Wick Rotation
in the
Tangent Space

Joseph Samuel

Theoretical Physics
Raman Research Institute

Fermi, Wick and Amaldi

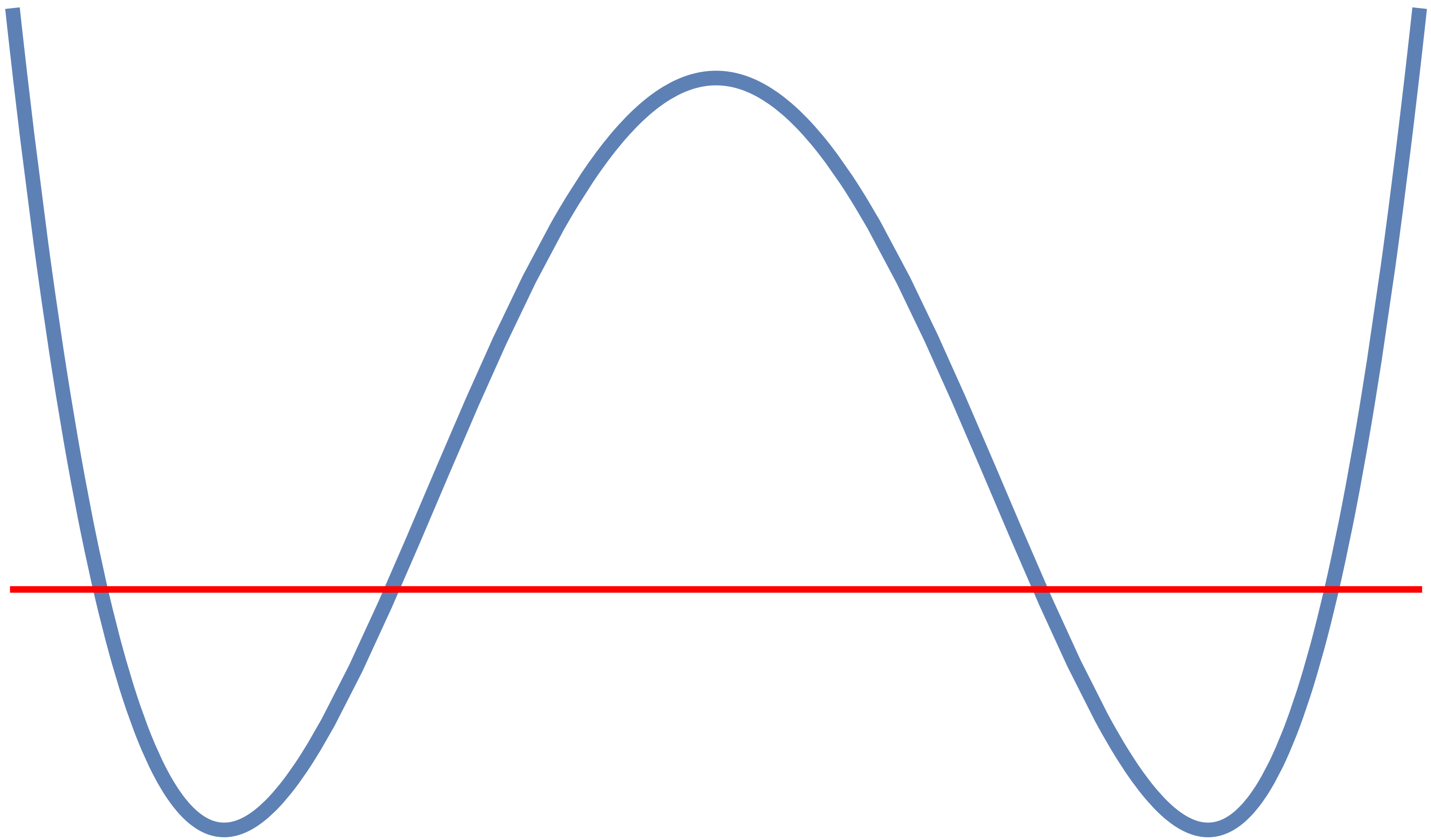
Figure 1: from Right Fermi, Wick and Amaldi in 1936

Introduction: motivation

- Euclidean Methods in Physics
  Connect Minkowskian problems to Euclidean ones
- Wick Rotation of the time coordinate to imaginary values $t \to -i\tau$
- Schrodinger equation to the Heat equation (Brownian Motion)
- Thermal fluctuations of DNA to Quantum fluctuations of Tops
- Field Theories to Stat Mech Problems
  $\exp -iHt \to \exp -\beta H$

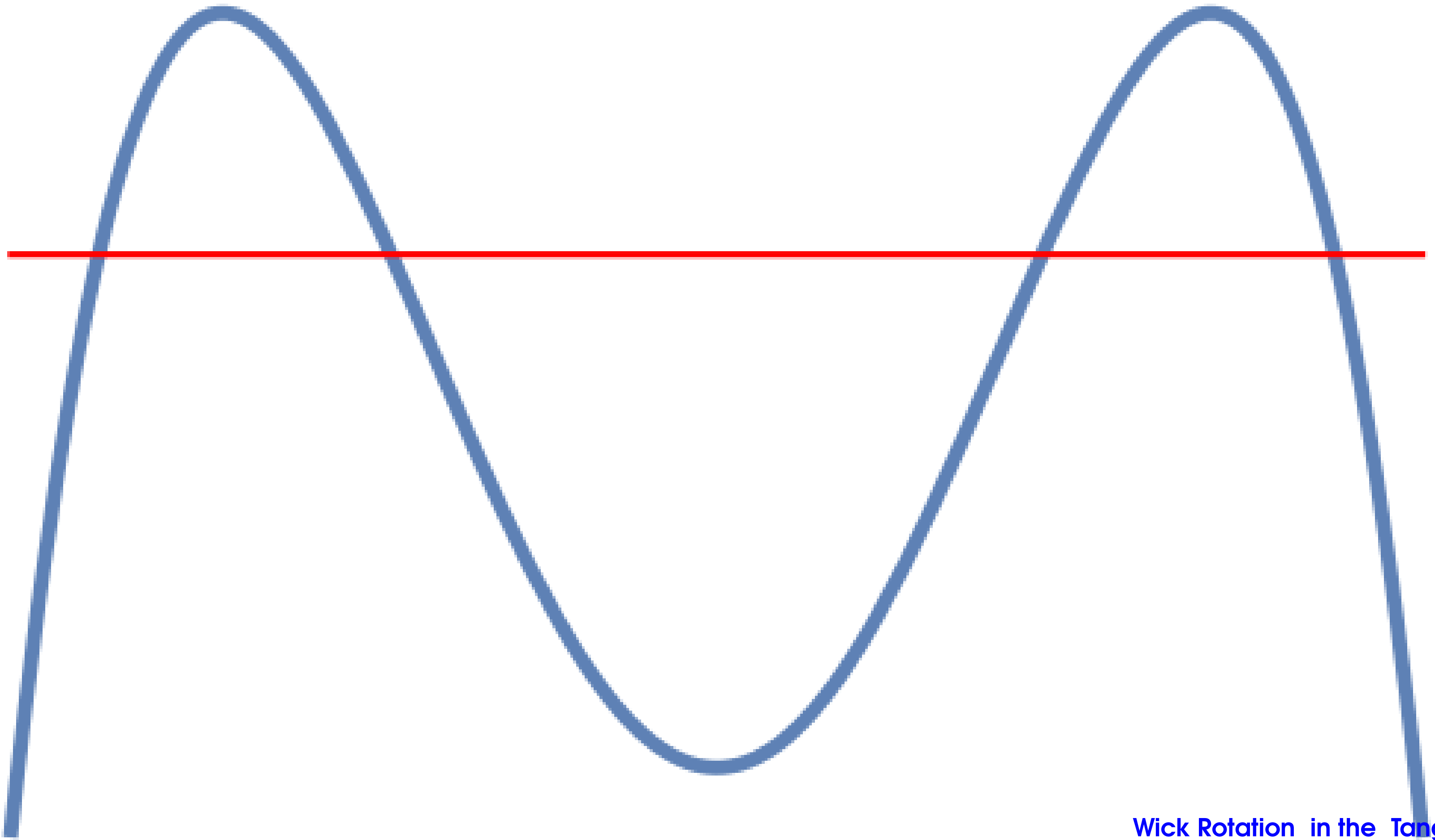
Tunneling over a barrier

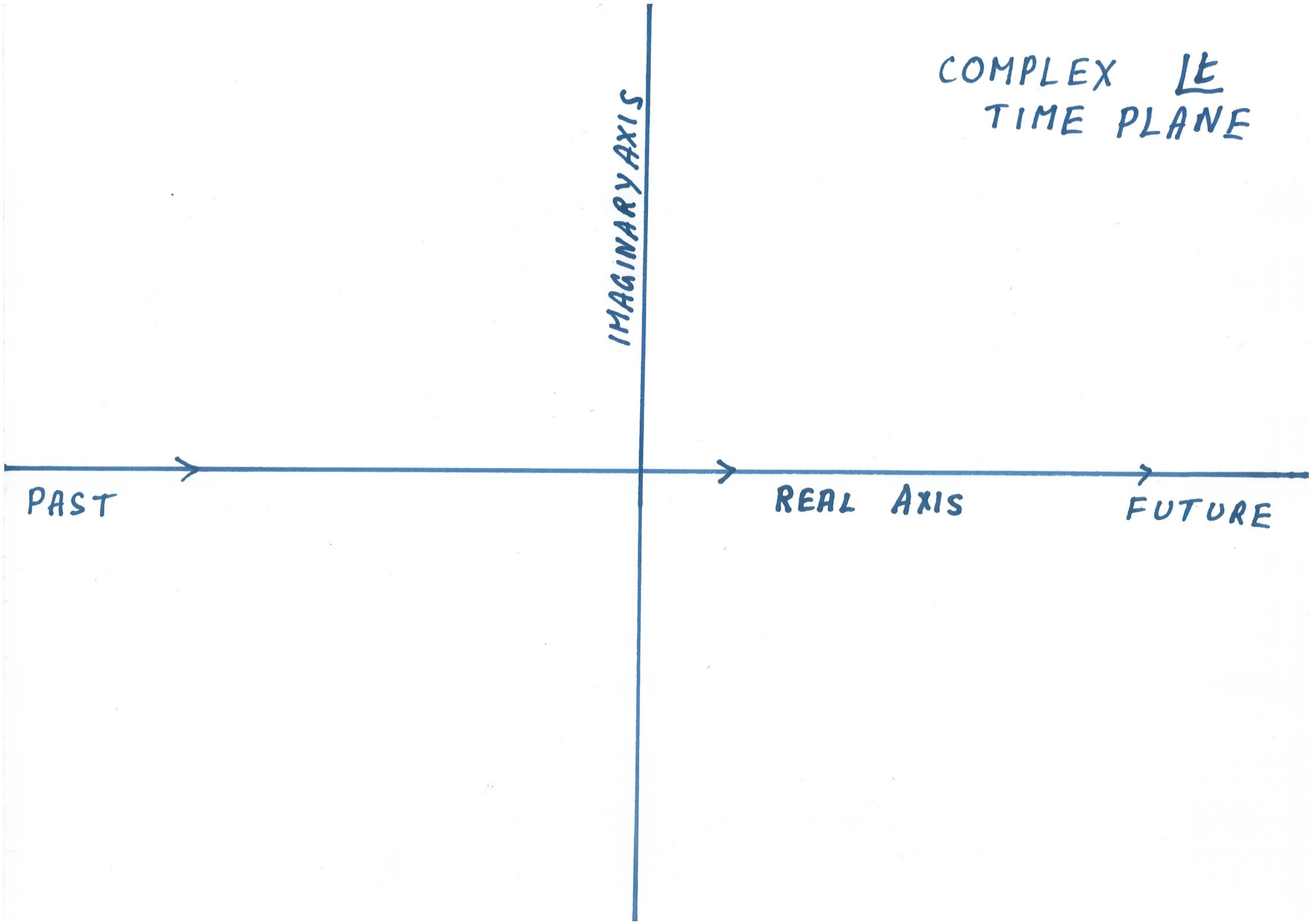
Euclideanisation inverts the barrier and makes it passable.



Inverted Potential

In imaginary time the particle problem is one in an inverted potential :Instanton gives the exponentially small tunneling amplitude




COMPLEX LE
TIME PLANE
IMAGINARY AXIS
PAST
REAL AXIS
FUTURE

Spherical Pendulum

- One can consider motion of a classical particle in an oscillator potential, get periodic functions
- A real pendulum has a potential $V(\theta) = (1 - g\cos\theta)$.
- in imaginary time it is possible, even with insufficient energy to go "over the barrier" and we get motions with an imaginary period.
- leads to Elliptic functions, doubly periodic in the complex time plane.

Instantons

- In quantum theory these solutions would be "Instantons" solutions of the Euler Lagrange equations in Euclidean time.
- For a potential $V(x) = V_0 \cos x$, we have the vacuum modified by such tunneling effects and the true vacuum is revealed only by these non perturbative instanton effects
- Yang Mills vacuum is also thus affected by Euclidean solutions of the Field equations, θ vacuua.

Special Relativity 1

- In special relativity time is not Universal but one can choose an inertial observer and use her time to Wick Rotate.
- Converts Oscillatory expressions to exponentially damped ones. Replace Fourier by Laplace.
- Fresnel Integrals with Gaussian ones
- Wick Rotation "commutes with" Lorentz transformations.
- This is the context in which Wick originally proposed the idea

Special Relativity2

- Quantum Field theory Feynman's $i\epsilon$ prescription for avoiding poles in the propagator.
- Deform the integration contour in momentum space to avoid the poles in the propagator.
- thermal Greens functions periodic with period $\beta = 1/T$ cyclic time
- β is inverse temperature

General Relativity

- Units $c = G = \hbar = 1$
- Einstein's GR gravity as the geometry of spacetime
- Predicts black holes (and many other things)
- Schwarzschild Black hole (Static)
- Kerr Black hole (rotating) (Stationary).
- Notion of "Time" takes a real beating in General Relativity.
- Time has an entirely different character in GR and QM)

Hawking Radiation

- Hawking found that black holes are "hot"
- Radiate like hot black bodies
- scatter incident radiation like hot black bodies absorption, spontaneous emission , stimulated emission.
- for a Schwarzschild black hole, $T = 1/(8\pi M)$ where M is the mass of the black hole.

Thermal Physics and Wick Rotation

- The idea that quantum mechanics and gravity lead to thermal physics and black hole entropy is a deep insight that takes us in interesting directions, which have still not been fully understood.
- Wick rotation is naturally suited to exploring thermal physics because of the connection between inverse temperature and imaginary time.

Geometrisation of Thermal Physics

Hawking Temperature from Euclidean Methods

- argument from Gibbons-Hawking
- Schwarzschild metric

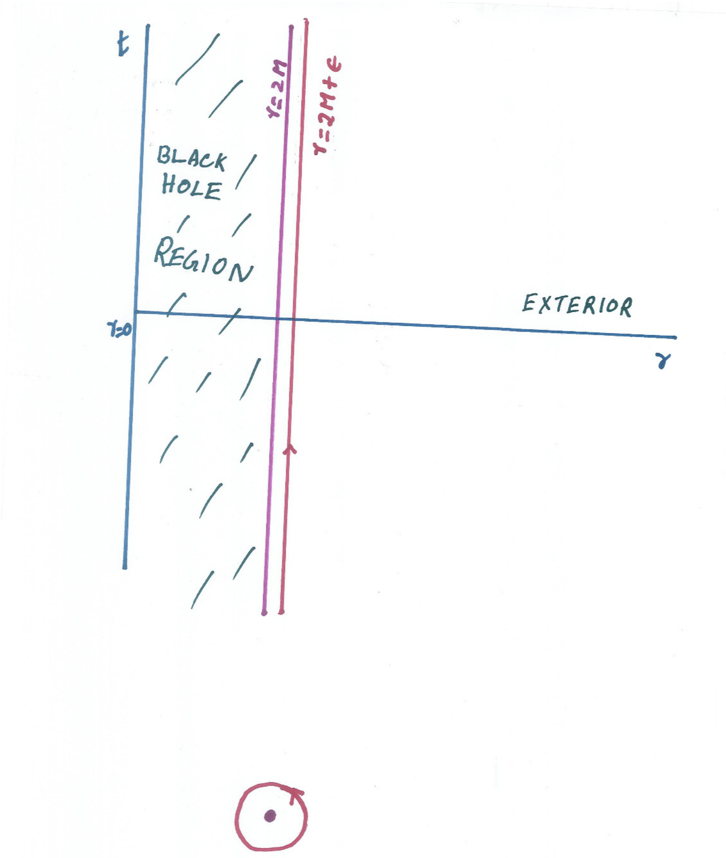
$$ds^2 = -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + d\Omega^2$$

- Use Schwarzschild time to Wick rotate
- Euclidean Schwarzschild metric

$$ds^2 = (1 - 2M/r)d\tau^2 + (1 - 2M/r)^{-1}dr^2 + d\Omega^2$$

Schwarzschild

- line $r = 2M$ is actually a point in the Euclidean space
- a line near the horizon is a circle in the Euclidean space



Conical Singularity

- Near the horizon, the geometry can be written (by change of coordinates) as

$$ds^2 = R^2 d\theta^2 + dR^2$$

  where $\theta = \tau/(4M)$.
- If theta is chosen periodic with period 2π, we get no singularity; Then τ is periodic with period $8\pi M$.
- Else conical singularity
- Gives $T_H = 0/\beta = 1/(8\pi M)$.
- Hawking temperature derived by Euclidean methods

Kerr Spacetime

- Apply the same idea to Kerr, find complex metrics because Kerr has cross terms $dtd\phi$ coming from rotation.
- In fact, no global time function exists in the Kerr spacetime.
- the timelike Killing vector is not hypersurface orthogonal
- the timelike Killing vector that is good at infinity becomes spacelike in the ergo region, so the metric has explicit time dependence, not even stationary.

de Sitter Spacetime2

- de Sitter shows the problems involved. Consider

$$ds^2 = -dt^2 + (H^{-2} \exp Ht)(dr^2 + r^2 d\Omega^2)$$

  where H is a constant

- Naive $t \to -it$ gives us

$$ds^2 = dt^2 + (H^{-2} \exp -iHt)(dr^2 + r^2 d\Omega^2)$$

  a complex metric not a Euclidean one.

de Sitter Spacetime 2

- Rewriting in the equivalent form

$$ds^2 = -dt^2 + H^{-2}\cosh Ht(dr^2/(1-kr^2) + r^2 d\Omega^2)$$

  and then performing a Wick Rotation gives

$$ds^2 = dt^2 + H^{-2}\cos Ht(dr^2/(1-kr^2) + r^2 d\Omega^2)$$

  a perfectly respectable Euclidean metric.

- Coordinate space Wick Rotation (CSWR) is coordinate dependent!
- Explore instead Tangent Space Wick rotation (TSWR)

Time in General Relativity

- subtleties stem from the nature of Time in relativity.
- time in Newtonian physics is a state function. Given a spacetime event, there is a global time function: one can read off the absolute time
- in Relativity, time is a path function, depending on your history. Twin paradox
- Minkowski time choose an inertial observer. Wick Rotation leads to the same results independent of observer.
- Schwarzschild time choose a static observer Killing vector $\xi_a = \lambda \nabla_a t$ defines a time function. Time as measured by stationary observers, suitably rescaled to allow for red shift

Tangent Space Wick Rotation

- generalise the idea of Wick rotation so that it works for a larger class of spacetimes
- Use a local tetrad frame e_i^a such that the metric is $g_{ij} = e_i^a e_j^b \eta_{ab}$ Local observers
- Wick rotation is done by rotating e^0 to $-ie^0$ or equivalently, η_{ab} to δ_{ab}, its Euclidean version.
- In usual CSWR in theories with tensor matter fields, one not only Wick rotates the metric, but also tensors, like say the electromagnetic (or Yang Mills gauge) potential, rotating tensor components to imaginary values

Spinor Fields

- In theories with Spinor matter fields Wick rotation also entails rotating the spinor fields.
- As is well known, to handle spinor fields in curved spacetime, one has to introduce tetrads. May as well introduce tetrads from the beginning.
- Cartan Structure equations

$$de^a + A^a_b \wedge e^b = 0$$

  determines A^a_b in terms of e^a $A_{ab} = -A_{ba}$.
- A^a_b is an $SO(3,1)$ connection, same info as the spin connection; the Newman-Penrose coefficients

Holonomy of the Connection

- The Holonomy of A is

$$H_\gamma(A) = P[\exp \int_\gamma A]$$

- The Wick rotated frames yield an $SO(4)$ connection which is the Wick rotation of the spin connection.
- Tetrads give us a way to Wick rotate spacetimes and arrive at *real* Euclidean metrics.

Wick Rotation without frames

- can also express the final result entirely in terms of metrics
- We do not need a global time coordinate, but make do with slightly less structure.
- what we *do* need is a local notion of 'time' in each tangent space: a timelike future pointing vector field u^i
- Euclidean metric given by

$$\mathcal{G}_{ij} = g_{ij} + 2u_i u_j$$

, u a unit timelike vector field.

Wick Rotation without Frames

- inverse metric given by

$$\mathcal{G}^{ij} = g^{ij} + 2u^i u^j$$

- in the special case that u^i is Killing and Hypersurface orthogonal, these local notions of time mesh together to give a global time coordinate and our method reduces to the usual Wick rotation
- We can apply the method to a larger class of spacetimes than before and get *real* Euclidean metrics

No time function for Kerr

- Kerr time no global time function exists. The Killing field at infinity is not timelike in the ergosphere. Locally Non rotating observers have a time in which the metric is not stationary.
- We use instead a local frame which defines time in each tangent space which defines a local time direction
- Local time directions *do not* mesh together to give a time function

Euclidean argument for Kerr

- Nevertheless can carry through the Euclidean argument.
- the null generator of the horizon projects down to a point in the Euclidean spacetime
- A curve near the horizon projects down to an infinitesimal circle near this point.
- trivial Holonomy of null generator of horizon
- forces us to make global identifications and gives the correct Hawking temperature.

The Kerr Horizon

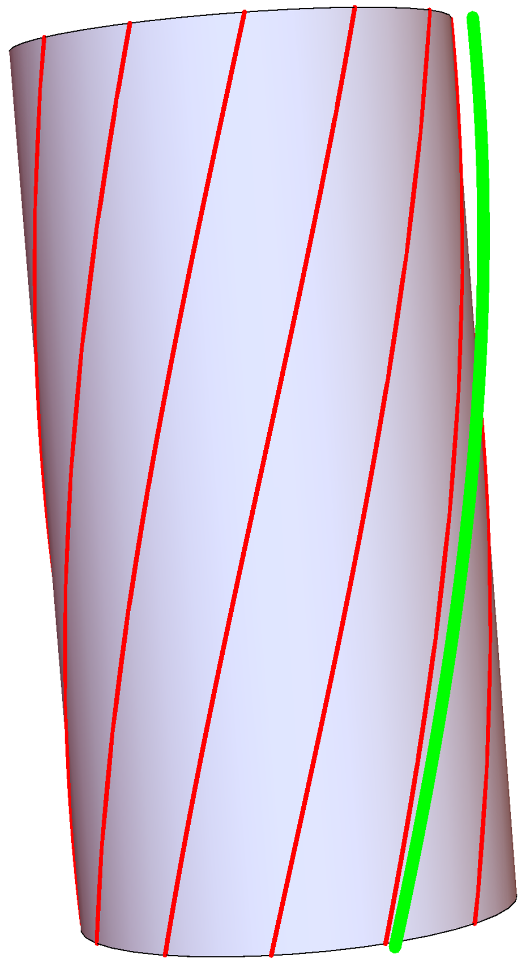


Figure 2: The Kerr Horizon

The Kerr Spacetime

- In particular for the Kerr metric, leading to a real Euclidean Kerr metric and the right Hawking temperature $T_H = (r_+ - M)/(2\pi(r_+^2 + a^2))$
- based on work published in *Classical and Quantum Gravity*, **33**, 2016, 015006

Conclusion

- Can generalise the notion of Wick Rotation to frames so that we make use of less structure. To allow a more general notion of Wick Rotation consistent with the nature of time in General relativity.
- All we need is a timelike vector field, it need not be Killing or Hypersurface orthogonal.
- Leads to *real* Euclidean metrics not complex ones.

Thank you!