The Scalar Field Propagator on a Causal Set

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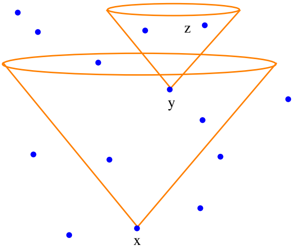
Outline

- The Causal Set Hypothesis - A Brief Review.
- Scalar Field Propagators in Causal Sets: The Flat Spacetime Ansatz
- An Extension to Small Causal Diamonds.
- Open Questions.

With Nomaan (RRI) and Fay Dowker (IC)

The Causal Set Hypothesis – L.Bombelli, J.Lee, D. Meyer and R. Sorkin, PRL 1987

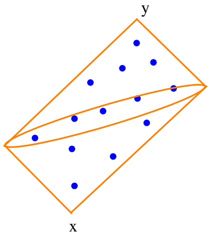
- The Causal Structure Poset $(M, \prec) \subset (M, g)$



- M: the **set** of events.
- $\prec$:
  - Acyclic: $x \prec y$ and $y \prec x \Rightarrow x = y$
  - Transitive: $x \prec y$ and $y \prec z \Rightarrow x \prec z$

The Causal Set Hypothesis – L.Bombelli, J.Lee, D. Meyer and R. Sorkin, PRL 1987

- The Causal Structure Poset $(M, \prec) \subset (M, g)$
- Spacetime Discreteness



Finite number of "atoms" of spacetime $\sim V/V_p$

The Causal Set Hypothesis

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The underlying structure of spacetime is a *causal set* or locally finite poset $(C, \prec)$

Motivation

Causal Structure + Volume Element = Spacetime

–Hawking-King-McCarthy-Malament Theorem

Causal Structure $\rightarrow$ Partially Ordered Set

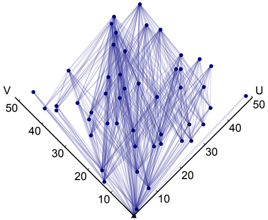
Spacetime Volume $\rightarrow$ Number

Order + Number $\sim$ Spacetime geometry

The Continuum Approximation

- Spacetime emerges as a "random lattice" generated via a Poisson process:

$$P_V(n) \equiv \frac{1}{n!} \exp^{-\rho V} (\rho V)^n$$



- Regular Lattice (eg. a diamond lattice): $n \not\sim V/V_p$ under boosts.
- In a Poisson sprinkling: $< n >= V/V_p$

The Continuum Approximation

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- Local Lorentz invariance: there are no preferred directions

– L.Bombelli, J.Henson, R. Sorkin, Mod.Phys.Lett. 2009

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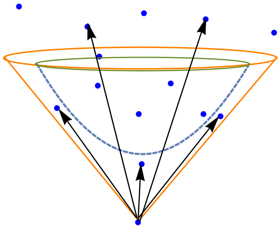
- Non-locality: A causal set need not be a fixed valency graph.



Non-Locality

- Non-finite/non-fixed valency $\Rightarrow \nabla \phi(x) \not\approx \frac{\phi(x+a)-\phi(x)}{a}$
- $\Box \phi(x)$ requires knowledge of whole causal past/future

– R. Sorkin, 2008, Henson, 2008, Dowker and Benincasa, 2010

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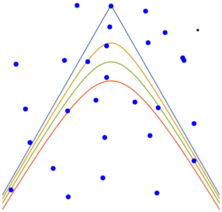

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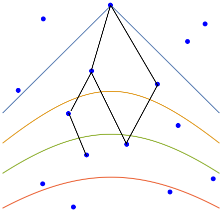


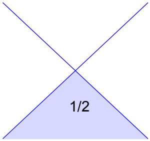
$$B\phi(x) = \frac{4\sqrt{\rho}}{\sqrt{6}}\left[-\phi(x) + \left(\sum_{L_1} - 9\sum_{L_2} + 16\sum_{L_3} - 8\sum_{L4}\right)\phi(y)\right] \quad (1)$$

$$\lim_{\rho\to\infty} \bar{B}\phi(x) = \langle B\phi(x)\rangle = \Box \phi(x) \quad (2)$$

Is there a way to construct the Green's function directly on a causal set?

Green's Function on a Causal Set : the 2D massless example

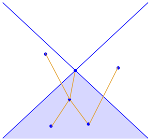
- $G(x,x') = \frac{1}{2}\Theta(x^0)\Theta(\tau^2)$



Green's Function on a Causal Set : the 2D massless example

- $G(x, x') = \frac{1}{2}\Theta(x^0)\Theta(\tau^2)$
- The Causal Matrix *is* the 2D massless Green's function:

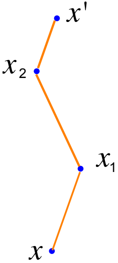
$$C_0 = \frac{1}{2}\begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 1 & 1 & 0 & 1 & \dots \\ \dots & 0 & 0 & 1 & 0 & 1 & 1 & \dots \\ \dots & 0 & 0 & 0 & 1 & 0 & 1 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix} \quad (3)$$



– A. Daughton (1993), R. Salgado(2008)

Massive Propagation: the "hop-and-stop" model – Steven Johnston, 2008, 2010

- "Causal Sequence" of k intervening elements $x \prec x_1 \prec x_2 \ldots \prec x_k < x'$.



- Amplitude for hopping $= a$ and Amplitude for stopping $= b$

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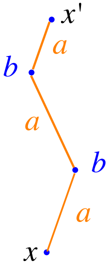

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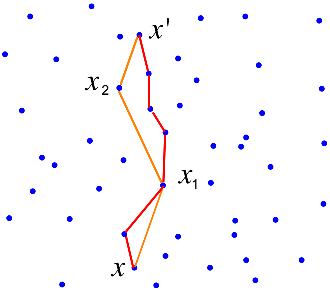
- “Causal Sequence” of k intervening elements $x \prec x_1 \prec x_2 \ldots \prec x_k < x'$.
- Amplitude for hopping $= a$ and Amplitude for stopping $= b$
- Ansantz:

$$K(x,x') = \sum_{k=0}^{\infty} a^{k+1} b^k \times \langle \# \text{ of k sequences} \rangle$$

- The choice of sequence is dimension dependent.

The 2D Ansatz: Counting Chains

- A sequence of elements $x \prec x_1 \prec x_2 \ldots \prec x_k < x'$ forms a *totally ordered* subset or k-chain.



The 2D Ansatz: Counting Chains

- A sequence of elements $x \prec x_1 \prec x_2 \ldots \prec x_k < x'$ forms a *totally ordered* subset or k-chain.
- $\langle C_k \rangle = \rho^k \int dx_1 \; \nu(x, x_1) \int dx_2 \; \nu(x_1, x_2) \ldots \int dx_k \; \nu(x_k, x')$

$$\nu(x, x') := \begin{cases} 1 & \text{if } x' \prec x \\ 0 & \text{otherwise} \end{cases}$$

- 2D Ansatz

$$K(x - x') = \sum_{k=0}^{\infty} a^{k+1} b^k \langle C_k \rangle$$

- $\langle C_k \rangle_\eta = \rho^k \chi_k \mathrm{Vol}(I[x, x'])^k = \rho^k \zeta_k \tau^{kn}$

—D. Meyer, 1988

Estimate for the Amplitudes

- Recursion Relation:

$$\langle C_{k+1}(x-x')\rangle = \rho \int dx_1 \ \nu(x-x_1)\langle C_k(x_1-x')\rangle$$

- Integral Equation:

$$K(x-x') = a\,\nu(x-x') + a\,b\,\rho \int dx_1\,\nu(x-x_1)K(x_1-x')$$

- $K(p) = a\,\nu(p) + a\,b\,\rho\nu(p)K(p) \quad \Rightarrow \quad K(p) = \frac{a\nu(p)}{1-ab\nu(p)}$
- Comparison with continuum $G(p) = \frac{1}{p^2+m^2} \quad \Rightarrow \quad a = \frac{1}{2}, b = -\frac{m^2}{\rho}$.
- The discrete propagator and the continuum propagator agree even for finite ρ.

Is $K(x-x') = \sum_{k=0}^{\infty} a^{k+1} b^k \langle C_k \rangle$ the propagator in all 2d spacetimes?

Extension to the Small Causal Diamond in 2d

- "Small" causal diamond $I[x, x']$.
- $g_{ab}(x-x') = \eta_{ab}(x') - \frac{1}{3}(x-x')^c(x-x')^d R_{acbd}(0) + O((x-x')^3), \qquad R\tau^2 << 1$
- $\langle C_1 \rangle = \rho \mathrm{Vol}([x, x']) \simeq \rho \mathrm{Vol}_\eta([x, x'])(1 + \alpha_1 R(x')\tau^2 + \beta_1 R_{00}(x')\tau^2)$.

J. Myrheim, 1978, G. W. Gibbons and S. N. Solodukhin, 2007.

- $$\begin{aligned} \langle C_k \rangle &= \rho^k \int dx_1 \sqrt{-g_1}\, \nu(0,1) \int dx_2 \sqrt{-g_2}\, \nu(1,2) \ldots \int dx_k \sqrt{-g_k}\, \nu(k, k+1) \\ &\simeq \langle C_k \rangle_\eta \left[1 + \tau^2 \alpha_k R(0) + \tau^2 \beta_k R_{00}(0) \right] \end{aligned}$$

$\alpha_k = -\frac{nk}{12(kn+2)((k+1)n+2)}, \quad \beta_k = \frac{nk}{12((k+1)n+2)}$.

– Mriganko Roy, Debdeep Sinha and Sumati Surya, 2013

Continuum Propagator in a Small Causal Diamond – Bunch and Parker, 1979

- $$(\Box - (m^2 + \xi R))G(x - x') = -\frac{1}{\sqrt{-g(x)}}\delta(x - x')$$
  $$\Downarrow$$
  $$(\Box - (m^2 + \xi R))\widetilde{G}(x - x') = -\delta(x - x')$$

  $$\widetilde{G}(x - x') = (-g(x))^{1/4} G(x - x')(-g(x'))^{1/4}$$

- $$\left(\Box_\eta + \frac{1}{3}(R^{a\ \ b}_{\ c\ \ d}(0) y^c y^d \partial_a \partial_b - R_d^{\ c}(0) y^d \partial_c) - (m^2 + (\xi - \frac{1}{6})R)\right)\widetilde{G}(x, x') = -\delta(x, x')$$

- $\widetilde{G}_0(k, x') = \frac{1}{k^2+m^2}$ and $\widetilde{G}_2(k, x') = -\frac{(\xi-1/6)R(x')}{(k^2+m^2)^2} = \frac{(\xi-1/6)R(x')}{2m}\partial_m(\widetilde{G}_0(k, x'))$.

- 2D: $\widetilde{G}(x - x') = \frac{1}{2}\Theta(y^0)\Theta(\tau^2)\left(J_0(m\tau) - \frac{\tau(\xi - \frac{1}{6})}{2m} J_1(m\tau)\right)$

The Discrete 2D case:

- Ansatz: $K(x-x') = \sum_{k=0}^{\infty} a^{k+1} b^k \langle C_k \rangle$
- In 2D $R_{acbd} = \frac{1}{2} R(g_{ab}g_{cd} - g_{ad}g_{bc})$
- $\langle C_k \rangle = \langle C_k \rangle_\eta (1 + R\tau^2(2\alpha_k - \beta_k)) = \langle C_k \rangle_\eta (1 - \frac{kR\tau^2}{24(k+1)})$.
- Flat spacetime choice of $a = 1/2$, $b = -m/\rho^2$ and $\xi = 0$

$$K(x-x') = \frac{1}{2}\Theta(y^0)\Theta(\tau^2)\left(J_0(m\tau) + \frac{R\tau^2}{24} J_2(m\tau)\right)$$

$$\updownarrow$$

$$G(x-x') = \frac{1}{2}\Theta(y^0)\Theta(\tau^2)\left(J_0(m\tau) + \frac{R\tau^2}{24} J_2(m\tau)\right)$$

The 4D Ansatz:

- $G(x, x') = \Theta(x^0)\Theta(\tau^2)[-\frac{1}{2\pi}\delta(\tau^2) + \frac{1}{4\pi\tau} m J_1(m\tau)]$
- Massless case
- Massive case

– Steven Johnston, 2008, 2010

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- $G(x, x') = \Theta(x^0)\Theta(\tau^2)[-\frac{1}{2\pi}\delta(\tau^2) + \frac{1}{4\pi\tau} mJ_1(m\tau)]$
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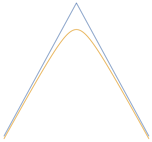
  - "First" Layer:



  - Nearest neighbours are "links": $x \prec y$ iff $\nexists z,\ x \prec z \prec y.$.
  - The "Link" Matrix $P_0(x, x') = \exp^{-\rho V(x,x')}$
  - $\lim_{\rho\to\infty} -\frac{\sqrt{\rho}}{2\pi\sqrt{6}} P_0(x, x') = -\frac{1}{2\pi}\delta(\tau^2).$
- Massive case

– Steven Johnston, 2008, 2010

The 4D Ansatz:

- $G(x, x') = \Theta(x^0)\Theta(\tau^2)[-\frac{1}{2\pi}\delta(\tau^2) + \frac{1}{4\pi\tau} m J_1(m\tau)]$
- Massless case
- Massive case – Steven Johnston, 2008, 2010
  - Choose the sequence $x \prec x_1 \prec x_2 \ldots \prec x_k < x'$ such that each $x_i \prec x_{i+1}$ is a link.
  - Ansatz:

$$K(x - x') = \sum_{k=0}^{\infty} a^{k+1} b^k \langle P_k \rangle$$

  - $\langle P_k \rangle = \rho^k \int\int\int dx_1 dx_2 \ldots dx_k \mu(x - x_1) \ldots \mu(x_k - x')$

    $\mu(x_i - x_{i+1}) = \exp^{-\rho \mathrm{Vol}([x_i, x_{i+1}])}$

  - Comparison with the continuum when $\rho \to \infty$ gives $a = \frac{\sqrt{\rho}}{2\pi\sqrt{6}}$ and $b = -\frac{m^2}{\rho}$.

Extension to a Small Causal Diamond: Massless, Conformally Coupled

- $G^{(4)}_{0,R}(x) = -\frac{1}{2\pi}\delta(\tau^2)\left(1 + \frac{1}{12}R_{ab}x^a x^b\right)$
- $K^{(4)}_{0,R}(x) = -\frac{\sqrt{\rho}}{2\pi\sqrt{6}}e^{-\rho V_\eta(x)}\left[1 + \frac{\rho V_\eta(x)}{180}R\tau^2 - \frac{\rho V_\eta(x)}{30}R_{ab}x^a x^b\right]$
- $\lim_{\rho\to\infty} K^{(4)}_{0,R}(x) = -\frac{1}{2\pi}\delta(\tau^2)\left(1 - \frac{1}{60}R_{ab}x^a x^b\right)$
- $\lim_{\rho\to\infty} K^{(4)}_{0,R}(x) = G^{(4)}_{0,R}(x)$ when $R_{ab} = \frac{1}{4}Rg_{ab}$.

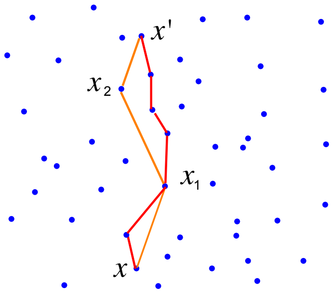
Why does this work .. sometimes?

A Convolution Argument

- $G_m(x, x') = G_0(x, x') + m^2 (G_0 * G_0)(x, x') + m^4 (G_0 * G_0 * G_0)(x, x') + \dots$
- $G_R(x, x') = G_0(x, x') - \frac{R}{6}(G_0 * G_0)(x, x') + (\frac{R}{6})^2 (G_0 * G_0 * G_0)(x, x') + \dots$
- This suggests that if there is a convolution form for $G(x, x')$ then knowing $G_0(x, x')$ is enough.
- Discrete case follows .. almost trivially.

Brand New Thoughts: 3d case

- $G(x,x') = -\Theta(x^0)\Theta(\tau^2)\frac{1}{2\pi}\frac{\cos m\tau}{\tau}$
- Causal set analog of $\tau(x,x')$: Length of the longest chain $\ell(x,x')$



- $K_0(x,x') = -\frac{1}{2\pi}\frac{1}{\ell(x,x')}$
- $K(x,x')$ generated via convolution.

Open Questions

- What is the propagator in $\not{3}, 5, 6.. \ldots$ dimensions?
- $K(x, x')$ for generic curved spacetime?
- Phenomenology: Finite ρ gives rise to violations of Huygen's principle.