# Thermal out-of-time-order correlators, KMS relations and spectral functions

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## Motivation

- Non-equilibrium statistical mechanics: Chaos and Thermalisation in Quantum Many-Body systems.
- Chaos Correlator: $\langle [V(0), W(t)]^2 \rangle$ grows like $e^{2\lambda_L t}$.
- Alternative diagnostic of quantum chaos.This is a 2-OTO correlator.
- Cosmology (QFT in non-equilibrium states), Precursors in Quantum Gravity and Black Holes [*Heemskerk, Marolf, Polchinski, Sully (2012)* - bulk reconstruction inside the BH horizon]

## Generalities

- Out-of-Time order (OTO) Wightman correlator (W.C.)
- Standard Feynman Path Integral computes time-ordered W.C. Choose canonical time ordering $t_1 > t_2 > .... > t_n$.
- $\langle O_1(t_1)O_2(t_2).....O_n(t_n)\rangle$ - just one item in a space of observables with $n!$ elements: $\langle O_{\sigma_1} O_{\sigma_2}......O_{\sigma_n}\rangle$
- Notation:
  $\langle 12...n\rangle \equiv \langle O_1(t_1)O_2(t_2).....O_n(t_n)\rangle \equiv Tr(\rho O_1(t_1).....O_n(t_n))$
- Consider correlators like $\langle 2\hat{1}4\hat{3}6....n....\rangle$.
- OTO number ($q$) of an $n$-pt. W.C. - no. of future turning point operators. $q \leq [\frac{n+1}{2}]$
- Hints that these are important observables in non-equilibrium dynamics: quantum chaos and scrambling.

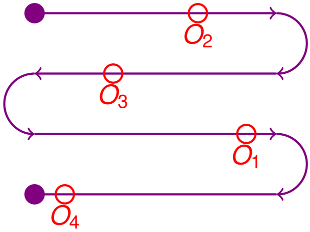
## Generalities

- OTOCs by analytic continuation of Euclidean correlators
- Begin with the Euclidean correlator: $G_E(\tau_1, \tau_2, ...., \tau_n)$
- Analytically continue to different domains to get different OTO W.C.

$$G_\sigma(t_1, t_2, ..., t_n) = \lim_{\epsilon_k \to 0} G_E(\tau_1, \tau_2, ...., \tau_n)|_{\tau_k = it_k + \epsilon_k},$$

with $\epsilon_{\sigma_1} > \epsilon_{\sigma_2} > ...\epsilon_{\sigma_n}$

- Generalised Schwinger-Keldysh Contour (Timefold)




computes $\langle O_4(t_4)O_1(t_1)O_3(t_3)O_2(t_2)\rangle$ [cf. Soumyadip's talk]

- $2k$-tuple d.o.f. : $\phi \to \phi_{iR,L}$

$$\mathcal{Z}[J_{iL}, J_{iR}] \equiv \int \prod_{i=1}^{k} \mathcal{D}\phi_{iR,L}\ exp\ i(S_{iR} + J_{iR}.\phi_{iR} - S_{iL} - J_{iL}.\phi_{iL})\rho_0$$

$$\mathcal{Z}[J_{iL}, J_{iR}] \equiv Tr(..U[J_{3R}]U^{\dagger}[J_{2L}]U[J_{1R}]\rho_0 U^{\dagger}[J_{1L}]U[J_{2R}]U^{\dagger}[J_{3L}]..)$$

- Functionally differentiate to get contour correlators - these encapsulate the OTOCs.
- Extended Hilbert Space $\otimes_{i=1}^{k}\mathcal{H}_{iR} \otimes_{i=1}^{k} \mathcal{H}^{*}_{iL}$ introduces redundancies

## Bases for presenting OTO-correlators

- Wightman Correlators: $\langle O_{\sigma_1} O_{\sigma_2} ...... O_{\sigma_n} \rangle$, $n!$ in number
- Nested Correlators: $\langle [...[[O_{\sigma_1}, O_{\sigma_2}]_{\pm}, ......, O_{\sigma_n}]_{\pm} \rangle$, $2^{n-2} n!$ in number (degeneracies captured by sJacobi relations).
- Contour Correlators: $\langle O_{1R} O_{2L} O_{3R} O_{1L} ........ \rangle$, $(2k)^n$ in number. Highly degenerate but computed naturally by Generalised SK Path Integrals.
- All of these are important in different settings - we will focus on the first two in thermal systems.

## Thermal OTO correlators and KMS relations

- Thermal Density Matrix is special: $\rho_T \sim e^{-\beta H}$
- $U(-i\beta)$: imaginary time evolution (of magnitude $\beta$)
- $A(t - i\beta) = \rho_T^{-1} A(t) \rho_T$ leads to the Kubo-Martin-Schwinger (KMS) relation for thermal Wightman correlators:

$$Tr[\rho_T A_1(t_1) A_2(t_2) ... A_n(t_n)] = Tr[\rho_T A_n(t_n - i\beta) A_1(t_1) ... A_{n-1}(t_{n-1})]$$

- Leads to analyticity conditions for Euclidean thermal correlators.
- KMS action simpler in frequency domain:

$$\langle 123.....n \rangle = e^{\beta\omega_1} \langle 23....n1 \rangle = e^{\beta(\omega_1+\omega_2)} \langle 34.....n12 \rangle$$
$$= ... = e^{\beta(\omega_1+\omega_2+...+\omega_(n-1))} \langle n12....(n-1) \rangle$$

## Thermal OTO correlators and KMS relations

- Breaks up set of $n!$ W.C. into $(n-1)!$ equivalence classes each with $n$ elements
- Start with any W.C. and repeated KMS action generates an orbit of $n$ elements
- Clearly for $\rho_T$ we need to compute $(n-1)!$ correlators (spectral functions) to span the whole space of OTO observables
- KMS action (involving cyclic shift of operators) clearly does not preserve OTO no. within the class (but changes it at most by one)

$$\langle \ldots \rangle_1^q \to \langle \ldots \rangle_2^{q+1} \to \langle \ldots \rangle_3^{q+1} \to \ldots\ldots\ldots \langle \ldots \rangle_n^q$$

- For thermal correlators $q \leq [n/2]$ : 4-pt. functions for 2-OTOCs, 6-pt. functions for 3-OTOCs etc.

## Spectral functions and the nested commutator basis

- $n!$ Wightman OTO $n$-pt. functions but for a thermal state - $(n-1)!$ spectral functions
- Can be chosen to be $\langle 1\sigma(23...n)\rangle$, each labels a KMS equivalence class

$$\langle \Sigma(123....n)\rangle = \sum_{\sigma\in S_{n-1}} a_\sigma \langle 1\sigma(23...n)\rangle$$

- There are $n!$ W.C. and $2^{n-2}n!$ nested correlators, but only $n!/2$ nested full commutators.
- Commutators capture linear response (Kubo)
- For thermal correlators $(n-1)!$ spectral functions, exactly $(n-1)!$ nested commutators $[1\sigma(23...n)]$:

$$\langle [...[[A,B],C]....,X]\rangle \equiv [ABC....X]$$

- Basis for any thermal correlator

$$\langle \Sigma[...[[1,2]_\pm,3]_\pm....,n]_\pm\rangle = \sum_{\sigma\in S_{n-1}} b_\sigma [1\sigma(23...n)]$$

## KMS relations and Fluctuation-Dissipation Relations (FDR)

- Fluctuations *in* equilibrium $\leftrightarrow$ Linear response *near* equilibrium. $\langle X_\omega X_\omega \rangle \sim Im(\chi_\omega)$
- Fluctuations: correlators (anti-commutators). Response captured by retarded Green functions/ causal commutators.
- Basic 2-pt. FDR in any thermal system (follows simply from KMS relation)

$$\langle AB \rangle = \frac{1}{2}\big(1 - \coth(\beta\omega_B/2)\big)\langle [A, B] \rangle$$

$$\langle \{A, B\} \rangle = -\coth(\beta\omega_B/2)\langle [A, B] \rangle$$

- Caveat: RHS can get corrections at certain singular points.

## KMS relations and Fluctuation-Dissipation Relations (FDR)

- Higher point extensions
- Observation 1: FDR amounts to expressing arbitrary nested thermal correlators (or W.C.) in terms of nested commutators.
- Observation 2 : FDR can be formulated for general higher OTOCs ($\mathbb{Z}_2$ involution - as in literature [eg. Wang, Heinz (98)] - to restrict to 1-OTOs not needed).
- $n = 3$ results: $\mathrm{N}_i \equiv \coth(\beta\omega_i/2)$

$$\begin{aligned} \langle\{[1,2],3\rangle\} &= -\mathrm{N}_3\,[123], \\ \langle\{\{1,2\},3\}\rangle &= -\mathrm{N}_3\mathrm{N}_1\,[123] + \mathrm{N}_3\,(\mathrm{N}_1+\mathrm{N}_2)\,[132], \\ \langle[\{1,2\},3]\rangle &= \mathrm{N}_1\,[123] - (\mathrm{N}_1+\mathrm{N}_2)\,[132] \end{aligned}$$

- Wightman Correlators $\mathfrak{f}_{i,j} \equiv 1/(e^{\beta(\omega_i+\omega_j)} - 1)$

$$\langle 123\rangle = (1+\mathfrak{f}_1)(1+\mathfrak{f}_{1,2})\,[123] + (1+\mathfrak{f}_1)\mathfrak{f}_{1,3}\,[132]$$

## KMS relations and FDRs: $n = 4$ results

- Nested correlators:

$$\langle\{\{[1,2],3\},4\}\rangle = \frac{N_4}{N_3+N_4}\Big((N_3N_4+1)[1234]+(N_3^2-1)\,[1243]\Big) + N_4\frac{N_1^2-1}{N_1+N_4}\,([1423]-[1432])$$

- Thermal W.C. in terms of the nested commutator basis:

$$\begin{aligned}\langle 1234\rangle = \Big(&(1+\mathfrak{f}_{1,2})(1+\mathfrak{f}_{1,2,3})\,[1234]+(1+\mathfrak{f}_{1,2})\mathfrak{f}_{1,2,4}\,[1243]\\ &+\mathfrak{f}_{1,3}(1+\mathfrak{f}_{1,2,3})\,[1324]+(1+\mathfrak{f}_{1,3})\mathfrak{f}_{1,3,4}\,[1342]\\ &+\mathfrak{f}_{1,4}(1+\mathfrak{f}_{1,2,4})\,[1423]+\mathfrak{f}_{1,4}\mathfrak{f}_{1,3,4}\,[1432]\Big)(1+\mathfrak{f}_1)\end{aligned}$$

## Harmonic Oscillator

- For a SHO (frequency $\mu$) can explicitly compute nested/Wightman correlators of Composite operators $X^a(t)$ in the thermal state:

$$(2\mu)^3\langle X(t_1)X^2(t_2)X^3(t_3)\rangle_\beta = \frac{3}{(e^{\beta\mu}-1)^3}\times$$
$$2e^{i\mu(3t_{13}-2t_{12})} + 2e^{3\beta\mu}e^{-i\mu(3t_{13}-2t_{12})}$$
$$+ e^{i\mu t_{13}}\left[3+8e^{\beta\mu}+e^{2\beta\mu}+e^{-2i\mu t_{12}}e^{\beta\mu}(4+2e^{\beta\mu})\right]$$
$$+ e^{-i\mu t_{13}}e^{\beta\mu}\left[1+8e^{\beta\mu}+3e^{2\beta\mu}+e^{2i\mu t_{12}}(2+4e^{\beta\mu})\right]$$

- Nested commutators are simpler, as expected

$$[X(t_1)X^2(t_2)X^3(t_3)] = -\frac{24}{(2\mu)^3}\coth(\beta\mu/2)\sin(\mu t_{12})\sin(\mu t_{23})$$

## Chaos Correlator

- $C(t) = \langle [V(0), W(t)]^2 \rangle$ - a 2-OTOC identified by [*Kitaev and Maldacena, Shenker, Stanford*] as a measure of chaotic behaviour in a thermal quantum system.
- For interacting many-body thermal quantum systems: $C(t) \sim e^{2\lambda_L t}$ with $\lambda_L < 2\pi/\beta$ (bound on chaos)
- Since it involves a thermal density matrix, it can be written in terms of our thermal spectral functions in the nested commutator basis. Work in frequency domain: $\langle [V(\omega_1), W(\omega_2)][V(\omega_3), W(\omega_4)] \rangle$
- $$C(t) = \frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} d\omega_k \frac{e^{-i(\omega_2+\omega_4)t}}{1 - e^{-\beta(\omega_1+\omega_2)}} ([1234] - [1243])$$

## Chaos Correlator

- Useful to consider the regulated correlator:

$$F(t_1, t_2, t_3, t_4) \equiv Tr[\rho_t^{\frac{1}{4}}\ V(t_1)\ \rho_t^{\frac{1}{4}}\ W(t_2)\ \rho_t^{\frac{1}{4}}\ V(t_3)\ \rho_t^{\frac{1}{4}}\ W(t_4)]$$

Letting $F(0, t, 0, t) = F(t)$

$$F(t) = \int_{-\infty}^{+\infty} \prod_{k=1}^{4} \frac{d\omega_k}{(2\pi)^4}\ e^{-i(\omega_2+\omega_4)t}\ e^{\frac{\beta}{4}\sum_{j=1}^{4} j\,\omega_j}(1+\mathfrak{f}_1)$$
$$\Big((1+\mathfrak{f}_{2,4})\mathfrak{f}_4\,[1324] + \mathfrak{f}_{2,4}\,(1+\mathfrak{f}_2)\,[1342]$$
$$-(1+\mathfrak{f}_{1,2})\,(\mathfrak{f}_4\,[1234] + (1+\mathfrak{f}_3)\,[1243])$$
$$-\mathfrak{f}_{1,4}\,(\mathfrak{f}_3\,[1423] + (1+\mathfrak{f}_2)\,[1432])\Big)$$

## Harmonic Oscillator

- "Chaos" Correlator:

$$F(t) \equiv \langle X(0)X^2(t+\frac{i\beta}{4})X(\frac{i\beta}{2})X^2(t+\frac{3i\beta}{4})\,\rangle_\beta$$

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$$F(t) = 6\mathfrak{f}\left[e^{\frac{\beta\mu}{2}}[1+12\mathfrak{f}(1+\mathfrak{f})]+4(1+\mathfrak{f})(1+2\mathfrak{f})\cos(2\mu t)\right]$$

- Bounded $F$ for large $\mu$ implies $|Im(t)| < \beta/2$

## Outlook

- Analyticity and causality of thermal nested correlators - alternative route to the chaos bound ?
- OTO transport/Kinetic theory: implications of KMS equivalence?
- Simple examples: Anharmonic, damped oscillators, scalar field theory
- Understanding further OTO aspects of far-from-equilibrium FDRs / Jarzynski relation
- OTOCs in Lorentzian CFT - Causality constraints : dispersion relations to reconstruct correlators from discontinuity in an OTOC [S. Caron-Huot (2017)]
- What information (about thermalisation) is captured by higher OTOCs? [cf. Halpern et al (2017)]