Group Recognition

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Constructive recognition

Problem

Let $\mathbb{G}$ be some ambient group and

$$M_1, \ldots, M_k \in \mathbb{G}.$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

- The group order $|G|$ and
- an algorithm that, given $M \in \mathbb{G}$,
  - decides, whether or not $M \in G$, and,
  - if so, expresses M as an SLP in the M_i.

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If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

GAP examples

see other window

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- H is in some sense "smaller"
- or at least "easier to recognise constructively"
- e.g. $H \leq \Sigma_m$ or $H \leq \mathrm{GL}_n(\mathbb{F}_q)$ with smaller m or n, q respectively.

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$\to$ Monte Carlo algorithm to compute N

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5. express $M \cdot M'^{-1}$ as SLP in the N_j,
6. get M as SLP in the M_i and N_j:
   $M' = \prod$ in the M_i, $\quad M \cdot M'^{-1} = \prod$ in the N_j
   $$\Rightarrow M = \left(\mathrm{SLP}(\{N_j\})\right) \cdot \left(\mathrm{SLP}(\{M_i\})\right).$$

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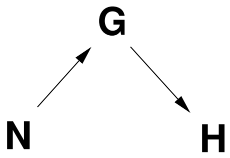
7. If $M \notin G$, then at least one step does not work.

Recursion: composition trees

We get a tree:



Up arrows: inclusions
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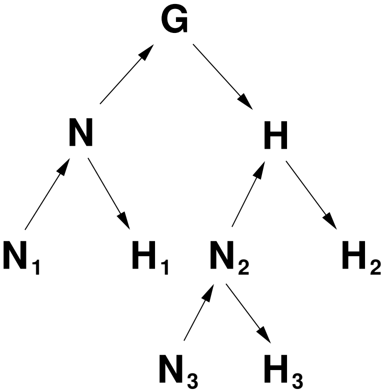

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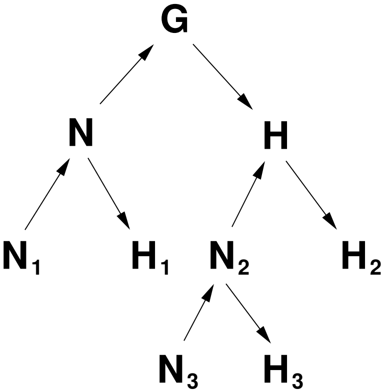

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Old idea, improvements are still being made.

Example: invariant subspace

Let $V = \mathbb{F}_q^{1\times d}$ and $G \leq \mathrm{GL}_d(\mathbb{F}_q)$, then G acts on V.
Let $W \leq V$ be an invariant subspace, i.e.:

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Choose basis $(w_1, \ldots, w_e)$ of W and extend to a basis

$$(w_1, \ldots, w_e, w_{e+1}, \ldots, w_d)$$

of V. After a base change the matrices in G look like this:

$$\left[\begin{array}{c|c} A & \mathbf{0} \\ \hline C & D \end{array}\right] \qquad \text{with } A \in \mathbb{F}_q^{e\times e}, C \in \mathbb{F}_q^{(d-e)\times e}, D \in \mathbb{F}_q^{(d-e)\times(d-e)}$$

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and

$$G \to \mathrm{GL}_{d-e}(\mathbb{F}_q), \left[\begin{array}{cc} A & \mathbf{0} \\ C & D \end{array}\right] \mapsto D$$

is a homomorphism of groups.

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Together with a reduction additional information is gained!

Long SLPs

Typical examples:

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Typical elements in W give SLPs of length $\approx$ 10000.

Learning from base and strong generators

The same groups with stabiliser chains:

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Stabiliser chain of length 55 with 434 strong generators.

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Stabiliser chain of length 55 with 434 strong generators.

Typical elements in W give SLPs of length $\approx$ 500.

Comparison

We compare lengths of SLPs:

	Stabiliser chain		Composition tree	
	in strong			in gens
G	15			900
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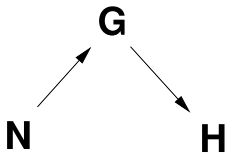
Problems with recursion



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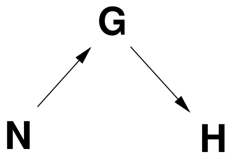

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Note: The first allows to compute N once H is recognised!

Constructive recognition revisited

Problem — new formulation

Let $\mathbb{G}$ be Σ_n or $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{PGL}_n(\mathbb{F}_q)$ and

$$M_1, \ldots, M_k \in \mathbb{G}.$$

Find for $G := \langle M_1, \ldots, M_k \rangle$:

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- another procedure that, given preimages $\hat{M}_1, \ldots, \hat{M}_k$ of the M_i under some homomorphism onto G, produces preimages of the nice generators.

If this problem is solved, we call

$\langle M_1, \ldots, M_k \rangle$ recognised constructively.

Recursion works again

Having recognised H in this sense, we can:

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If we remember how we created the generators for N, then we have recognised G constructively:

- Using H and N we can test membership in G,
- express elements as SLPs in the nice generators,
- and, given preimages of the original generators of G under some homomorphism, we can find preimages of the nice generators.

Recognising permutation groups

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This approach implements the asymptotically best known algorithms for permutation groups in the composition tree framework.

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Theorem (Aschbacher, 1984)

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- *$T \subseteq G/Z \subseteq Aut(T)$ for a non-abelian simple group T, and*
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The classes C8 and C9 have to be dealt with as leaves of the composition tree.

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In the GAP implementation, we have

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The End

Bibliography

William M. Kantor and Ákos Seress.
Computing with matrix groups.
In *Groups, combinatorics & geometry (Durham, 2001)*, pages 123–137. World Sci. Publ., River Edge, NJ, 2003.

Charles R. Leedham-Green.
The computational matrix group project.
In *Groups and computation, III (Columbus, OH, 1999)*, volume 8 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 229–247. de Gruyter, Berlin, 2001.

C. R. Leedham-Green and E. A. O'Brien.
Constructive recognition of classical groups in odd characteristic.
J. Algebra, 322(3):833–881, 2009.

Bibliography

F. Lübeck, K. Magaard, and E. A. O'Brien.
Constructive recognition of $\mathrm{SL}_3(q)$.
J. Algebra, 316(2):619–633, 2007.

Martin W. Liebeck and E. A. O'Brien.
Finding the characteristic of a group of Lie type.
J. Lond. Math. Soc. (2), 75(3):741–754, 2007.

Kay Magaard, E. A. O'Brien, and Ákos Seress.
Recognition of small dimensional representations of general linear groups.
J. Aust. Math. Soc., 85(2):229–250, 2008.

Bibliography

Max Neunhöffer and Ákos Seress.
A data structure for a uniform approach to computations with finite groups.
In *ISSAC 2006*, pages 254–261. ACM, New York, 2006.

E. A. O'Brien.
Towards effective algorithms for linear groups.
In *Finite geometries, groups, and computation*, pages 163–190. Walter de Gruyter, Berlin, 2006.

Ákos Seress.
A unified approach to computations with permutation and matrix groups.
In *International Congress of Mathematicians. Vol. II*, pages 245–258. Eur. Math. Soc., Zürich, 2006.