Finitely Presented Groups

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Introduction

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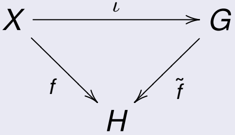
- There is a map $\iota : X \to G$ and $G = \langle \iota(X) \rangle$, and
- for every map $f : X \to H$ into a group H with $H = \langle f(X) \rangle$ such that $f(R) = \{1\}$ there is a unique group homomorphism $\tilde{f} : G \to H$ such that the following diagram commutes:



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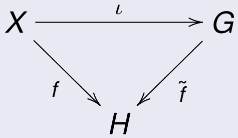

Example

$F := \langle X \mid \emptyset \rangle$ the free group, $G := \langle a, b \mid a^2, b^3, aba^{-1}b \rangle$.

Facts about finitely presented groups

Lemma

Every finitely presented group $G := \langle X \mid R \rangle$ *is a quotient of the free group* $F := \langle X \mid \emptyset \rangle$.

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Given $G = \langle X \mid R \rangle$. Find an algorithm that decides about any given word $w \in F := \langle X \mid \emptyset \rangle$, whether its image in G is equal to the identity or not.

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Problem (Word problem)

Given $G = \langle X \mid R \rangle$. Find an algorithm that decides about any given word $w \in F := \langle X \mid \emptyset \rangle$, whether its image in G is equal to the identity or not.

Nasty fact of life: Novikov (1955) and Boone (1959) have proved that no such algorithm exists!

GAP examples

see other window

Abelianisation

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We can add all commutators $C := \{xyx^{-1}y^{-1} \mid x, y \in X\}$ to R and get a factor group

$$G = \langle X \mid R \rangle \longrightarrow \tilde{G} := \langle X \mid R \cup C \rangle$$

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($\tilde{G}$ is the largest abelian factor group G/G').
We also get $\tilde{G}$ as factor group of the free abelian group:

$$A := \langle X \mid C \rangle \longrightarrow \tilde{G} = \langle X \mid R \cup C \rangle .$$

However, $A \cong \mathbb{Z}^n$ if $|X| = n$, so $\tilde{G}$ is a quotient of $\mathbb{Z}^n$.

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In particular, if $\tilde{G}$ contains infinite factors, we have proved that G is infinite.

Todd-Coxeter: the idea

Let $G := \langle X \mid R \rangle$ and $H = \langle h_1, \ldots, h_k \rangle < G$.

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A "name" of a coset is a number and a word representing the coset.

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We give names to right cosets and make sure that

- multiplication by elements of R fixes all cosets, and
- multiplication of H by elements of H fixes this coset.

A "name" of a coset is a number and a word representing the coset.

We make up new names and draw conclusions as we go and hope. . .

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

Events:

#	coset	a	a^{-1}	b	b^{-1}
1	H				

We start with an empty table like this.

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2			
2	Ha		1		

Events:

Def. 2 := Ha

We call the coset Ha number 2, a definition.

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2		
2	Ha	1	1		

Events:

Def. $2 := Ha$
Ded. $Haa = H$

Note $Haa = H$, since $a^2 = 1$, this is a deduction.

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2		2
2	Ha	1	1	1	

Events:

Def. $2 := Ha$
Ded. $Haa = H$
Ded. $Hab = H$

Note $Hab = H$, since $ab \in H$, this is a deduction.

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2	3	2
2	Ha	1	1	1	
3	Hb	4			1
4	Hba		3		

Next we define $3 := Hb$ and $4 := Hba$.

Events:

Def. $2 := Ha$
Ded. $Haa = H$
Ded. $Hab = H$
Def. $3 := Hb$
Def. $4 := Hba$

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2	3	2
2	Ha	1	1	1	
3	Hb	4	4		1
4	Hba	3	3		

Deduce $Hbaa = Hb$.

Events:

Def. $2 := Ha$

Ded. $Haa = H$

Ded. $Hab = H$

Def. $3 := Hb$

Def. $4 := Hba$

Ded. $Hbaa = Hb$

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2	3	2
2	Ha	1	1	1	
3	Hb	4	4	5	1
4	Hba	3	3		
5	Hbb				3

Define $5 := Hbb$.

Events:

Def. $2 := Ha$
Ded. $Haa = H$
Ded. $Hab = H$
Def. $3 := Hb$
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

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2	Ha	1	1	1	
3	Hb	4	4	5	1
4	Hba	3	3		
5	Hbb			1	3

Events:

[...]
Ded. $Hab = H$
Def. $3 := Hb$
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$
Ded. $Hbbb = H$

Deduce $Hbbb = H$. Thus Hb^{-1} is both 2 and 5!

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
1	H	2	2	3	2
2	Ha	1	1	1	3
3	Hb	4	4	2	1
4	Hba	3	3		
5	Hbb	–	–	–	–

Events:

[. . .]
Def. $3 := Hb$
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$
Ded. $Hbbb = H$
Coi. $5 = 2$

Conclude $Ha = Hbb$, replace 5 by 2: a coincidence.

An example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$.

#	coset	a	a^{-1}	b	b^{-1}
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Def. $5 := Hbb$
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Coi. $5 = 2$

Note $Habab = H$, a deduction that is already known.

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2	Ha	1	1	1	3
3	Hb	4	4	2	1
4	Hba	3	3	2	
5	Hbb	–	–	–	–

Use $Ha \cdot abab = Ha$, deduce $Hbab = Ha$.

Events:

[. . .]
Def. $4 := Hba$
Ded. $Hbaa = Hb$
Def. $5 := Hbb$
Ded. $Hbbb = H$
Coi. $5 = 2$
Ded. $Hbab = Ha$

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Events:

[. . .]
Ded. $Hbaa = Hb$
Def. $5 := Hbb$
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Coi. $5 = 2$
Ded. $Hbab = Ha$
Table closed.

Conclude $Hba = Hb$, replace 4 by 3. Table is closed.

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Events:

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Ded. $Hbaa = Hb$
Def. $5 := Hbb$
Ded. $Hbbb = H$
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Indeed, we have found a permutation representation on 3 points. The subgroup H fixes the first point.

Since we have checked all relations we have found a group homomorphism from G to Σ_3.

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- If $[G : H]$ is finite, then the algorithm terminates eventually (with the right strategy).

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- A completed coset enumeration with $H = \{1\}$ proves G to be finite and determines the order.

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- If it terminates, it proves that $[G : H]$ is finite and it constructs the permutation action of G on the right cosets of H.
- If $[G : H]$ is finite, then the algorithm terminates eventually (with the right strategy).
- No limit on memory and runtime is known a priori.
- A completed coset enumeration with $H = \{1\}$ proves G to be finite and determines the order.
- A not-yet-terminated coset enumeration proves nothing whatsoever.

Finding low index subgroups: the idea

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We start with an empty coset table with k rows and

- try out all possibilities to fill it in (finite!),
- check that all elements of R act trivially,
- use backtrack search,
- determine the point stabiliser H in each case, and
- remove equivalent actions (conjugate subgroups H).

$\Longrightarrow$ This very quickly becomes impractical for larger k.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1			
2			
3			

Guesses:

We start with an empty table like this.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1	1		
2			
3			

Guesses:

$1a = 1$

We first assume $1a = 1$. Nothing follows.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1	1	2	
2			1
3			

Guesses:

$1a = 1$
$1b = 2$ (wlog)

$1b = 1$ would be intransitive, so (wlog) $1b = 2$.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1	1	2	2
2		1	1
3			

Guesses:

$1a = 1$
$1b = 2$ (wlog)
$2b = 1$

From $2b = 1$ would follow $1bbb = 2$, a contradiction.

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Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

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1	1	2	
2		3	1
3			2

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So be backtrack and conclude $2b = 3$.

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2		3	1
3		1	2

Guesses:

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It follows that $3b = 1$ for $1bbb = 1$ to hold.

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#	a	b	b^{-1}
1	1	2	3
2	2	3	1
3	3	1	2

Guesses:

$1a = 1$
$1b = 2$ (wlog)

$2a = 2$ would imply $3a = 3$ and then $1abab = 3$.

Low index: an example

Let $G := \langle a, b \mid a^2, b^3, abab \rangle$ and $k = 3$.

#	a	b	b^{-1}
1	1	2	3
2	3	3	1
3	2	1	2

Guesses:

$1a = 1$
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Thus we have $2a = 3$ and everything is good.

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2	3	3	1
3	2	1	2

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For this solution $H = \langle a \rangle$.

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Guesses:

$1a = 1$
$1b = 2$ (wlog)

Thus we have $2a = 3$ and everything is good.

For this solution $H = \langle a \rangle$.

To go on, we would change the assumption $1a = 1$ to $1a = 2$ (wlog) and continue the search.

Overview over the situation for FP groups

For a finitely presented group $G = \langle X \mid R \rangle$ we can

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- in general not solve the word problem.
- prove finiteness, if a Todd-Coxeter coset enumeration terminates.
- prove that it is infinite, if we find some Abelian invariant to be 0.
- find a presentation for a subgroup given by some generators.
- compute lots of nice things if it is polycyclic.

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For a finitely presented group $G = \langle X \mid R \rangle$ we can

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- prove finiteness, if a Todd-Coxeter coset enumeration terminates.
- prove that it is infinite, if we find some Abelian invariant to be 0.
- find a presentation for a subgroup given by some generators.
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The End

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