Questions on Weak Gravity Conjecture

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Weak Gravity Conjecture (WGC)

Weak Gravity Conjecture can be motivated in a few ways, we will do it as follows:

We accept that extremal black holes, unless they are protected by symmetry (like supersymmetry), must be unstable in any correct theory of quantum gravity. If that is the case, there must exist states to which they can decay.

$$(M, Q) \to (M', Q - q) + (m, q), \text{ with } M > M', m. \tag{1}$$

Together with the extremality condition $M = Q$, this forces that one of the two pieces in the decay product must have charge to mass ratio ≥ 1. We will take $q/m \geq 1$.

This claim, that there exists a state in the theory which has charge bigger than mass in Planck units is interpreted as the statement that gravity is the weaker of the two forces.

It is not very clear to me what precisely qualifies as a particle/state in the WGC paradigm. (Eg: what about eigenstates of the quantum gravity Hamiltonian?) For instance, black holes qualify as states. Even though imprecise, WGC has triggered quite a bit of work/ideas, so there is probably something there.

WGC was orginally motivated by hopes to distinguish the landscape (of UV consistent gravity theories) from the swampland (of those that are not), by looking at the spectrum of the low energy EFT. [Arkani-Hamed, Motl, Nicolis, Vafa].

However, we will only be interested in the theory as a statement about the UV theory.

Aside: There exists a paper by a Prashant Sarawat from Maryland, where it is claimed that a WGC respecting UV theory's spectrum from an EFT point of view can change depending on whether the EFT is Higgsed or not. And this affects whether the EFT spectrum respects WGC or not. If true, this throws a spanner in the hope of using WGC as an EFT diagnostic of UV completion.

Convex Hull Condition [Cheung-Remmen]

One can write down the decay equations for a charged black hole with multiple $U(1)$'s:

$$Q^a = \sum_i n_i m_i \xi_i^a, \;\; M > \sum_i n_i m_i \tag{2}$$

Here i is the species of particle,

$$\vec{\xi}_i = (\xi_i^a), \;\; \xi_i^a = q_i^a/m_i. \tag{3}$$

One can rewrite the first set of equations as:

$$Z_a = \sum_i \sigma_i \xi_i^a, \;\; 1 > \sum_i \sigma_i. \tag{4}$$

where $Z_a = Q^a/M$ and $\sigma_i = n_i m_i/M$.

This latter condition is the condition for decay of the vector Z_a: it means that if that vector is inside the convex hull of the charge-to-mass-ratio vectors of the species of particles in the theorym then it can decay.

Note also that the extremality condition for the black hole takes the form

$$Z_a^2 = 1, \tag{5}$$

which means that if the unit circle is contained in the Convex Hull, all (extremal and sub-extremal) black holes can decay.

We will focus on one particular generalization of the Weak Gravity Conjecture, called the Lattice Weak Gravity Conjecture (LWGC). The basic idea here is to demand that a theory should respect WGC even afer Kaluza-Klein reduction.

Lets start with a theory that has a $U(1)$ gauge field and graviton. We will also assume that theory respects WGC, which means that there exists a particle of charge $Q_F = q$ and mass m_0 suh that

$$Z_0 \equiv |q|/m_0 > 1. \quad (6)$$

Lets reduce on a circle of radius R. We obtain a tower of KK modes with masses and charges

$$m^2 = m_0^2 + n^2/R^2, \;\; Q_F = q, \;\; Q_{KK} = n. \quad (7)$$

The Z-vectors of this spectrum of particles is given by

$$Z_n = \frac{(q, n/R)}{\sqrt{m_0^2 + n^2/R^2}} \tag{8}$$

One can straightforwardly check that these Z_n lie on the ellipsoid

$$Z_F^2/Z_0^2 + Z_{KK}^2 = 1 \tag{9}$$

Now, the sub-extremal black holes in the reduced theory are defined by

$$Q_{BH,F}^2 + Q_{BH,KK}^2 < M_{BH}^2 \tag{10}$$

or equivalently,

$$Z_F^2 + Z_{KK}^2 < 1. \tag{11}$$

This means, because $Z_0 > 1$, that the ellipsoid on which the reduced particles lie, is super-extremal. But this is not enough.

For WGC to hold, we also need that the black hole region in the reduced theory should be contained in the Convex Hull of the Z-vectors of the particles in the reduced theory. This is the necessary and sufficient condition for black holes of all charges to decay.

Draw picture on chalkboard for $m_0 > 0$ and $m_0 = 0$. The $m_0 = 0$ case is of particular interest for our purposes.

For any finite value of m_0 we can show that CHC is violated for some small enough compactification radius R. We will treat the $m_0 = 0$ case as a limiting version of the finite m_0 case.

How can one avoid the problem that KK-reduction results in violation of the WGC?

Heidenreich, Reece and Rudelius suggest that the way out is not to relax the GWC, but instead to make it more stringent. They require that-

LatticeWGC: For every point $\vec{Q}$ on the charge lattice, there is a particle of charge $\vec{Q}$ such that its charge-to-mass ratio is at least as large as that of a semi-claassical extremal (non-rotating) black hole with charge $\vec{Q}_{BH} \propto \vec{Q}$.

They give evidence for this by showing that it implies the Convex Hull condition and is preserved under torus reductions, in various examples (from Kaluza-Klein/string theory). Also, some circumstantial evidence from higher derivative corrections and modular invariance, etc.

A New Spin on Weak Gravity Conjecture

So far we discussed the Weak Gravity Conjecture (WGC) in the context of $U(1)$ charged black holes and particles, where it first arose.

But the arguments based on extremal black hole instability that are used to motivate it, apply equally well to rotating black holes and states with spin.

First observation: the basic WGC when appropriated to spin is trivially satisfied in any theory of quantum gravity because the graviton has non-zero spin but zero mass.

Question 1: What about the more stringent Lattice WGC for spinning black holes?

Question 2: Are there any constraints arising from dimensional reduction?

Question 3: Do these constraints force the existence of a higher spin tower?

Question 4: Can we see some hints of string theory in such a spectrum?

In the remainder of this talk, we will dicuss some of these questions, but will not answer them.

Analogous to the case of the convex hull with multiple charges, one can write down the decay equations for a charged rotating black hole:

$$J = \sum_i n_i m_i \xi_i^J, \;\; Q = \sum_i n_i m_i \xi_i^Q, \;\; M > \sum_i n_i m_i \tag{12}$$

Here again i is the species of particle,

$$\vec{\xi}_i = (\xi_i^J, \xi_i^Q), \;\; \xi_i^J = j_i/m_i, \;\; \xi_i^Q = q_i/m_i. \tag{13}$$

One can reqrite the first set of equations as before:

$$Z_J = \sum_i \sigma_i \xi_i^J, \;\; Z_Q = \sum_i \sigma_i \xi_i^Q, \;\; 1 > \sum_i \sigma_i. \tag{14}$$

where $Z_J = J/M$, $Z_Q = Q/M$, and $\sigma_i = n_i m_i/M$.

Straightforwardly, this again leads us to the convex hull condition. If the vector $\vec{Z}$ is in the convex hull of the vectors $\vec{\xi}_i$, then the former can decay:

$$\vec{Z} = \sum_i \sigma_i \vec{\xi}_i, \quad 1 > \sigma_i. \tag{15}$$

So far, the situation is as it was for the case with multiple charges. But there is one big difference: rotating black holes satisfy a different extremality condition.

$$M^2 = Q^2 + J^2/M^2. \tag{16}$$

I am being schematic, and not careful about factors of two etc. One can of course also consider more general configurations with more spins and charges etc in appropriate number of dimensions.

In terms of the Z's, the extremality condition no longer looks like a circle, instead it looks like

$$Z_J^2/M^2 + Z_Q^2 = 1. \tag{17}$$

Now lets consider the natural generalization of the LatticWGC for spin.

LatticeWGC including spin: For every point $\vec{Q} = (j, q)$ on the charge lattice, there is a particle of charge $\vec{Q}$ such that its charge-to-mass ratio is at least as large as that of a semi-claassical extremal (rotating) black hole with charge $\vec{Q}_{BH} \propto \vec{Q}$.

Note that this a word-for-word adaptation of the original Lattice WGC.

The key question howver is, what does one mean by "large" in the $\vec{Q}$ space? In the case of $U(1)$ charges, there was a natural quadratic norm that one could define in the space of charges, which matched with the quadratic form that appears in the correponding black hole extremality bound.

Here on the other hand, no such immediate norm is clear to me: The rotating black hole extremality conditions depends not just on the charge to mass ratio, but also on the actual mass of the black hole, as we noted already.

However, we will conclude by noting one curiosity. Lets write out the LatticeWGC including spin explicitly to the extent that we can. It yields-

For every point $\vec{Q} = (j, q)$ on the lattice, there exists a particle with charge $\vec{Q}$ such that

$$|(j, q)/m| \geq |\frac{(J_{BH}, Q_{BH})}{\sqrt{J_{BH}^2/M_{BH}^2 + Q_{BH}^2}}| \tag{18}$$

with

$$(j, q) \propto (J_{BH}, Q_{BH}). \tag{19}$$

The trouble we noted with this statement is that we don't know what is a suitable norm to use.

But lets note anyway that the inequality is most stringent, if one were to let $M \to \infty$.

(Draw picture of degenerate ellipse on chalkboard.)

This leads to

$$|(j,q)/m| \geq |(J_{BH}/Q_{BH},1)| = |(j/q,1)| \tag{20}$$

Now, lets assume that there exists a norm of (j,q) that is quadratic homogeneous. This might be a striong assumption.

In fact, I think we have already assumed this, in drawing the $M \to \infty$ limit of the ellipsoid in the vector space of $\vec{Q}$.

Interestingly, with this choice of the norm, the Lattice WGC with spin is satisfied, provided

$$q \geq m, \tag{21}$$

in other words, the Lattice WGC with spin is satisfied, when the WGC (or LWGC) for the $U(1)$ charge is satisfied. Not quite sure if we put in what we wanted or not with the norm assumption, but at least superficially, it looks like one can make a statement about the existence of a whole tower of higher spin states in the theory.

This is perhaps encouraging, but also somewhat trivial. A sufficiently arbitrarily large collection of particles can have an arbitrarily high spin. To argue that what we are seeing is a vestige of string theory, we need evidence for something like a new scale α' (which controls the scale of non-locality in the theory). Stay tuned.

Concluding Questions

- Basic question: What is the analogue of the Convex Hull Condition for charges when the black hole has spin? To answer this, one needs to have a notion of norm in the vector space defined by the Z's.
- What is the connection between the spinning black hole extremality condition $M^2 = J^2/M^2 + Q^2$, with the string spectrum formula $M^2 \sim J/\alpha' + Q^2$? The latter can also be derived from dimensional reduction of an uncharged rotating state.
- Speculation (probably not original): Can one understand this difference as a renormalization effect?
- Possibly items 1 and 2 are related. But see also "Spinning Strings as Small black Rings", Dabholkar et al.

- Modular invariance was used as an argument for motivating LWGC. That should go through here also.
- Higher derivative corrections change $Q = M$ of extremal black holes to $Q < M$. This is viewed by AMNV as evidence that black holes should be viewed as particles that respect WGC. What about rotating black holes? Preliminary encouraging results for Gauss-bonnet theory in papers of [Kleihaus, Radu, et al.]
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Thank you!