Yang-Mills Matrix Model Coupled to Fermions

Superselection Sectors and Quantum Phases

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Bangalore Area String Meeting
ICTS, Bangalore
26 July 2016

- Joint work with Mahul Pandey
  arXiv:1606.05466

Introduction

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1. Pure Yang-Mills Theory
2. Matrix Model for Yang-Mills
3. Adding Fermions
4. Born-Oppenheimer Approximation
5. Fermion Energies
6. Quantum Phases of $SU(2)$ Yang-Mills-Dirac Theory

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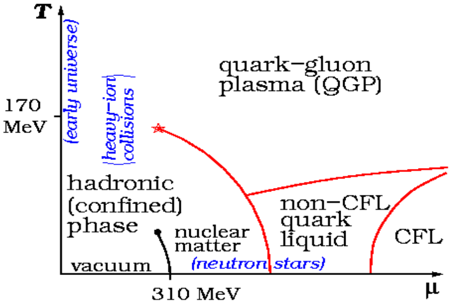
Review of YM Theory

- What are the physical states of Yang-Mills-Dirac theory?
- Wide implications: confinement, chiral symmetry breaking, color superconductivity, ...
- Even at zero temperature, QCD displays diverse phases: vacuum/hadronic, nuclear superfluid, quark liquid, color-flavor locking,. . ..



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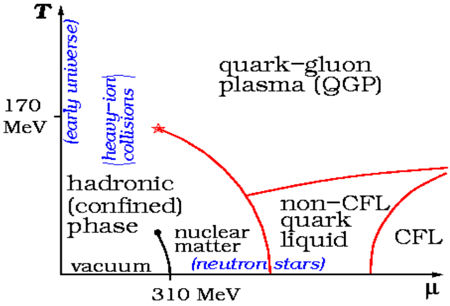

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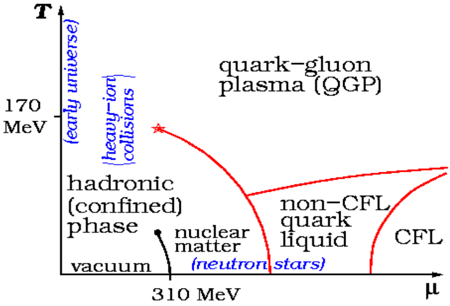

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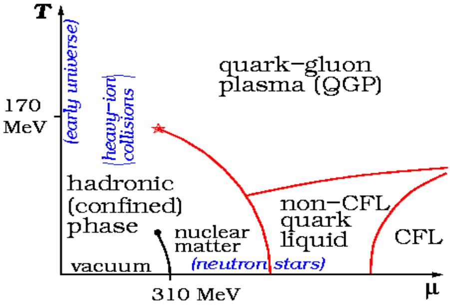

YM Matrix Model

- Many of these phases have uniform chromo-magnetic (or electric) fields, and fermions with uniform density (condensates).
- We will try to describe this phase structure by quantizing these *spatially homogeneous* degrees of freedom.
- More precisely, we study YM theory on $S^3 \times \mathbb{R}$ and restrict to the zero mode sector.
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Case of $SU(2)$

- An intrinsic derivation of the matrix model comes from the work of Narasimhan and Ramadas (1979).
- The key idea for $SU(2)$ YM theory on $S^3 \times \mathbb{R}$:
- They consider a special subset of left-invariant connections

$$\omega = (\mathrm{Tr}\, \tau_i u^{-1} du) M_{ij} \tau_j, \quad u \in SU(2), M \in M_3(\mathbb{R}) \equiv \mathcal{M}_0.$$

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- The action of $SO(3)$ on $\mathcal{M}_0$ is free for all matrices with rank 2 or 3.
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$$\Omega = \mathrm{Tr}\left(\frac{\lambda}{2} u^{-1} du\right) M_{ab}\lambda_b, \quad u \in SU(3).$$

- Here M is a 8×8 real matrix.
- Map the spatial S^3 diffeomorphically to $SU(2) \subset SU(3)$.
- $X_i \equiv$ vector fields for right action on $SU(3)$ representing λ_i ($i = 1, 2, 3$), then $[X_i, X_j] = i\epsilon_{ijk}X_k$.
- $\Omega(X_i) = -M_{ib}\frac{\lambda_b}{2}$.
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$SU(3)$ Yang-Mills

- The M's parametrize a submanifold of connections $\mathcal{A}$.
- They have no spatial dependence: we have completely gauge-fixed the "small" gauge transformations.
- Only the global transformations are left – the ones responsible for the Gribov problem.
- Global color $SU(3)$ acts on the vector potential:

$$A_j \to hA_jh^{-1}, \quad \text{or} \quad M \to M(Ad\,h)^T, \quad h \in SU(3)$$

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Configuration space of $SU(N)$ YM Matrix Model

- More generally, The configuration space $\mathcal{C}$ for pure $SU(N)$ is $M_{3,N^2-1}(\mathbb{R})/Ad\,SU(N)$.
- This space has dimension $3(N^2-1)-(N^2-1)=2(N^2-1)$ (but not so at fixed points).
- Wavefunctions are sections of vector bundles on $\mathcal{C}$ that transform according to representations of $Ad\,SU(N)$.
- Those transforming according to the trivial representation are colorless, which those transforming nontrivially are coloured.
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YM Matrix Model

- Recall that the YM Hamiltonian is

$$H_{YM} = \frac{1}{2} \int d^3x \, \mathrm{Tr} \left(g^2 E_i E_i + \frac{1}{g^2} F_{ij}^2 \right) .$$

- For the matrix model, M_{ia} are the dynamical variables, and the (Legendre transform of) $\frac{dM_{ia}}{dt}$ the conjugate of M_{ia}.
- We identify this conjugate operator as the matrix model chormoelectric field E_{ia}.
- Quantization: $[M_{ia}, \Pi_{jb}] = i\delta_{ij}\delta_{ab}$. $(\Pi_{jb} = E_{jb})$
- The above can easily be generalised to $SU(N)$ YM theory.

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Quantization of the Matrix Model

- The reduced matrix model Hamiltonian is

$$H = \left(\frac{g^2 E_{ia} E_{ia}}{2} + V(M) \right).$$

- The quantum Hamiltonian is

$$H = -\frac{g^2}{2} \sum_{i,a} \frac{\partial^2}{\partial M_{ia}^2} + V(M)$$

  It acts on the Hilbert space of functions $\psi(M)$ with scalar product

$$(\psi_1, \psi_2) = \int \prod_{i,a} dM_{ia} \, \bar{\psi}_1(M) \psi_2(M)$$

- Physical states obey Gauss law constraint: $G_a|\psi\rangle = \epsilon_{abc} \Pi_{ia} M_{ib} |\psi\rangle = 0$.

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Rotations, Gauge transformations and SVD

- Return to $SU(2)$:
- Physical rotations act on the left $M \to gM$, gauge transformations act on the right $M \to Mh^T$.
- A natural decomposition of $M_3(\mathbb{R})$ is thus in terms of Singular Value Decomposition:
- $M = RAS^T$, where R and S are orthogonal matrices, and A is a diagonal matrix with non-negative entries a_i, which we arrange as $a_1 \geq a_2 \geq a_3 \geq 0$.
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- The fundamental fermions couple to the gauge field via

$$H^{ff} \equiv \frac{1}{g^2}\left(-(\lambda^A)^\dagger\lambda^A - \frac{1}{2}(\tau_b)_{AC}(\lambda^A)^\dagger\sigma_i\lambda^C M_{ib}\right) = (\lambda_A)^\dagger\mathcal{H}^{ff}_{AB}\lambda_B.$$

- The first term is curvature term on S^3. We ignore it henceforth, it only contributes an additive constant to the energy.
- The second term is the analog of $\psi^\dagger\gamma_i A_i\psi$.

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- The total Hamiltonian is $H = H_{YM} + H^{ff}$.
- Solve $H\psi^E = E\psi^E$.
- We look at $g << 1$, but rather than do perturbation theory, quantize in two steps:
- First treat the gauge field as a (background) fixed field and quantize the fermions.
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Born-Oppenheimer quantization

- Total Hilbert space $\mathcal{H} \simeq \mathcal{H}^{slow} \otimes \mathcal{H}^{fast}$.
- First solve $H^{ff}|n(M); M\rangle = E_n(M)|n(M); M\rangle$
- Basis vectors of $\mathcal{H}$ are of the form $|M\rangle \tilde{\otimes} |n(M)\rangle$.
- So $\langle M; n(M)|\Pi_{ia}|\Psi\rangle \neq -i\frac{\partial}{\partial M_{ia}}\langle M; n(M)|\Psi\rangle$.
- Instead, we must derive the matrix element of $\Pi_{ia} \equiv \Pi_{ia} \otimes \mathbf{1}_{fast}$.
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- Related to the real part of the *quantum geometric tensor*

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\begin{aligned}
G_{IJ} &= \frac{1}{g_0}\mathrm{Tr}[P(\partial_I P)(\partial_J P)P] = g_{IJ} + \frac{i}{2}F_{IJ}, \\
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\end{aligned}
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- g_{IJ} is a Riemannian metric, a measure of distance between pure states represented by projectors $P(x_I)$ and $P(x_I + dx_I)$.
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Φ for YM fermions

- We will compute Φ for fundamental fermions coupled to the Yang-Mills field M_{ia}.
- The 1-fermion states $|\psi^{(1)}\rangle = \sum_A c_A(M)(\lambda_A)^\dagger|0\rangle$, $A = (\alpha, a)$.
- The $c_A(M)$ are complex functions of M_{ia}; $\lambda, \lambda^\dagger$ obey $\{\lambda_A, (\lambda_B)^\dagger\} = \delta_{AB}$, $\lambda_A|0\rangle = 0$.
- For 1-fermion states, $H^{ff}|\psi^{(1)}\rangle = E|\psi^{(1)}\rangle$ becomes:

$$\mathcal{H}^{ff}_{AB}c_B = Ec_A, \quad \mathcal{H}^{ff} = -\frac{1}{2}\sigma_i \otimes \tau_a M_{ia}$$

- That is, we want to investigate

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Fundamental Fermions

- The eigenvalue equation $H^{ff}|\psi\rangle = E|\psi\rangle$ is now an eigenvalue equation for a 4×4 matrix.
- The characteristic equation (with $x = \frac{g^2 E}{(\frac{1}{3}\operatorname{Tr} M^T M)^{1/2}}$) is

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- Since $\mathcal{H}^{ff}$ is manifestly Hermitian, it has only real roots.
- The conditions for this come from Sylvester's theorem: one condition is that the discriminant Δ of $x^4 - \frac{3}{2}x^2 - \mathbf{g}x + \mathbf{h}$ must be non-negative.
- This gives us another unexpected identity obeyed by 3×3 real matrices:

$$-\frac{1}{4}729\left[108(\det M)^4 + \left(\mathrm{Tr}(M^T M)^2 - 2\,\mathrm{Tr}(M^T M)^2\right)\left(\mathrm{Tr}(M^T M)^2 - \mathrm{Tr}(M^T M)^2\right)^2 - 4\,(\det M)^2\left(5\,\mathrm{Tr}(M^T M)^3 - 9\,\mathrm{Tr}(M^T M)\,\mathrm{Tr}(M^T M)^2\right)\right] \geq 0$$

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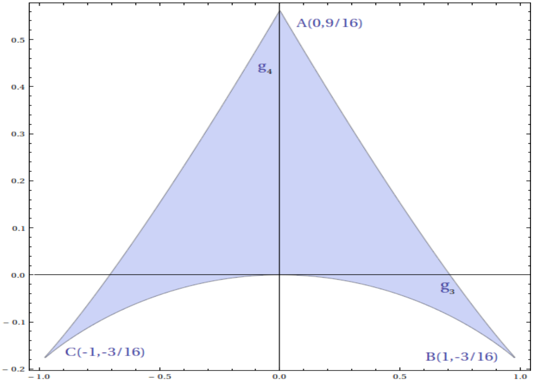
- Any 3×3 matrix lies inside the bounded region.
- At the top corner, the degeneracy structure is $(2, 2)$.
- At the two corners at the bottom, the degeneracy structure is $(3, 1)$.



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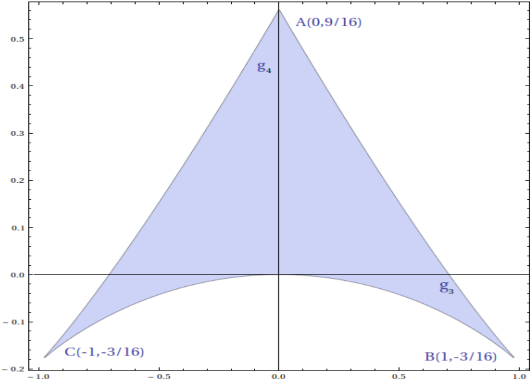

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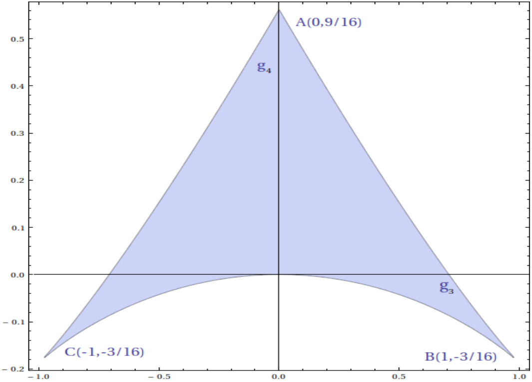

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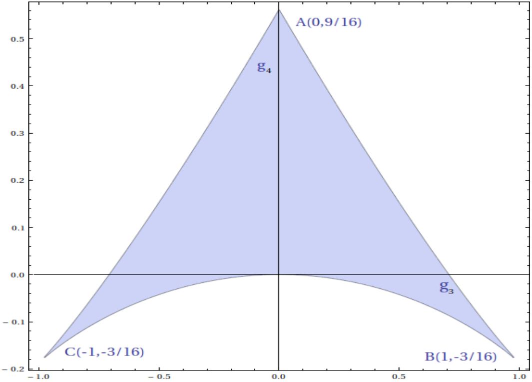
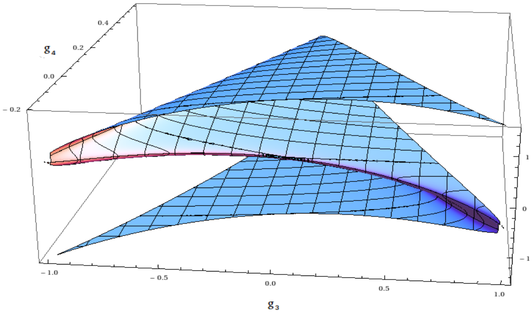


Weyl 1-fermion spectrum



- Actually, the theory with a single fermion is has a gauge anomaly.
- The physical theory has two fermions (with either chirality).
- We can compute the effective potential for this case as well.
- It shows a divergent behaviour whenever the ground state degeneracy jumps.
- The fundamental fermion is more sensitive than an adjoint fermion:
- For example, it can distinguish between situations when only two of the singular values coincide, as opposed to all three.
- The edges/corners are places where fermion eigenmodes condense.
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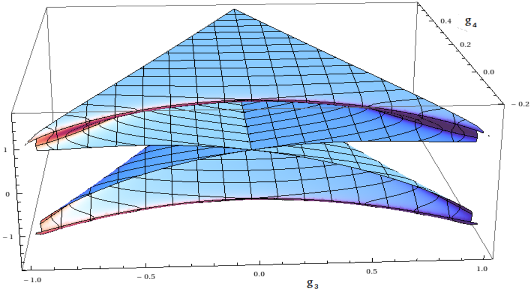
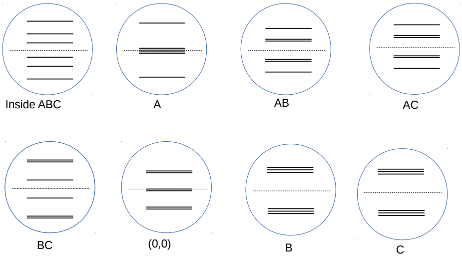
Weyl 2-fermion spectrum



2 Weyl fermions



- The characteristic polynomial of the two fermion Hamiltonian is

$$P_2(x) = x^6 - 3x^4 + 4x^2\left(\frac{9}{16} - \mathbf{h}\right) - \mathbf{g}^2 = 0$$

- This gives us the effective potential

$$\Phi^{(2)}_{bulk} = \frac{6}{\mathbf{f}^2}\frac{-x_1^6 + 5x_1^4 + 4(9/16 - \mathbf{h})(1 - 7x_1^2/3)}{(3x_1^4 - 6x_1^2 + 4(9/16 - \mathbf{h}))^2}, \quad \mathbf{f}^2 = \frac{1}{3}\mathrm{Tr}M^T M.$$

where $x_1(\mathbf{g}, \mathbf{h})$ is the smallest root of P_2.

- The ground state degeneracy changes from 1 to 2 at the edge BC, and to 3 at the corner B. At the edge BC:

$$\Phi^{(2)}_{edge} = \frac{2}{9\mathbf{f}^2}\frac{9 - 6x_1^2 + 5x_1^4}{x_1^2(1 - x_1^2)^2} \to \frac{2}{9a^2}\frac{1}{(1 + x_1)^2}$$

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- Finally we can also compute

$$\Phi^{(2)}_{corner} = \frac{1}{a^2}$$

- We see that the Hilbert space for gauge dynamics has split into 3 regions:
- Inside the bulk, it is governed by $\Phi^{(2)}_{bulk}$, which diverges as we approach the edge BC or the corner B.
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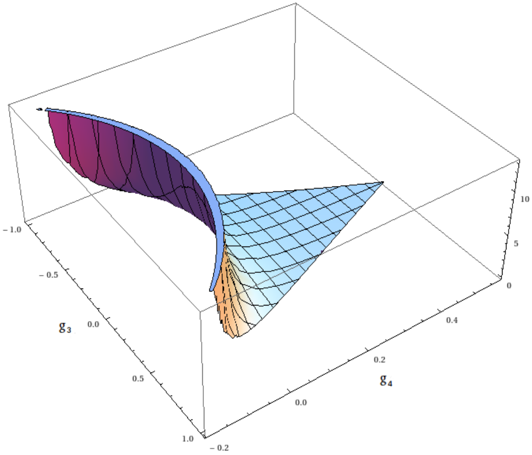
Scalar potential for 2 Weyl fermions



- There are therefore three distinct phases of $SU(2)$ gauge theory (with Weyl fermions).
- These are superselected: states in one phase cannot be obtained as superpositions of states from other sectors.
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- We can identify the phase as color-spin locked phase. These are known to exist in 3-color QCD.

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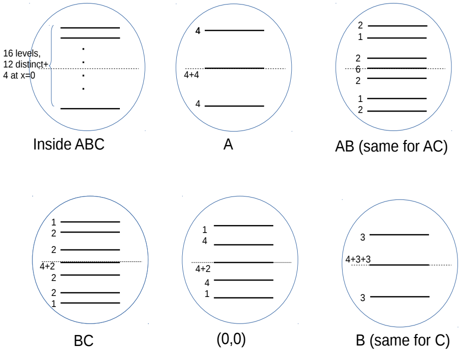

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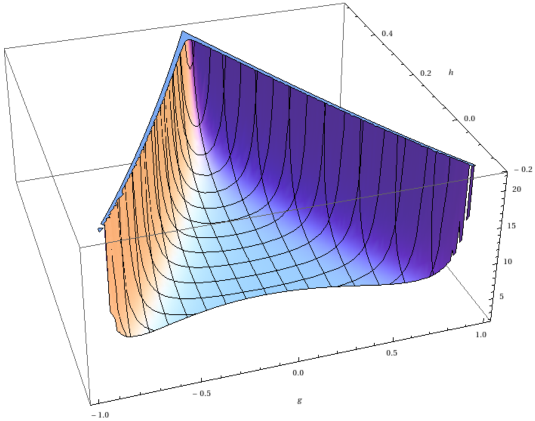

- For massless Dirac fermions, the situation is similar.
- Now, we can identify four distinct phases.
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- The singularities of the effective potential arise from fermion eigenvalue repulsion.
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- Extension to $\mathcal{N} = 2, 4$ SUSY YM (in progress, with Mahul Pandey and Veronica Errasti Diez).

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References

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