A New Poisson Bracket Identity for Gravity

(or an excuse to review LQG!)

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I will motivate and display a new classical PB identity concerning the dynamics of GR.

Motivation for a search for this identity comes from considerations in Loop Quantum Gravity so most of talk will provide a quick review of LQG.

Briefly:
General Cov of GR implies dynamics generated by constraints. Ham constr generates dynamics normal to 3d slice, (spatial) diffeo constraint generates dynamics parallel to slice.

In LQG sptl diffeos are under control. Construction of Ham constraint oprtr is key open issue.

The new identity relates PB between pair of Ham constr to PB of pair of diffeo constraints suggesting certain avenues for progress towards defining Ham constr operator in LQG.

Plan of Talk:

1. **Review of the Hamiltonian formulation of General Relativity**

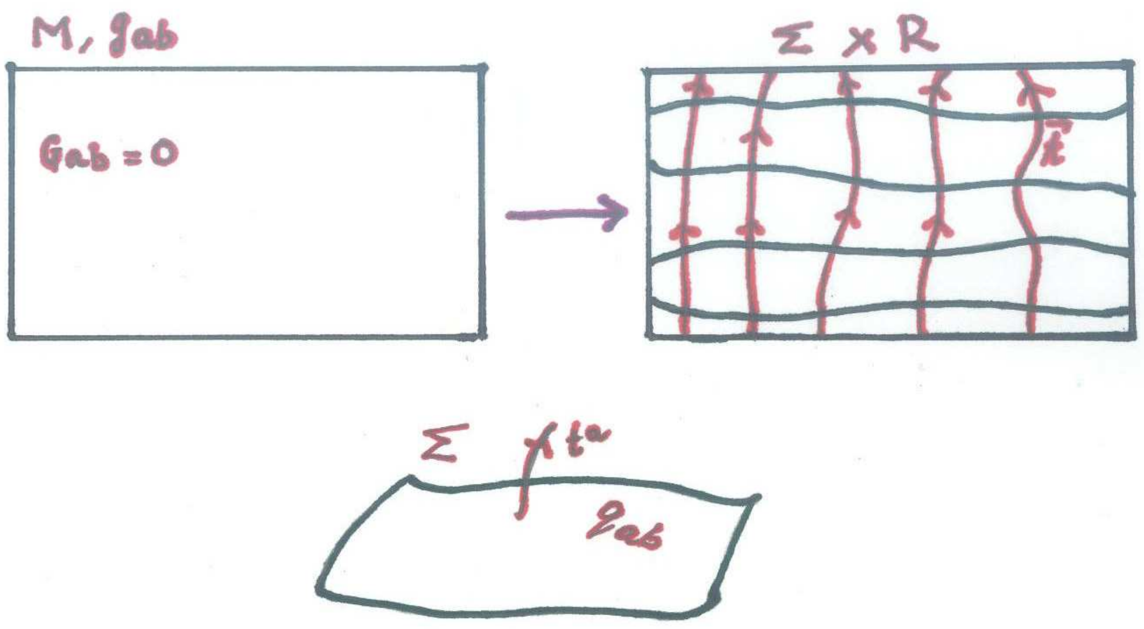
2. **A change of variables**

3. **Loop Quantum Gravity**

4. **Spacetime Covariance**

5. **The New Identity**

1. Hamiltonian GR



- $q_{ab}, \dot{q}_{ab} \rightarrow q_{ab}, p^{ab}$. $\{q, p\} \sim \delta$.
  $q \sim$ intrinsic geometry of Σ.
  $p \sim$ extrinsic geometry of Σ in M.
- Useful to decompose time flow into components normal, tangential to slice. $t^a = Nn^a + N^a$ $N \sim$ Lapse $N^a \sim$ Shift

- Hamiltonian density which evolves (q, p) normal to slice:
  $\mathcal{H}(q, p)$ `**Scalar /Hamiltonian Constraint**.
  Hamiltonian for evoltn along $N\vec{n}$ is $H(N) = \int_\Sigma N\mathcal{H}$.
- Hamiltonian density for evolution tangential l to slice:
  $D_a(q, p)$. `**Vector/Diffeomorphism Constraint**'.
  Hamiltonian for evoln along $\vec{N}$ is $D(\vec{N}) = \int_\Sigma N^a D_a$
- Evoltn along $\vec{t} = N\vec{n} + \vec{N}$:
  $\dot{q} = \{q, H(N) + D(\vec{N})\}$
  $\dot{p} = \{p, H(N) + D(\vec{N})\}$
- Einstein Eqns also constrain values of q, p.
- Constraints + Evolution equations $\equiv G_{ab} = 0$.
  Solns $q(x,t), p(x,t) \equiv$ Soln $g_{ab}(x,t)$
- Change time flow by changing N, N^a. Get same sptime geometry.

2. A Change of Variables:

- Instead of 3-metric use a triplet of orthonormal fields $E^a_i, i = 1, 2, 3$ so that $\sum_{i=1}^{3} E^a_i E^b_i \sim q^{ab}$. Rotated triad gives same metric. Under such 'internal rotations' 'i' transforms as $S0(3)$ Lie algebra valued index (equivalently, $SU(2)$ Lie algebra valued index).
- E^a_i is a $SU(2)$ Yang Mills Electric field! Conjugate variable is an $SU(2)$ connection A^i_a, $\{A, E\} \sim \delta$
- Due to triad rotation gauge, extra degrees of freedom in (E, A) relative to (q, p). Can ensure equivalence by imposing 'Gauss Law' constraint $\mathcal{D}_a E^a_i = 0$ so that (E, A) + $\mathcal{D}_a E^a_i = 0 \quad \equiv \quad (q, p)$
- Finally, can rexpress: $\mathcal{H}(q, p), D_a(q, p) \rightarrow \mathcal{H}(E, A), D_a(E, A)$.

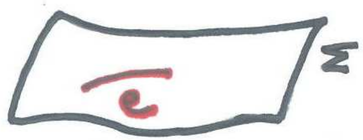
3. Loop Quantum Gravity:

- Canonical Quantization Approach:

- Choose set of functions on phase space. Represent each function f as operator $\hat{f}$ on Hilbert space of wave functions such that $\widehat{\{f,g\}} = \frac{[\hat{f},\hat{g}]}{i\hbar}$. Quantum Kinematics

- Construct quantum correspondents of generator(s) of evolution. Quantum Dynamics.

- Basic connection dep fns associated with **edges** and are called 'holonomies', $h_e(A)$.



- $h_e = P\exp\int_e A_a$
- $A^i_a \sim SU(2)$ Lie algebra $\Rightarrow h_e(A) = SU(2)$ matrix $\equiv h_e(A)^B_C$.

- Similarly, Electric field dep fns are electric fluxes through surfaces in Σ.
- Holonomies, fluxes do not require any background metric for their construction.

Quantum Kinematics

- Construct a repn of PB algebra between holonomies, electric fluxes.
- "Connection Repn": $\hat{f}(A)\psi(A) = f(A)\psi(A)$
  $\hat{g}(E)\psi = g(i\hbar\frac{\delta}{\delta A})\ \psi(A)$, f=holonomies, g=fluxes.
- Can construct a diffeomorphism invariant measure $d\mu(A)$ on space of (quantum) connections for which holonomy, flux operators satisfy correct adjointness conditions.
- As a result, spatial diffeomorphisms are represented unitarily on the kinematic Hilbert space of square integrable (wrto $d\mu(A)$) functions on the (quantum) conf space.

Quantum Dynamics:

- Dynamics along Σ generated by $D(\vec{N})$. Classicaly, this generates diffeomorphisms of (A, E). Since diffeos represented unitarily, turns out that kernel of diffeo constraint can be constructed.
- Dynamics normal to Σ is the KEY OPEN ISSUE. Need to construct $\hat{H}(M) = \int_\Sigma M\widehat{\mathcal{H}}(E, A)$.
  Problem: $\mathcal{H}(E, A)$ depends on local fields like connection. Basic connection operators nonlocal holonomies. Need to construct $\hat{A}$ from $\hat{h}_e$.
  Classically: $A \sim \frac{h_{\text{small edge}} - \mathbf{1}}{\delta}$ with $\delta \to 0$.
  QMly: Limit does not exist on operators due to diffeo inv of measure $d\mu(A)$.

Hamiltonian Constraint

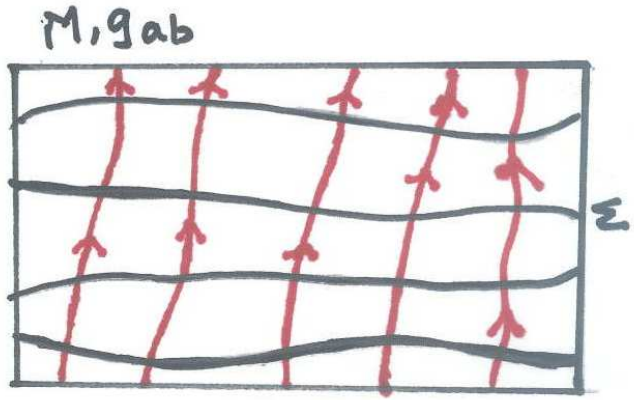
- Strategy:
  - Consider a triangulation T_δ of Σ i.e. divide Σ into coordinate cells of size $\sim \delta^3$
  - Within each cell, approximate local quantities in $\mathcal{H}$ to order δ by holonomies and suitable triad fns.
  - Obtain approximant $\mathcal{H}_I$. Approximate $H(N)$ by $H_\delta(N) \sim \sum_{cells} \delta^3 N_I \mathcal{H}_I$.
  - Replace fns in $\mathcal{H}_I$ by operators and get $\hat{H}_\delta(N)$.
  - Take $\delta \rightarrow 0$. Hope that while individual operator approximants dont have limit, maybe conglomeration of such operators which define $\hat{H}_\delta(N)$ does have limit in some operator topology.
- Thiemann was able to construct such continuum operator. But operator action depends on choices made at finite δ such as shape of edges which label holonomies. Infinite Ambiguity.

- Idea is to then confront any candidate operator with physical requirements. One such requirement is that the dynamics generated by the operator be spacetime covariant.

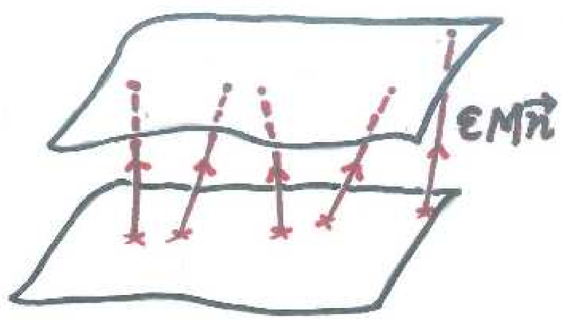
A brief digression....

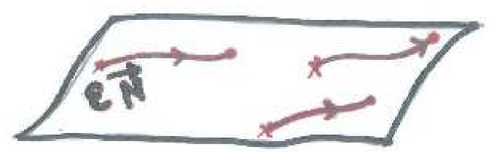
- **Spacetime Covariance**: Theory is sptime covariant if its solns describe spacetime geometries.
- GR is manifestly sptime covariant since solns to $G_{ab} = 0$ are sptime metrics.
- Since underlying theory is GR, the Ham formulation of GR is also sptime covariant. However the sptime metric is constructed out of the primary variables of the formulation which are evolving 3d objects. So sptime covariance is not as explicit in the Ham formulation. This raises the following general question: **Does any structure in the Ham formulation of a sptime cov theory acquire a characteristic form deriving from sptime covariance?**
  Answer:(H-K-T) Yes! The P.B. algebra between generators of evolution obtains a characteristic form deriving from spacetime covariance.

- Idea: Ham form of *any* theory of sptime geometry, g_{ab}, has sptime $=\Sigma \times R$. Evolution pushes Σ along time flow and generates sptime.



  - Consider infinitesmal normal and tangential displacements of Σ in M.





- Work out algebra of hypersurface deformations:
  $[\textbf{Ta}\,(\vec{N}_1), \textbf{Ta}\,(\vec{N}_2)] = \textbf{Ta}(\mathcal{L}_{\vec{N}_2}\vec{N}_1), \;\; [\textbf{No}(M), \textbf{Ta}(\vec{N})] = \textbf{No}(\mathcal{L}_{\vec{N}}M)$

  $[\textbf{No}(M_1)\textbf{No}(M_2)] = \textbf{Ta}(\vec{N}_{M_1,M_2,q_{ab}})$
- Can show this imples that PB between evolution generators must have this structure.
  For e.g. in GR:
  $\{D(\vec{N}_1), D(\vec{N}_2)\} = D(\mathcal{L}_{\vec{N}_2}\vec{N}_1) \quad \{H(M), D(\vec{N})\} = H(\mathcal{L}_{\vec{N}}M)$
  $\{H(M_1), H(M_2)\} = D(\vec{N}_{M_1,M_2,q_{ab}})$.

- In the classical theory, Sptime Covariance implies a characteristic PB structure between the generators of evolution.

- If we want some sort of a spacetime structure to emerge in quantum theory, impose condition that PB algebra be represented in quantum theory. Refer to this condition as **"Quantum Spacetime Covariance Condition"**

- **Idea**: Use `Quantum Sptime Cov' to restrict choices i.e. require PB algebra of evolution generators be appropriately represented in quantum theory.
- Recall: $\{H(M_1), H(M_2)\} = D(\vec{N}_{M_1,M_2,q_{ab}})$.

- In LQG want:
  $[\hat{H}(M_1), \hat{H}(M_2)] = i\hbar \hat{D}(\vec{N}_{M_1,M_2,\hat{q}_{ab}})$
- Two operators on LHS. 2 limits $\delta_1, \delta_2 \rightarrow 0$.
  On RHS one Diffeo operator so one limit.
  Need to choose finite triangulation operators so that continuum LHS= RHS. Easier to analyse if finite triangultn LHS, RHS structurally similar.
  Question: RHS written as commutator of 2 opertrs? RHS is diffeo, so commutator of 2 diffeos?
- Classical precursor to this question: Can we write classical RHS as **Poisson Bracket** between 2 diffeos ?
- Look for a **new PB identity**:
  $D(\vec{N}_{M_1,M_2,q_{ab}}) = \{D(\ ?\ ), D(\ ?\ )\}$
  What can these shifts be?

4. The New Identity

- From lapses M_1, M_2, define Electric Shifts as combination:
  $M_1 E^a_i \sim N^a_{1i}$, $M_2 E^a_i \sim N^a_{2i}$
- **New Identity** (Tomlin, MV):

  $\sum_{i=1}^{3} \{D(N^a_{1i}), D(N^a_{2i})\} = -2\alpha D(\vec{N}_{M_1,M_2,q})$

  ${N^c}_{M_1,M_2,q} \sim q^{cd}(M_1 \partial_d M_2 - M_2 \partial_d M_1)$
- This in turn implies:
  $\sum_{i=1}^{3} \{D(N^a_{1i}), D(N^a_{2i})\} = -2\alpha \{H(M_1), H(M_2)\}$

  Triads **crucial**- not pble with 3-metrics!

  α is numerical factor related to density wt of constraint.

  Identity relates PB of diffeos with PB of Ham constr.

  Recall that LQG handles sptl diffeos well....

- Since Identity relates PB of 2 diffeos to PB of 2 Ham constraints, suggests following question:

  Can we write evolution gen by $H(N)$ in terms of such diffeos ?

- Indications are that this is not hopeless Ashtekar...

Summary

My research is on Ham constraint in LQG. Many choices possible.

Restrict these choices thru non-trivial requirements. One such is sptime cov (very hard due to str functions).

Other requirements:
- Physical states encode propagation effects.
- Emergent sptime should be locally Lorentz inv.

At the moment dont know if any choices exist which satisfy all these requirements (not even sure how to pose the latter) but there is slow progress I think...