can be tested by calculating the 4PF for different configurations having the same u and v.

Curve Collapse for Four-Point Functions (contd.)

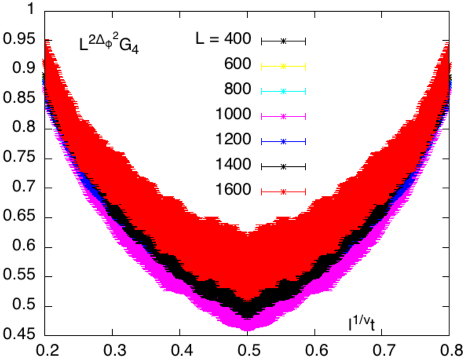
- Based on all of this, we may write

$$G_4(\mathbf{r_1}, \ldots, \mathbf{r_4}, t, L) = \frac{1}{r_{12}^{\Delta_\varphi} r_{34}^{\Delta_\varphi}} \tilde{\mathcal{G}}_4\left(\frac{r_{12}}{\xi}, \frac{r_{13}}{\xi}, \frac{r_{14}}{\xi}, \ldots, \frac{L}{\xi}\right), \tag{14}$$

  where the ... represent the angles.

- Previously, we had studied FSS for two sets of configurations: Squares of sides $l \times l$ and strips of length $1 \times l$. Of the two however, curve collapse can only be observed for squares since the angles change in the case of the strip as the length l is increased.

Curve Collapse for 4PF of Squares



The above figure shows that curve collapse works for squares of size $l \times l$. To achieve this, we had to put $\Delta_{\phi^2} = 0.25$. Again, within our current precision, this can be varied within a few percent.

Conclusions

- We have shown that calculating anomalous dimensions and OPE coefficients using lattice techniques is possible, although quite challenging.
- Two of the challenges that we face are:
  1. The correlators are quite sensitive to finite-size effects and one needs to work with rather large lattices, and
  2. We also need to be quite close to the critica temperature T_c, otherwise our calculations will be dominated by non-universal and non-conformal contributions.

Conclusions

- We have shown thatFinite-Size Scaling (FSS) is a promising method to obviate some of these finite-size effects. This is because we replace the conformal predictions by a finite-volume (but still universal) function, which we obtain by rescaling variables.
- Nevertheless, the method gets quite challenging as one goes to higher orders. It will be interesting to see how far these techniques can be pushed, and how far they will work for more complicated models ($3d$ Ising, $O(N)$ models, etc.).