Factorization $\Longrightarrow$ Evolution $\Longrightarrow$ Resummation

easy to see in UV case

$$G_B(p, \Lambda, g_s) = Z\left(\frac{\mu}{\Lambda}, g_R(\mu)\right) G_R\left(\frac{p}{\mu}, g_R(\mu)\right)$$

$$\mu \frac{d}{d\mu} G_B = 0$$

$$\Rightarrow \quad \mu \frac{d}{d\mu} \ln G_R\left(\frac{p}{\mu}, g_s(\mu)\right) = -\mu \frac{d}{d\mu} Z\left(\frac{\mu}{\lambda}, g_s(\mu)\right)$$

$$\equiv \gamma(g_R(\mu))$$

$$\Rightarrow \quad G_R\left(\frac{p}{\mu}, g_R(\mu)\right) = G_R(1, g_R(p))$$

$$\times \exp\left[\int_p^\mu \frac{d\lambda}{\lambda} \gamma_{g_R(\lambda)}\right]$$

$\rightarrow$ single scale two fn factorization

Generalize this to 2 scale 3 fn. multiple

factorization of our new

factorization relation

Contopanagos, E Laenen & Sterman:

2-scale, multi function factorization is characteristic of physical quantities that describe hard scatterings & which are sensitive to 3 regions of momentum space associated with 3 sorts of quanta

off shell (hard) partons at short distances

fast moving (collinear) partons near light cone

long wave length (soft) partons

Separation of a cross section into a separate functions for each type of excitation requires multiple factorizations

Independence of final expression from details of these factorizations

$\Rightarrow$ consistency equation

$\Rightarrow$ Sudakov resummation

Let us consider $A\,B \to \mu^+\mu^- X$ at fixed Q_T

$$\frac{d^2\sigma_{AB\to\mu^+\mu^- X}(Q, p_1, p_2)}{dQ^2\, d^2Q_T}$$

$$= \sum_{a=q,\bar q} \int d\xi_1\, d\xi_2\, \frac{d\hat\sigma_{a\bar a\to\mu^+\mu^- X}(Q, \mu, \xi_1 p_1, \xi_2 p_2, \vec Q_T)}{dQ^2\, d^2Q_T}$$

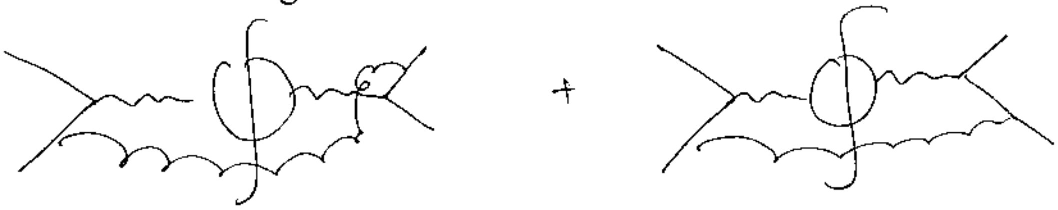
$$f_{a/A}(\xi_1, \mu)\, f_{b/A}(\xi_2, \mu)$$

μ → factorization scale (μ_F)

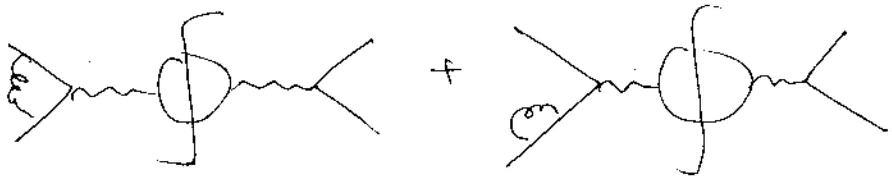
that separates IR (f) from UV($\hat\sigma$)

Diagrams at $O(\alpha_s)$

At $Q_T \neq 0$ gluon emissions contribute



Virtual corrections contribute only at $Q_T = 0$



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The ~~so~~ result is finite for $Q_T \neq 0$

$$\frac{d\hat{\sigma}^{(1)}}{dQ^2\, dQ_T} = \sigma_0 \frac{\alpha_s C_F}{\pi^2} \left[1 - \frac{4Q_T^2}{(1-z)^2 \xi_1 \xi_2 S} \right]^{-1/2}$$

$$\times \left[\frac{1}{Q_T^2} \frac{1+z^2}{1-z} - \frac{2z}{(1-z)Q^2} \right]$$

diverges as $z \to 1$, or $Q_T \to 0$

So for $z \neq 1$, $Q_T \neq 0$ everything is fine

Integrate over $Q_T \Longrightarrow \frac{\ln(1-z)}{1-z} \to$ threshold log

" " $z \Longrightarrow \cancel{\frac{\ln Q_T^2}{Q_T^2}}$ $\ln\cancel{\left(\frac{Q_T^2}{Q^2}\right)} \frac{\ln Q_T^2}{Q_T^2}$

$\longrightarrow$ Recoil log

(Both singularities cancel in inclusive x-section z

Transverse Momentum Resummation

$\to$ resums of logs of $\frac{Q_T}{Q}$ to all orders

Requires a different kind of factorization

$$\frac{d\sigma_{AB\to QX}}{dQ\, d^2Q_T} = \int d\xi_1\, d\xi_2\, d^2k_{1T}\, d^2k_{2T}\, d^2k_{ST}\; \delta^2(Q_T - k_{1T} - k_{2T} - k_{ST})$$

$$\times H(\xi_1 p_1, \xi_2 p_2, Q, n)_{a\bar a \to Q, X}$$

$$P_{a/A}(\xi_1, \cancel{p_1}\, p_1\cdot n, k_{1T})\; P_{\bar a/B}(\xi_2, p_2\cdot n, k_{2T})$$

$$U_{a\bar a}(k_{ST}, \hat n)$$

new transverse momentum dependent pdf's

$P \longrightarrow$ new transverse dependent pdf

δ-fn $\longrightarrow$ states that ~~the~~ measured transverse momentum results from recoil against radiation from jets & soft funct.

Q. What is $\hat n$? $\longrightarrow$ splits up the phase space

new factorization variable

If a gluon has a $p_i\cdot k$ which is less than $n\cdot k$ then k is included in P_i

$p_i\cdot k < n\cdot k \Rightarrow k \in P$

$p_a\cdot k,\ p_{\bar a}\cdot k > n\cdot k \Rightarrow k \in U$

n^μ tells us what it means to have a gluon $\parallel$ to p.

$U_{ab}(k_{ST}, \hat{n}) \to$ soft fn for wide angle soft gluon quanta <u>not</u> part of jet

Take a fourier transform $e^{i\bar{Q}_T\cdot\bar{b}}$ (move to impact parameter space)

$$\frac{d\sigma_{AB}(Q,b)}{dQ} = \int d^2Q_T \int d\xi_1\, d\xi_2\, e^{i\bar{Q}_T\cdot\bar{b}}$$

$$d^2k_{1T}\, d^2k_{2T}\; \delta^2(\bar{Q}_T - \bar{k}_{1T} - \bar{k}_{2T} - \bar{k}_{ST})$$

$$H(\xi_1 p_1, \xi_2 p_2, Q, n)\; \mathcal{P}_{a/A}(\xi_1\; p_1, n, k_T)$$

$$\mathcal{P}_{\bar{a}/B}(\qquad)\quad U(\qquad\qquad)$$

$$= \int d\xi_1\, d\xi_2\; H(\xi_1 p_1, \xi_2 p_2\; Q, \hat{n})$$

$$\mathcal{P}_{a/A}(\xi_1\, p_1, n, b)\; \mathcal{P}_{\bar{a}/B}(\xi_2 p, n, b)$$

$$U_{a\bar{a}}(b, n)$$

LHS independent of $\mu_{ren}, \hat{n}$. Resum by ~~Sepot~~ Separation of variables

$\Rightarrow$ Two equations

$$\Rightarrow \boxed{\begin{aligned} &\mu_{Ren} \frac{d\sigma}{d\mu_{Ren}} = 0 \\ &n^{\alpha} \frac{d\sigma}{dn^{\alpha}} = 0 \end{aligned}}$$

$p_1 \cdot n \frac{\partial}{\partial p_1 \cdot n} (RHS) = 0 \Rightarrow$ change in $\mathcal{P}$ must cancel change in H & U

$$\left(p_1 \cdot n \frac{\partial}{\partial p_1 \cdot n} H \right) \mathcal{P}_{a/A} \, \mathcal{P}_{\bar{a}/B} \, U_{a\bar{a}}$$

$$+ H \left(p_1 \cdot n \frac{\partial}{\partial p_1 \cdot n} \mathcal{P}_{a/A} \right) \mathcal{P}_{\bar{a}/B} \, U_{a\bar{a}}$$

$$+ H \, \mathcal{P}_{a/A} \, \bar{\mathcal{P}}_{\bar{a}/B} \left(p_1 \cdot n \frac{\partial}{\partial (p_1 \cdot n)} U_{a\bar{a}} \right) = 0$$

($\mathcal{P}_{\bar{a}/B}$ indep of $p_1 \cdot n$)

Divide by $\mathcal{P} \, \mathcal{P} \, H \, U$

$$\Rightarrow p_1 \cdot n \frac{\partial}{\partial p_1 \cdot n} \ln \mathcal{P}_{a/A} = - p_1 \cdot n \frac{\partial}{\partial p_1} \ln H - p_1 \cdot n \frac{\partial}{\partial p_1 \cdot n} \ln U$$

$$\boxed{p_1\cdot n \frac{\partial}{\partial p_1\cdot n} \ln \beta = G(p_1 n/\mu) + K(b\mu)}$$

depends only on variables common to β & H

depends on variables common to β & U

consistency condition that leads to Sudakov exponentiation

Renormalization independence of n^μ implies that

$$\mu \frac{\partial}{\partial \mu}\left[G(n\cdot p/\mu) + K(b\mu)\right] = 0$$

Separation of variables

$$\Rightarrow \quad \mu \frac{\partial}{\partial \mu} G(p\cdot n/\mu) = \underbrace{A(\alpha_s(\mu))}_{\text{Sudakov anomalous dim}} = -\mu \frac{\partial}{\partial \mu} K(b\mu)$$

$$K(b\mu) = -\int_{1/b}^{\mu} \frac{A(\alpha_s(\mu'))}{\mu'} d\mu' + K(1)$$

$$K\left(\frac{\mu}{p\cdot n}\right) = -\int_{p\cdot n}^{\mu} \frac{A(\alpha_s(\mu'))\, d\mu'}{\mu'} + K(1)$$

$$\Rightarrow \quad K(b\mu) = K\left(\frac{\mu}{p\cdot n}\right) - \int_{\frac{1}{b}}^{p\cdot n} \frac{A_a(\alpha_s(\mu'))\,d\mu'}{\mu'}$$

Finally

$$G(p\cdot n/\mu) + K(b\mu) = G\left(\frac{p\cdot n}{\mu}\right) + K\left(\frac{\mu}{p\cdot n}\right) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'}\, A_a(\alpha_s(\mu'))$$

$\therefore$

$$(p\cdot n)\frac{\partial}{\partial(p\cdot n)} \ln P\left(\frac{p\cdot n}{\mu}, b\mu\right) = G\left(\frac{p\cdot n}{\mu}\right) + K\left(\frac{\mu}{p\cdot n}\right) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'}\, A(\alpha_s(\mu'))$$

$$\Rightarrow \quad \ln P = \int_{1/b}^{p\cdot n} \frac{d(p\cdot n)}{(p\cdot n)} \left[G\left(\frac{p\cdot n}{\mu}\right) + K\left(\frac{\mu}{p\cdot n}\right) - \int_{1/b}^{p\cdot n} \frac{d\mu'}{\mu'}\, A(\alpha_s(\mu')) \right] + \ln P(1,1)$$

Integrating one gets double logs

To see this expand $A(\alpha_s(\mu))$ & consider only first term

$$\ln \tilde{P} - \ln \tilde{P}(1,1)$$

$$= \underbrace{\int_{1/b}^{p\cdot n} \frac{d(p\cdot n)}{p\cdot n}}_{\ln b} \left[G\left(\frac{p\cdot n}{\mu}\right) + K\left(\frac{\mu}{p\cdot n}\right) \right.$$

$$\left. - \underbrace{\int_{1/b}^{p\cdot n} \frac{d\mu}{\mu}}_{\ln b} \frac{\alpha_s}{2\pi} A_a^{(1)} \right]$$

$\longrightarrow$ $A_a^{(1)} \frac{\alpha_s}{2\pi} (\ln b)^2$ in exponent.

Transform back to Q_T space

$$\longrightarrow \frac{\log Q_T}{Q_T}.$$

$$\frac{d\sigma_{AB}}{dQ^2\, d^2Q_T} = \sum H_{a\bar{a}} \int \frac{d^2b}{(2\pi)^2} e^{i\vec{Q}\cdot\vec{b}} \exp\left[E_{a\bar{a}}(b, Q, \mu)\right]$$

$$\sum_{a\cdot q\bar{q}} \frac{d\hat{\sigma}}{dQ^2}(Q,\mu)\, f_{a/A}\left(\xi, \tfrac{1}{b}\right) f_{\bar{a}/B}\left(\xi, \tfrac{1}{b}\right)$$

Sudakov exponent

$$E_{a\bar{a}} = -\int_{1/b^2}^{Q^2} \frac{dk_T^2}{k_T^2} \left[2 A_q(\alpha_s(k_T)) \ln \frac{Q^2}{Q_T^2} + 2 B_q(\alpha_s(k_T)) \right]$$