Exploring perturbative CFTs in Mellin Space

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joint work with R. Gopakumar, A. Rudra, S. Sarkar and M. Verma
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The Mellin space representation for CFT correlators

- Proposed by G. Mack (2009) [also Symanzik (1972)]
- Mellin transform:

$$\mathcal{M}\{f(x)\} \equiv F(s) = \int_0^\infty x^{s-1} f(x)\, dx$$

- Inverse

$$f(x) = \frac{1}{2\pi i} \oint_{c-i\infty}^{c+i\infty} ds\; x^{-s} F(s)$$

- Particularly useful in the presence of scale covariance.
- Mellin space rep. of CFT correlator [Mack (2009)]:

$$A(x_i) = \oint_{-i\infty}^{i\infty} [ds_{ij}]\; M(s_{ij}) \prod_{1\leq i<j\leq n} \left(\Gamma(s_{ij})(x_i - x_j)^{-2s_{ij}}\right)$$

Mellin Momenta

- Constraint from conformal invariance: $\sum_{j \neq i} s_{ij} = \Delta_i$
- $\frac{n(n-3)}{2}$ independent Mellin variables (equals no. of cross-ratios and Mandelstam invariants).
- Solve constraints by : $s_{ij} \equiv k_i . k_j$ with $k_i^2 = -\Delta_i$ and $\sum_i k_i = 0$
- 4-point function:

$$s = -(k_1 + k_2)^2 = \Delta_1 + \Delta_2 - 2s_{12}$$

$$t = -(k_1 + k_3)^2 = \Delta_1 + \Delta_3 - 2s_{13}$$

- $(u, v) \leftrightarrow (s, t)$ and $A(u, v) \leftrightarrow M(s, t)$

Properties of the Mellin Space CFT correlator $M(s_{ij})$:

- $M(s_{ij})$ is meromorphic in s_{ij}: only isolated simple poles (discrete spectrum of CFTs)
- Residue at poles: OPE coefficients
- Analogy with scattering amplitudes:

$$M(s,t) = \sum_{l,n} \frac{C^n_{12l} C^n_{34l} Q_l(t)}{s - \tau_l - 2n}$$

- Crossing symmetry $\leftrightarrow$ Channel Duality

Mellin Space and AdS/CFT

- large-N CFT correlators at strong coupling computed by Witten diagrams in AdS supergravity.
- In Mellin space: *Penedones* (2010), *Paulos* (2011), *Kaplan, Fitzpatrick, Penedones, Raju, van Rees* (2011)
- Diagrammatic (tree level, for scalars) rules developed for computing $M(s_{ij})$.
- Considerable simplification compared to position space computations.
- A definition of the flat space S-matrix (of n massless particles) in terms of an integral transform of the CFT Mellin amplitude [*J. Penedones* (2010)]:

$$\mathcal{T}(\mathcal{S}_{ij}) = \Gamma\left(\tfrac{\sum \Delta_i - d}{2}\right) \lim_{R\to\infty} \oint_{-i\infty}^{i\infty} \frac{d\alpha}{2\pi i} \, \frac{e^{\alpha} \alpha^{(d-\sum \Delta_i)/2}}{R^{n(1-d)/2+d+1}} \, M\left(s_{ij}=\tfrac{R^2 \mathcal{S}_{ij}}{2\alpha}\right)$$

Perturbative CFTs in Mellin space

- Develop conformal perturbation theory in Mellin space.
- Feynman rules in Mellin space (tree level, for scalars, to start with).
- Mellin representation of dual conformal integrals for perturbative $\mathcal{N} = 4$ SYM amplitudes. [*Paulos, Spradlin, Volovich* (2012)]
- $$A(x_i) = \int \prod_{i,a} [du_a] \; \frac{1}{(x_i^a - u^a)^{2\Delta_i^a} (u^a - u^{a+1})^{2\gamma_a}}$$

  is the tree level position space CFT correlator.
- What is its Mellin space representation?
- For QFTs in Momentum space, at tree level, the amplitude factorises. Does this happen for Mellin space CFT correlators?

Outline of computation [*Symanzik, Paulos et al*]

- Schwinger parametrisation:

$$\frac{1}{(x-y)^{2\Delta}} = \frac{1}{\Gamma(\Delta)} \int_0^\infty d\alpha\, \alpha^{\Delta-1} \exp\left[-\alpha(x-y)^2\right]$$

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$$I = \prod_{i=1}^{N} \left[\int_0^\infty d\alpha^i (\alpha^i)^{\Delta^i - 1}\right] \int [du] \exp\left[-\textstyle\sum_{i=1}^{N} \alpha^i (x^i - u)^2\right]$$

- Do u integrals (Gaussian)

Outline of computation

- Further manipulations: introduce partition of unity - $1 = \int_0^\infty dv\, \delta(v - \sum_i \alpha_i)$, v dependent rescaling of Schwinger parameters, impose conformality conditions: $\sum_i \Delta_i^a - D = 0$
- Introduce the Cahen-Mellin integral:

$$e^{-x} = \frac{1}{2\pi i} \oint_{-i\infty}^{i\infty} ds\, \Gamma(s) x^{-s} \leftrightarrow \Gamma[s] = \int_0^\infty dx\ x^{s-1} e^{-x}$$

  for each exponential factor [*Symanzik*(1972)], thereby introducing Mellin variables:

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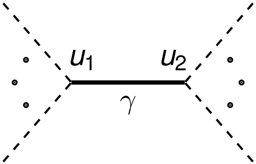
$$I = \oint_{-i\infty}^{i\infty} [ds_{ij}] \prod_a \int dt_a \mathcal{F}(t_a, s_{ij}) \prod_{1 \le i < j \le n} \left(\Gamma(s_{ij})(x_i - x_j)^{-2s_{ij}} \right)$$

- For a contact diagram: $M = 1$.
- s-channel tree level diagram in Mellin space:



$$M(s_{12}) = \frac{1}{2\Gamma(\gamma)}\beta\left(\frac{\gamma - s_{12}}{2}, \frac{D}{2} - \gamma\right) = \frac{-1}{\Gamma(\gamma)}\sum_{n=0}^{\infty}\frac{(n+\gamma-\frac{D}{2})_n}{s_{12}-\gamma-2n}$$

- Poles at $s_{12} = \gamma + 2n$, corresponding to exchanged primary + all descendants.

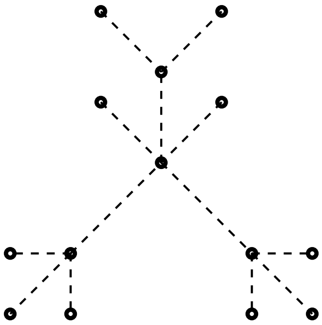
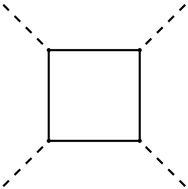


Figure: Example of a general tree level (skeleton) Feynman diagram

- At tree level, *factorisation* of $M(s_{ij})$ in Mellin space:
- $$\prod_I \frac{1}{2\Gamma(\gamma_I)} \beta\left(\frac{\gamma_I - s_I}{2}, \frac{D}{2} - \gamma_I\right) \qquad s_I \equiv -(\sum_i k_i^{(I)})^2$$
- Product over propagators: similar to tree level momentum space QFT amplitudes.

One loop diagram



- We can express 1-loop Mellin amplitude in terms of Schwinger integrals.
- Can it be recast as $\oint dv \prod \beta(..v..)$: integral of product of propagators over an undetermined Mellin loop variable?
- We could then systematically develop conformal perturbation theory in Mellin space.

Preliminary results on non-conformal Mellin representation.

- Utility in non-conformal settings?
- Consider a small generic (scalar) deformation of a CFT.
- Mellin rep. of correlators in the perturbative field theory ?
- Relax conformality conditions: $\lambda_a = D - \sum_i \Delta_a^i \neq 0$
- Tree level propagator $\propto {}_3F_2(...) \to \beta(...)$ as $\lambda_a \to 0$

$$M = {}_3F_2\left(\gamma - \frac{D}{2} + \lambda_1 + \lambda_2\,,\, \frac{R_1+\rho_1-\lambda_1}{2}\,,\, \frac{R_1+\rho_2-\lambda_2}{2}\,;\, \frac{R_1+\rho_1+\lambda_1}{2}, \frac{R_1+\rho_2+\lambda_2}{2}\,;\, 1\right) \times \frac{1}{2\Gamma(\gamma_{12})} \prod_{a=1}^{2}\left[\prod_{i\in a}\Gamma(\rho_a^i)\right]\left[\frac{\Gamma\left(\frac{R_1+\rho_1-\lambda_1}{2}\right)\Gamma\left(\frac{R_1+\rho_2-\lambda_2}{2}\right)}{\Gamma\left(\frac{R_1+\rho_1+\lambda_1}{2}\right)\Gamma\left(\frac{R_1+\rho_2+\lambda_2}{2}\right)}\right]$$

- Witten diagram for AdS/CFT tree level (scalar) propagator (correlator at strong coupling) [Paulos (2011)]

$$M(s_{12}) = \frac{1}{2}\frac{g^2}{s_{12}-\gamma}\,{}_3F_2\left(\frac{2-\Delta_1-\Delta_2+\gamma}{2}, \frac{2-\Delta_3-\Delta_4+\gamma}{2}, \frac{\gamma-s_{12}}{2}; \frac{2+\gamma-s_{12}}{2}, 1+\gamma-\frac{D}{2}; 1\right) \times \frac{\Gamma(\frac{\Delta_1+\Delta_2+\gamma-\frac{D}{2}}{2})\Gamma(\frac{\Delta_3+\Delta_4+\gamma-\frac{D}{2}}{2})}{\Gamma\left(1+\gamma-\frac{D}{2}\right)}$$

Summary and Outlook

- Mellin Space - natural setting to study CFTs. Analogies with momentum space for QFTs; Mellin Momenta.
- Tree level Feynman rules (for scalars) in Mellin space for perturbative CFTs.
- Incorporate spin - tensor and spinor exchanges.
- Feynman rules for loop diagrams.
- Further exploration of utility of Mellin space in non-conformal settings.