Breakdown of Perturbation theory and the Black Hole Information Problem

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Introduction

- Perturbation theory is a good approximation for calculating most observables in weakly coupled QFT's.
- Generically asymptotic behaviour of perturbative expansion at very large orders in loop expansion parameter.
- However, perturbation theory can break down much earlier for certain observables, e.g.,
  Large point connected correlation functions or scattering amplitudes for large number of particles.
- This can be due to coherent contribution of large number of Feynman diagrams.

Example: multipoint amplitudes in $\lambda\phi^4$ theory

- Consider an $n+1$ point connected correlation function of ϕ's at tree level. For large n the number of Feynman grows factorially leading to

$$\mathcal{A}_n^{tree} = \langle \phi(x_1)\phi(x_2)...\phi(x_{n+1})\rangle \sim \lambda^{(n-1)/2} n!$$

- For simplicity consider a $1 \to n$ scattering process near threshold. The cross section for such a process then can be estimated as, [Rubakov, '97]

$$\sigma_{1\to n}^{tree} \sim \lambda^n n! \mathcal{E}^n$$

where $\mathcal{E}$ is the average energy outgoing particles.

Multiparticle amplitudes in field theory

- Now amplitudes can grow too fast because of unitarity.
- For a $1 \to n$ process unitarity of the S-matrix yields

$$\mathrm{Im}\mathcal{M}(1 \to 1) = m_\phi \sum_k \Gamma(1 \to k)$$

  where $\mathcal{M}(1 \to 1)$ denotes the onshell 2-point function and $\Gamma(1 \to n)$ is the decay rate for producing n particles in the final state.
- Assuming that the L.H.S can be efficiently computed perturbatively, the cross section for producing a large number of particles in the final state should be suppressed.

Multiparticle amplitudes in field theory

- But from above estimates for $n \geq 1/\lambda$, the $n!$ growth completely dominates the small factor of λ^n.
- An estimate of the 1-loop contribution gives

$$\mathcal{A}_n^{1-\text{loop}} \sim \lambda^2 n^2 \mathcal{A}_n^{\text{tree}}$$

- Thus perturbation theory is completely invalidated even at tree level for $n \geq 1/\lambda$.

Breakdown of perturbation theory in gravity

- Consider defining gravitational amplitudes with respect to a path integral

$$\mathcal{Z} = \int_{g_i,\phi_i}^{g_f,\phi_f} \mathcal{D}g\mathcal{D}\phi e^{i(S[g]+S[g,\phi])}$$

- Perturbative amplitudes are evaluated around some semiclassical saddle point.
- Breakdown of perturbation theory in this case implies a breakdown of locality.

Implications for Black Hole Information Problem

- Hawking's formulation of the information problem is based on the assumption that perturbative effective field theory is valid along "nice slices" in an evaporating black hole geometry.
- However this is reliable only for simple observables. Locality, a key input in Hawking's conclusion breaks down for correlation functions involving $\mathcal{O}(\mathcal{S}_{\mathcal{BH}})$ Hawking quanta.
- Non-local and non-perturbatively small corrections can restore unitarity of Hawking evaporation process. [PR, 2013]

Breakdown of $1/N$ expansion in AdS/CFT

- Large N factorization fails for connected correlation functions with $\mathcal{O}(N^2)$ no. insertions.
- Breakdown of $1/N$ perturbation expansion for such correlation functions. By AdS/CFT, this implies breakdown of the bulk G_N perturbation expansion.
- No smooth semiclassical bulk dual description for such observables. Cannot be computed within the approximation of local, perturbative effective field theory in the bulk.

Black Hole Complimentarity

- Resolution of recent formulation of information problem in AdS/CFT by restricting the algebra of bulk effective field theory observables to a small subsector of the CFT such that $dim(\mathcal{A}_{EFT})$ does not scale with N^2. [PR, '13]
- Realization of black hole complimentarity. CFT duals of local fields in black hole interior actually complicated superposition of operators outside the black hole. However effectively commute within low point correlation functions.
- General implications for bulk reconstruction in AdS/CFT and understanding the emergence of effective notion of locality in bulk.

Scaling of large n amplitudes

- The goal is to estimate the scaling behaviour of scattering amplitudes in the large n limit and see when perturbation theory can break down.
- BCFW recursion relations provide an efficient technique for estimating scaling behaviour of large n graviton scattering amplitudes in field theory.
- For this talk we will however consider the large n behaviour of bosonic string scattering amplitudes in flat space. This turns to be an interesting problem in its own right.

String scattering amplitudes

Polyakov path integral with insertions

$$\mathcal{A}^{(n)} = \int_{\mathcal{G}_{g,n}} \left(\frac{\mathcal{D}g}{V_{diff \times Weyl}} \right) \int \mathcal{D}X^{\mu} e^{-S_m - \lambda_\chi} \left(V_1 V_2 \ldots V_n \right)$$

$$= \int_{\mathcal{G}_{g,n}} \left(\frac{\mathcal{D}g}{V_{diff \times Weyl}} \right) \left(\frac{8\pi^2}{\int_{\Sigma_{g,n}} d^2\sigma \sqrt{g}} \det{}' \Delta_g \right)^{-13} \langle V_1 V_2 \ldots V_n \rangle$$

- Fix Diff $\times$ Weyl by choosing a gauge slice S through $\mathcal{G}_{g,n}$.
- S is parametrised by $3g - 3 + n$ complex modular parameters m_α.
- Here we treat metric moduli and position of vertex operators on equal footing. Hence $\dim(\mathcal{M}_{g,n}) = 3g - 3 + n$.

String scattering amplitudes

- General metric variation

$$\delta g_{ab} = (P_1 \delta V) + \sum_{\alpha} \delta m^{\alpha} \phi^{\alpha}_{ab} + \delta\Omega g_{ab}$$

- ϕ_α are quadratic differentials, providing a basis for cotangent space to $\mathcal{M}_{g,n}$.
- δV and $\delta\Omega$ denote infinitesimal diffeomorphisms and Weyl rescalings respectively.
- Measure $\mathcal{D}g$ decomposes as

$$\mathcal{D}g = \mathcal{D}\Omega \det(P_1^\dagger P_1)^{1/2} \mathcal{D}V \prod_{\alpha,\beta}^{3g-3+n} dm_\alpha d\bar{m}_\beta \left(\frac{\det(\mu_\alpha, \phi_\beta)}{\det(\phi_\alpha, \phi_\beta)^{1/2}} \right)$$

- μ_α are the Beltrami differentials corresponding to modular deformations ; tangent vectors to chosen slice S.

String scattering amplitudes

- n point closed string scattering amplitudes are then given by

$$\mathcal{A}^{(n)} = \sum_g \int_{\mathcal{M}_{g,n}} d\mu_{WP} \,\det(P_1^\dagger P_1)^{1/2} \Delta^{-13} \langle V_1 V_2 \ldots V_n \rangle$$

where,

$$d\mu_{WP} = \left(\prod_{\alpha,\beta}^{3g-3+n} dm_\alpha d\bar{m}_\beta \left(\frac{\det(\mu_\alpha, \phi_\beta)}{\det(\phi_\alpha, \phi_\beta)^{1/2}} \right) \right)$$

is the Weil-Petersson measure on $\mathcal{M}_{g,n}$

$$\Delta = \frac{8\pi^2}{\int_{\Sigma_{g,n}} d^2\sigma \sqrt{g}} \det{}' \Delta_g$$

Weil-Petersson measure

- The Weil -Petersson metric admits a useful explicit expression in terms of legth, twist coordinates $(\ell_\alpha, \tau_\alpha)$ on moduli space.
- The Weil-Petersson Kahler form in these coordinates is [Wolpert]

$$\omega_{WP} = \sum_{\alpha}^{3g-3+n} d\ell_\alpha \wedge d\tau_\alpha$$

- D'Hoker and Phong have used this parametrisation in explicit evaluation of the Polyakov path integral for closed bosonic string theory.
- Also, useful for computations of volumes of $\mathcal{M}_{g,n}$.

Asymptotics of Weil-Petersson volumes

- The volumes of $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson measure have been recently computed. Also the asymptotic behaviour in the large g and large n limit is known. [Mirzakhani, Zograf, 2011]
- Some useful results for us are

$$V_{g,n} = \left(4\pi^2\right)^{2g+n-3} (2g-3+n)! \frac{1}{\sqrt{g\pi}} \left(1 + \frac{c_n}{g} + \mathcal{O}(\frac{1}{g^2})\right)$$

as $g \to \infty$, and $n \geq 0$;

$$V_{g,n} = n! C^n n^{(5g-7)/2} \left(c_g^{(0)} + \frac{c_g^{(1)}}{n} + ...\right)$$

for $n \to \infty$, and $g \geq 0$

$$\lim_{g+n\to\infty} \frac{\log(V_{g,n})}{(2g+n)\log(2g+n)} = 1$$

Bound on growth of functional determinants

- The product of functional determinants appearing in the string scattering amplitude can be expressed as

$$\det(P_1^\dagger P_1)^{1/2}\Delta^{-13} = Z(2)Z'(1)^{-13}$$

- $Z(s)$ is the Selberg Zeta function.

$$Z(s) = \prod_\gamma \prod_{k=1}^{\infty} \left(1 - e^{-(s+k)\ell_\gamma}\right)$$

where γ denotes simple closed geodesics on the underlying Riemann surface endowed with a constant curvature $= -1$ metric, ℓ_γ is the hyperbolic length of γ.

Bound on growth of functional determinants

- When $\ell_\gamma \to 0$, the Riemann surface degenerates. Let us introduce a lower cut-off on geodesic lengths, say ℓ_0.
- Close to the degeneration limit the Selberg Zeta functions have the following behaviour [D'Hoker, Phong, '88]

$$Z(2) \sim \ell_0^{-3} e^{-\pi^2/3\ell_0}, \quad Z'(1) \sim \ell_0^{-1} e^{-\pi^2/3\ell_0} \prod_{0<\lambda_n<1/4} \lambda_n$$

- λ_n are eigenvalues of the Laplacian on scalars on the Riemann surface.

Breakdown of string perturbation theory

- In terms of Green's function for the scalar laplacian on the Riemann surface

$$\langle V_1(k_1)V_2(k_2)...V_n(k_n)\rangle \approx e^{-\sum k_i k_j G(\zeta_i,\zeta_j)} \quad (1)$$

- With the lower cut-off on geodesic lengths the correlation function can be bounded as

$$\langle V_1(k_1)V_2(k_2)...V_n(k_n)\rangle \approx e^{-n^2 E^2/\ell_0} \quad (2)$$

since there are roughly n^2 terms in the exponent and E is taken to be the average energy per particle.

- Thus as long as $E < 1/n$, the $n!$ growth of the volume of moduli space $\mathcal{M}_{0,n}$ provides the dominating scaling factor in the n-point tree level amplitude at large n.

Breakdown of string perturbation theory

- Also, In the large n limit,

$$\mathcal{V}_{1,n} \approx n^2 \mathcal{V}_{0,n} \tag{3}$$

  where $\mathcal{V}_{1,n}$ and $\mathcal{V}_{0,n}$ are the Weil-Petersson volumes of $\mathcal{M}_{1,n}$ and $\mathcal{M}_{0,n}$ respectively.

- This naively implies that as $n \to \infty$

$$\mathcal{A}_n^{\text{torus}} \approx g_s^2 n^2 \mathcal{A}_n^{\text{sphere}} \tag{4}$$

- This indicates that string perturbation theory can break down when $n \sim 1/g_s$.

Summary

- Perturbation theory can break down even at lowest orders for high point correlation functions.
- The breakdown of perturbation theory in gravity has important implications for the black hole information problem.
- In a sense, the notions of locality and smooth semiclassical spacetime breakdown if spacetime is probed at "too many" points.
- Large n scaling of string scattering amplitudes suggests an interesting breakdown of string perturbation theory.
- However our present scaling arguments are somewhat crude. Need to make it more rigorous.
- Hopefully, these issues will be understood soon.

THANK YOU