α_i integral → linear → no pinch
but pole may migrate to end point $\alpha_i = 0$
or ~~maybe~~ $\cancel{k^2 - m^2} = 0$

Inside D we have $\alpha_j (k_j^2(l,p) - m^2)$
There are two ways in which it can be zero

1. $\alpha_j = 0$: end point singularity
2. $k_j^2 = m^2$

To summarize

$$\boxed{\begin{array}{c} k_j^2 = m^2 \quad \text{or} \quad \alpha_j = 0 \\ \\ \displaystyle\sum_{j = \text{loops}} \alpha_j \, l_j \, \epsilon_{js} = 0 \end{array}}$$

$\epsilon_{js} = \pm 1$ when the momentum l_j flows in the same (opposite) direction as loop momentum k_s

Let us apply this to our 3 pt. vertex

$$G = \int_0^1 d\alpha_1\, d\alpha_2\, d\alpha_3$$

$$\int d^4 l \; \frac{N(l, p_1, p_2)}{D^3}$$

where $D = \left[\alpha_1 l^2 + \alpha_2 (p_1+l)^2 + \alpha_3 (p_2-l)^2\right]$

Landau eq^n

$$\boxed{\alpha_1 l^\mu + \alpha_2 (p_1+l)^\mu + \alpha_3 (p_2-l)^\mu = 0}$$

One sol^n : corresponding to <u>soft limit</u> (of zero gluon mom)

$$\boxed{l^\mu = 0 \,, \quad \frac{\alpha_2}{\alpha_1} = \frac{\alpha_3}{\alpha_1} = 0}$$

Another sol^n : corresponding to <u>collinear limit</u> (of $l \parallel$ to p_1/p_2)

$$\boxed{\begin{array}{l} 1)\;\; l^\mu = -z\, p_1^\mu \,, \quad \alpha_1 z = \alpha_2 (1-z) \,, \quad \alpha_3 = 0 \\ 2)\;\; l^\mu = z'\, p_2^\mu \,, \quad \alpha_1 z' = \alpha_3 (1-z) \,, \quad \alpha_2 = 0 \end{array}}$$

Not hard for simple cases

How do you do it for higher orders?

<u>Coleman & Norton interpretation</u>

Consider a term $\alpha_j l_j^\mu$ $\quad \boxed{\sum \alpha_j k_j^\mu = 0}$

define $\Delta x_j^\mu = \underbrace{\lambda \alpha_j} k_j^\mu$ with λ some scale

Interpret $\lambda \alpha_j = \dfrac{\Delta x_j^0}{k_j^0}$ (Lorentz inv.)

s.t.

$$\Delta x_j^\mu = \Delta x_j^0 \, v_j^\mu$$

$v_j^\mu = \left(1, \dfrac{\vec{k}_j}{k_j^0}\right) \leftarrow$ 4 vel.

Landau Eq^n $\boxed{\sum_j \Delta x_j^\mu = 0}$

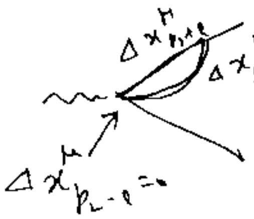
∴ Soln 1) reads (with $\lambda = 1$)

$$\Delta x_\ell^\mu = -\Delta x_{p_1+\ell}^\mu \,, \quad \Delta x_{p_2-\ell}^\mu = 0$$

Consider the vertices of the original diagram as space time points separated by Δx_j^μ

∴ Sol^n to Landau eq^{n}s corresponds to reduced diagrams

Sol 1)



2)

⟹ associate a vector Δx_j^μ with each line then (in (1)) the vectors associated with the two on-shell lines at the pinch satisfy $\Delta x^\mu_{\ell+p_1} = \Delta x^\mu_\ell$ i.e. are equal

In the reduced diagram one contracts the single off shell line $(p_2 - l)$ to a point corresponding to $\alpha_3 = 0$, i.e no propagator for this line

Reduced Diagram : any diagram in which all off shell lines are contracted to points

→ describe a physical process of freely moving particles → only interaction at the vertices

For this pinch the reduced diagram describes a physical process in which two on-shell massless particles of momenta l^μ & $(p_1 + l)^\mu$ are created at vertex 1 & propagate freely to vertex 2 where they combine to form the outgoing massless particle p_1

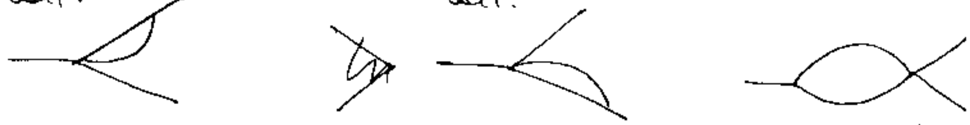
(kinematically possible because lines are massless)

This collinear pinch surface describes a physical process in which vertices may be identified with points in spacetime between which particles propagate on mass shell

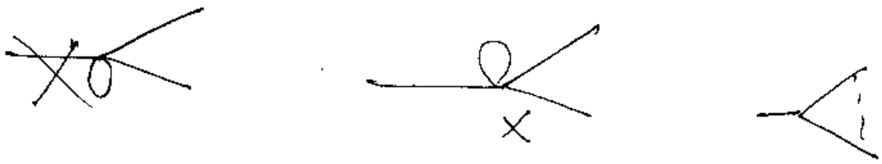
Coleman & Norton (1965) $\rightarrow$ turn this around

- Make all possible reduced diagrams
- Discard those that cannot be ~~described by~~ classified as scattering
- Remainder are solutions.

coll. coll.



not physically possible



X

IR

Trivial reduced diagram

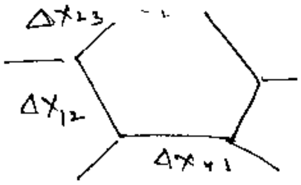
corresponds to soft sol^n.

Soft gluons with infinite wave length can attach at any point in spacetime

$l^\mu = 0$ can be thought of as $\lim\limits_{\lambda\to\infty} \frac{\bar{l}^\mu}{\lambda}$

$$\Rightarrow \Delta x_l^\mu = \lambda \alpha_l l^\mu = \lambda \alpha \frac{\bar{l}^\mu}{\lambda} \not\to 0$$

Generalizat^n to completely arbitrary diagram requires only schematic subdiagram



$\Delta x_{12} + \cdots \Delta x_{n1} = 0$

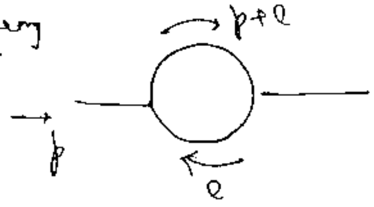
identify $\Delta x_i^\mu = \alpha_i l_i^\mu$

$\Rightarrow$ Landau condition

IR power counting

Recall power counting for UV divergences

ϕ^3 theory



$$\sim \int d^4 l \, \frac{1}{l^2 - m^2} \, \frac{1}{(p+l)^2 - m^2}$$

$$\sim \int d\Omega_3 \int |l|^3 \, d|l| \, \frac{1}{|l|^4}$$

$$\sim \int \frac{d|\bar{l}|}{|\bar{l}|}$$

$$\omega = 0$$

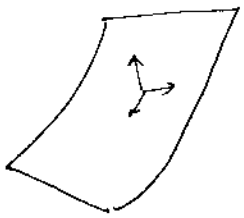
In general $\omega = 4 - E_b - \frac{3}{2} E_f$

all internal information is removed

Collinear & IR divergences :

Solutions to Landau equations define surfaces in multi-dimensional loop momentum space.

At each point on such a pinch surface we can choose intrinsic co-ordinates & normal co-ordinate



Intrinsic $\rightarrow$ co-ordinates that lie in the surface

Normal $\rightarrow$ co-ordinates that parametrize directions out of the surface

Integrand is a singular function of normal co-ordinates only

→ need to power count only the behaviour of the integral for normal co-ordinates.

let K_i → normal co-ordinate

3 denominators
l^2, $(p_1+l)^2$, $(p_2-l)^2$

Choose a frame $p_1^\mu = \frac{Q}{\sqrt{2}} \delta^{\mu +}$

$p_2^\mu = \frac{Q}{\sqrt{2}} \delta^{\mu -}$

$$D_1 = l^2 = 2l^+l^- - l_\perp^2$$
$$D_2 = (p_1+l)^2 = 2p_1^+l^- + 2l^+l^- - l_\perp^2$$
$$D_3 = (p_2-l)^2 = -2p_2^- l^+ + 2l^+l^- - l_\perp^2$$

Soft pinch surface → $D_1 = D_2 = D_3 = 0$
$\Updownarrow$
$l^\mu \to 0$

All four l^μ's are normal co-ordinates
no intrinsic "

($D_2 \sim 2p_1^+l^-$, $D_3 \sim -2p_2^-l^+$

$I \to \int \frac{d^4 l}{l^2\, 2p_1^+ l^-\, 2p_2^- l^+}$ → logarithmic IR div.

↳ singular fn. of all four l^μ's)

Collinear pinch surface

for l^μ parallels to p_1^μ pinch surface

l^-, $l_\perp^2$ are normal, l^+, ϕ intrinsic

because

$$D_2 \sim 2(p_1^+ + l^+) l^- - l_\perp^2$$

$$D_3 \sim -2 p_2^- l^+ \quad (l^+ \parallel p_1^+ \neq 0)$$

$$I \to \int \frac{dl^+ dl^- dl_\perp^2 d\phi}{l^2 \; D_2 \, (-2 p_2^- l^+)} \longrightarrow \text{also log. div}$$

(note $l^+ \parallel p_1^+ \Rightarrow l^+ \neq 0$)

$\Downarrow$

singular fn of l^- & $l_\perp^2$ only.

Use Landau eq^{n}s (with Coleman-Norton trick) & use power counting for each solution to classify all collinear & IR div. to all orders

(Collins, Soper, Sterman)

Power Counting

1) Redefine each of normal variables k_j in terms of a scaling variable λ

$$k_j = \lambda^{a_i} k_j'$$

- Determine the behaviour of the integral when λ vanishes for fixed values of the ratios. $\frac{k_j'}{k_j}$

Go back to 

Recall scaling eq^n. 1) Soft $\to$ $k^\mu \sim \lambda\sqrt{q^2}$

2) Collinear $\to$ $k^{\pm} \sim \sqrt{q^2}$

$k^{\mp} \sim \lambda\sqrt{q^2}$

$k_\perp \sim \lambda\sqrt{q^2}$

In the collinear region where k is in the direction of p_1

$$(p_1-k)^2 \to 2p_1.k \sim p_1^+ k^- \sim \lambda$$

$$k^2 \sim 2k^+k^- \sim \lambda$$

$$p_2 k \sim p_2^- k^+ \sim 1.$$

for soft scaling $p_1.k \sim \lambda$, $p_2.k \sim \lambda$, $k^2 \sim \lambda^2$

k^2 is quadratic & $(p_1-k)^2$, $(p_2-k)^2$ are linear

(2) Given a set of powers a_j we retain only terms of lowest powers λ^{A_i} in λ for each perturbative denominator

$$k_i^2(k_j, \lambda) - m_i^2 = \lambda^{A_i} f(k_j) + \dots$$

$\rightarrow$ homogeneous integral

(3) The homogeneous integral for pinch surface S is proportional to λ^{n_s}

$$n_S = \sum_j a_j - \sum_i A_i + S_I$$

S_I → powers of λ from moments in numerator

If $n_S > 0$ integral is finite

" $n_S = 0$ logarithmic divergence

" $n_S < 0$ diverges as a power

(4) Check for pinch surfaces in the homogeneous integral. If there are further pinch surfaces → find bounds for those regions as well.