# Local Langlands correspondence and Weil restriction (joint work with Roger Plymen)

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Let $E/F$ be a finite and Galois extension extension of non-archimedean local fields, with degree denoted by $d$. Review of the ramification of the Galois group $\Gamma = \Gamma_{E/F} := \mathrm{Gal}(E/F)$ in the lower numbering : We have a decreasing sequence of normal subgroups of $\Gamma$ :

$$(*) \qquad \Gamma = \Gamma_{-1} \supseteq \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_i = \Gamma_{i+1} = \cdots = \{1\}$$

where $\Gamma_i$ is the group of elements in $\Gamma$ acting trivially on the quotient of the ring of integers of $E$ by the $(i+1)$th power of its maximal ideal. If $t$ is a real number $\geq -1$, let $\Gamma_t$ denote the ramification group $\Gamma_i$, where $i$ is the smallest integer $\geq t$.
We set $\Gamma_{t+} := \bigcap_{t'>t} \Gamma_{t'}$ and we call $t$ a **break** (or **jump**) of the filtration if $\Gamma_{t+} \neq \Gamma_t$. Let $b$ denote the largest ramification break in the filtration.

## Remark

- $E/F$ is unramified if and only if $\Gamma_0 = 1$. In that case, $b = -1$.
- $E/F$ is tamely ramified if and only if $\Gamma_1 = 1$. In that case, $b = 0$.

For $t \geq 0$, we denote by $(\Gamma_0 : \Gamma_t)$ the index of $\Gamma_t$ in $\Gamma_0$. For $t \in [-1, 0)$, we set $(\Gamma_0 : \Gamma_t) := (\Gamma_t : \Gamma_0)^{-1}$. Let $e = e(E/F)$ denote the ramification index of $E/F$.

The **Hasse-Herbrand function** $\varphi_{E/F} \colon [-1, \infty) \to \mathbb{R}$ is defined by

$$\varphi_{E/F}(u) := \int_0^u \frac{1}{(\Gamma_0 : \Gamma_t)} dt.$$

It is continuous, piecewise linear, increasing and concave.

Remark : Let the ramification breaks occur at $b_1$, $b_2$, …, $b_k = b$. We have

$$\begin{aligned} \frac{b}{e} &= \int_0^b \frac{1}{(\Gamma_0 : \{1\})} dt \\ &= \int_0^{b_1} \frac{1}{(\Gamma_0 : \{1\})} dt + \cdots + \int_{b_{k-1}}^b \frac{1}{(\Gamma_0 : \{1\})} dt \\ &\leq \int_0^{b_1} \frac{1}{(\Gamma_0 : \Gamma_{b_1})} dt + \cdots + \int_{b_{k-1}}^b \frac{1}{(\Gamma_0 : \Gamma_b)} dt \\ &= \varphi_{E/F}(b), \end{aligned}$$

so that $\varphi_{E/F}(b) - \frac{b}{e} \geq 0$, with equality if and only if $b = 0$.

## Proposition

Let $r > 0$. Then we have

$$\varphi_{E/F}(er) = r + \varphi_{E/F}(b) - \frac{b}{e} \geq r,$$

where $e = e(E/F)$ denotes the ramification index of $E/F$. In particular, $\varphi_{E/F}(er) = r$ if and only if $E/F$ is tamely ramified.

## Corollary

We have

$$\frac{\varphi_{E/F}(er)}{r} \to 1 \quad \text{as} \quad r \to \infty.$$

For later use : Define an upper numbering on ramification groups by setting

$$\Gamma^s := \Gamma_t, \quad \text{if } s = \varphi_{E/F}(t).$$

Upper numbering behaves well with respect to quotients, i.e., if $H$ is a normal subgroup of $\Gamma$, then $(\Gamma/H)^s = \Gamma^s H/H$.

The ramification in the upper numbering of the absolute Galois group $\Gamma_F := \mathrm{Gal}(\overline{F}/F)$ is defined as follows :

$$\Gamma(F)^t := \operatorname{proj\,lim} \mathrm{Gal}(L/F)^t$$

with $L$ running through the set of finite Galois extensions of $F$.

The ramification in the upper numbering of the Weil group $W_F$ of $F$ is then inherited from the ramification in the upper numbering of the absolute Galois group :

$$W_F^t \hookrightarrow \Gamma(F)^t.$$

Let $X$ be an $E$-group scheme of finite type. The Weil restriction $\mathfrak{R}_{E/F}\, X$ is a finite type $F$-scheme satisfying the universal property :

$$(\mathfrak{R}_{E/F}\, X)(A) = X(E \otimes_F A),$$

for any $F$-algebra $A$. The Weil restriction exists as a $F$-scheme of finite type, and it has a natural $F$-group structure.

## Notation

Let $\mathbf{G}$ be a split connected reductive group over $E$ and let $\mathbf{M} := \mathfrak{R}_{E/F}\, \mathbf{G}$.

Let $G^\vee$ be the **Langlands dual group** of $\mathbf{G}$, that is, the complex Lie group with root system dual to that of $\mathbf{G}$.

## Examples

- $G^\vee = \mathrm{GL}_n(\mathbb{C})$ if $G = \mathrm{GL}_n(F)$,
- $G^\vee = \mathrm{PGL}_n(\mathbb{C})$ if $G = \mathrm{SL}_n(F)$ and $G^\vee = \mathrm{SL}_n(\mathbb{C})$ if $G = \mathrm{PGL}_n(F)$,
- $G^\vee = \mathrm{Sp}_{2n}(\mathbb{C})$ if $G = \mathrm{SO}_{2n+1}(F)$,
- $G^\vee = \mathrm{SO}_{2n+1}(\mathbb{C})$ if $G = \mathrm{Sp}_{2n}(F)$,
- $G^\vee = \mathrm{SO}_{2n}(\mathbb{C})$ if $G = \mathrm{SO}_{2n}(F)$.

Next goal : to define the Langlands dual group $M^\vee$ of $\mathbf{M}$.

## Notation

Let $A$ be a group, $A'$ be a subgroup of $A$ of finite index, and let $D$ be an $A'$-module. Let $D^*$ be the group of maps $f$ of $A$ into $D$ such that $f(a'a) = a'f(a)$ for all $a \in A, a' \in A'$. Make $D^*$ into a $A$-module by setting $(sf)a = f(as)$ with $a, s \in A$. Write $D^* = \mathrm{Ind}_{A'}^{A}(D) := \mathrm{I}_{A'}^{A}(D)$.

## Definition

The Shapiro isomorphism is defined as follows :

$$\mathrm{Sh}_A^{A'} \colon \mathrm{H}^1(A, D^*) \simeq \mathrm{H}^1(A', D), \qquad \alpha \mapsto (\theta \circ \alpha)|A',$$

where $\theta \colon D^* \to D$ is the morphism defined by $\theta(f) = f(1_A)$.

## Remark

Let $B$ be a group, $\mu \colon B \to A$ a homomorphism. Let $B' = \mu^{-1}(A')$ and assume that $\mu$ induces a bijection $B/B' \to A/A'$. Then $f \mapsto \mu \circ f$ induces an isomorphism $\mu' \colon \mathrm{I}_{A'}^{A}(D) \simeq \mathrm{I}_{B'}^{B}(D)$.

## The absolute Weil group $W_F$ of $F$

- It is a subgroup of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$.
- We have an exact sequence

$$1 \to I_F \to \mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1$$

  where $I_F$ is the inertia group and $\mathbb{F}_q$ is the residual field of $F$.
- Then $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$.
- $W_F$ is defined to be the inverse image of $\mathbb{Z}$ under $\mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$.

Let $W'_F$ denote the Weil-Deligne group of $F$ defined as $W'_F := W_F \ltimes \mathbb{C}$ with $w \cdot z = ||w|| \cdot z$ for all $w \in W_F, z \in \mathbb{C}$, where $w$ acts on $\mathbb{F}_q$ as $x \mapsto x^{q^{||w||}}$. Define similarly $W'_E$.

We will apply the previous Remark thrice.

• With $A = W_F$, $A' = W_E$, $B = W'_F$, $\mu\colon W'_F \to W_F$ then $B' = W'_E$. The condition $B/B' \simeq A/A'$ is satisfied. We infer that

$$\mathrm{Ind}_{W'_E}^{W'_F} G^\vee \simeq \mathrm{Ind}_{W_E}^{W_F} G^\vee.$$

• With $A = \Gamma_F$, $A' = \Gamma_E$, $B = W_F$, $\mu\colon W_F \to \Gamma_F$ then $B' = W_E$. The condition $B/B' \simeq A/A'$ is satisfied. We infer that

$$\mathrm{Ind}_{W_E}^{W_F} G^\vee \simeq \mathrm{Ind}_{\Gamma_E}^{\Gamma_F} G^\vee.$$

• With $A = \mathrm{Gal}(L/F)$, $A' = \mathrm{Gal}(L/E)$, $B = \Gamma_F$, $\mu$ given by the projection $\Gamma_F \to \mathrm{Gal}(L/F)$ then $B' = \Gamma_E$. The condition $A/A' \simeq B/B'$ is satisfied. We infer that

$$\mathrm{Ind}_{\Gamma_E}^{\Gamma_F} G^\vee \simeq \mathrm{Ind}_{\mathrm{Gal}(L/E)}^{\mathrm{Gal}(L/F)} G^\vee.$$

## Definition

The common value will be denoted $M^\vee$ :

$$M^\vee := \mathrm{Ind}_{W_E}^{W_F} G^\vee \simeq \mathrm{Ind}_{W_E'}^{W_F'} G^\vee \simeq \mathrm{Ind}_{\Gamma_E}^{\Gamma_F} G^\vee \simeq \mathrm{Ind}_{\mathrm{Gal}(L/E)}^{\mathrm{Gal}(L/F)} G^\vee.$$

## Theorem [A-Plymen]

Assume that $\mathbf{G} = \mathrm{GL}_n$ and let $\lambda_G \colon \mathrm{Irr}\,\mathbf{G}(E) \simeq \mathrm{Hom}(W'_E, G^\vee)$ denote the LLC for $\mathbf{G}(E)$.
Set $\mathrm{Sh} := \mathrm{Sh}_{W'_E}^{W'_F}$ and let $\iota$ denote the natural bijection from $\mathrm{Irr}\,\mathbf{M}(F)$ to $\mathrm{Irr}\,\mathbf{G}(E)$.
Then

$$\lambda_M := \mathrm{Sh}^{-1} \circ \lambda_G \circ \iota$$

defines a bijection from $\mathrm{Irr}\,\mathbf{M}(F)$ onto $\mathrm{H}^1(W'_F, M^\vee)$.

The bijection $\lambda_M$ should provide the LLC for $M$. When $\mathbf{M}$ is a torus, it is indeed the case (Langlands).

We will consider in this section the case $n = 1$, that is $\mathbf{G} = \mathrm{GL}_1$. The group $\mathrm{GL}_1(E) = E^\times = \mathbb{G}_\mathrm{m}$ is an affine algebraic variety with defining equation $xy - 1 = 0$ in affine space $\mathbb{A}^2$. Upon restriction of scalars, $E^\times$ becomes a connected commutative algebraic group with defining equation in the affine space $\mathbb{A}^{2d}$. This algebraic group $\mathbf{M} = \mathfrak{R}_{E/F}\,\mathbb{G}_\mathrm{m}$ is a torus defined over $F$, called an *induced torus*. We have $\mathbf{M}(F) = E^\times$.

**Theorem (Langlands)**

Let $\mathbf{T}$ be a torus defined over $F$.
There is a unique family of homomorphisms
$\lambda_T \colon \mathrm{Hom}(\mathbf{T}(F), \mathbb{C}^\times) \to \mathrm{H}^1(W_F, T^\vee)$ with the following properties :
(1) $\lambda_T$ is additive functorial in $T$, i.e., it is a morphism between two additive functors from the category of tori over $F$ to the category of abelian groups ;
(2) For $T = \mathfrak{R}_{E/F}\,\mathbb{G}_\mathrm{m}$ where $E/F$ is a finite separable extension, $\lambda_T$ is the isomorphism described above.

The Weil group $W_F$ carries an upper number filtration $\{W_F^r : r \geq 0\}$ and

Remark : The upper number filtration of $W_F$ is compatible with the filtration of $F^\times$ via the local class field theory isomorphism :

$$(W_F^r)^{ab} \simeq F_r^\times .$$

## Depth

Recall that $\mathbf{M} = \mathfrak{R}_{E/F}\,\mathrm{GL}_n$. Define the depth of a co-cycle $a \in \mathrm{H}^1(G, M^\vee)$ to be

$$\mathrm{dep}(a) := \inf \{r \geq 0 \;:\; W_F^s \subset \ker(a) \;\text{ for any } s > r\} .$$

Recall that $\iota$ denotes the natural bijection from $\mathrm{Irr}\,\mathbf{M}(F)$ to $\mathrm{Irr}\,\mathbf{G}(E)$.

**Proposition (Mishra-Pattanayak for $\mathbf{M} = \mathbf{T}$, A-Plymen for $\mathbf{M}$ arbitrary)**

We have

$$\mathrm{dep}(\lambda_M(\pi)) = \varphi_{E/F}(\mathrm{dep}(\lambda_G(\pi_E))),$$

where $\pi_E := \iota(\pi)$.

The multiplicative group $F^\times$ admits, for $r > 0$, a standard filtration :

$$F_r^\times = \{x \in F^\times : \mathrm{val}_F(x-1) \geq r\}$$

On the other hand $\mathbf{T}(F)$ carries a Moy-Prasad filtration

$$\{\mathbf{T}(F)_r \,:\, r \geq 0\}.$$

When $\mathbf{T} = \mathfrak{R}_{E/F}(\mathbb{G}_m)$, we have

$$\mathbf{T}(F)_r = \{t \in \mathbf{T}(F) : \mathrm{val}_E(t-1) \geq er\} = E_{er}^\times$$

where $\mathrm{val}_E$ is the valuation on $E$ normalized so that $\mathrm{val}_F(F^\times) = \mathbb{Z}$.

The *depth* of a character $\chi\colon T(F) \to \mathbb{C}^\times$ is

$$\mathrm{dep}(\chi) := \inf\{r \geq 0 \,:\, \mathbf{T}(F)_s \subset \ker(\chi) \text{ for } s > r\}.$$

**Comparison of depths**

Theorem (J.-K. Yu)

In the LLC for $T$, depth is preserved if $E/F$ is tamely ramified.

Theorem (Mishra-Pattanayak)

$$\mathrm{dep}(\lambda_T(\chi)) = \varphi_{E/F}(e \cdot \mathrm{dep}(\chi))$$

where $\chi$ is any character of $\mathbf{T}(F)$.

Mishra and Pattanayak constructed examples with $E/F$ wildly ramified in which the depth is not preserved by the LLC.

As a combination of the theorem above and of the property of the Hasse-Herbrand function, we obtain :

**Theorem (A-Plymen)**

In the $\mathrm{LLC}$ for $T$, depth is preserved (for positive depth characters) if and only if $E/F$ is tamely ramified.
If $E/F$ is wildly ramified then, for each character $\chi$ of $\mathbf{T}(F)$ such that $\mathrm{dep}(\chi) > 0$, we have

$$\mathrm{dep}(\lambda_T(\chi)) > \mathrm{dep}(\chi),$$

and

$$\mathrm{dep}(\lambda_T(\chi))/\mathrm{dep}(\chi) \to 1 \quad \text{as} \quad \mathrm{dep}(\chi) \to \infty.$$

A Langlands parameter for $M$ is a homomorphism

$$\phi\colon W_F' \to M^\vee \rtimes W_F =: {}^L M$$

where $\phi$ satisfies several conditions, one of which is

$$\phi(w) = (a(w), \nu(w)) \in M^\vee \rtimes W_F$$

for all $w \in W_F'$, where $\nu$ is the projection map $W_F' \to W_F$. The crossed product rule for $M^\vee \rtimes W_F$ implies that

$$a(ww') = a(w)(\nu(w) \cdot a(w')) \in M^\vee$$

so that the map $w \mapsto a(w)$ is a 1-cocycle of $W_F'$ with values in $M^\vee$.

This determines an injective map

$$j\colon \Phi(M) \to \mathrm{H}^1(W'_F, M^\vee), \qquad \phi \mapsto a$$

where $\Phi(M)$ is the set of ($M^\vee$-conjugacy classes) of Langlands parameters for $M$.

### Depth of $L$-parameters

For $\phi \in \Phi(M)$, we set

$$\mathrm{dep}(\phi) := \inf \left\{ r \geq 0 \;:\; W_F^s \subset \ker(\phi) \;\text{ for any } s > r \right\}.$$

We have $\mathrm{dep}(\phi) = \mathrm{dep}(j(\phi))$.

Take $d = [E : F] = 2$. Recall that $\mathbf{G} = \mathrm{GL}_n$. We denote by $V$ the $n$-dimensional $\mathbb{C}$-vector space $\mathbb{C}^n$. The $L$-group of $G = \mathbf{G}(F)$ is ${}^L G = G^\vee \rtimes W_F$ where $G^\vee = \mathrm{GL}_n(\mathbb{C}) = \mathrm{GL}(V)$. We have

$$M^\vee = \mathrm{Ind}_{W_E}^{W_F}({}^L G^\circ) = \mathrm{Ind}_{W_E}^{W_F}(\mathrm{GL}_n(\mathbb{C})) = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$$

and the $L$-group of $M$ is ${}^L M = M^\vee \rtimes W_F$, where $W_F$ permutes the two factors $\mathrm{GL}_n(\mathbb{C})$. We will write

$$w \cdot (g_1, g_2) = (g_{w(1)}, g_{w(2)}), \quad \text{for } g_i \in \mathrm{GL}(V) \text{ and } w \in W_F.$$

Let $\widetilde{\mathbf{G}}$ denote the group $\mathrm{GL}_N$ where $N = n^2$. The $L$-group of $\widetilde{G} = \widetilde{\mathbf{G}}(F)$ is ${}^L \widetilde{G} = \widetilde{G}^\vee \rtimes W_F$ where $\widetilde{G}^\vee = \mathrm{GL}_N(\mathbb{C}) = \mathrm{GL}(V^{\otimes 2})$.

Let $r\colon {}^L M \to {}^L\widetilde{G}$ denote the map defined by

$$r(g_1, g_2, w) := (g_{w(1)} \otimes g_{w(2)}, w) \quad \text{for } g_1,\ g_2 \text{ in } \mathrm{GL}(V) \text{ and } w \in W_F,$$

where

$$(g_{w(1)} \otimes g_{w(2)})(v_1 \otimes v_2) = g_{w(1)}(v_1) \otimes g_{w(2)}(v_2),$$

for $v_1,\ v_2 \in V$.

Let $\mathfrak{o}_E$ denote the ring of integers of $E$ and let $\mathfrak{p}_E$ be the maximal ideal of $\mathfrak{o}_E$. Let $\psi_E$ be a continuous nontrivial additive character of $E$ and let $c(\psi_E)$ be the largest integer $c$ such that $\psi$ is trivial on $\mathfrak{p}_E^{-c}$.
For $\Pi_E \in \mathrm{Irr}(GL_N(E))$, let $\varepsilon(s, \Pi_E, \psi_E)$ denote the Godement-Jacquet local constant. It takes the form

$$\varepsilon(s, \Pi_E, \psi_E) = \varepsilon(0, \Pi_E, \psi_E)\, q^{-f(\Pi_E, \psi_E)s},$$

where $\varepsilon(0, \Pi_E, \psi_E) \in \mathbb{C}^\times$.
The integer $f(\Pi_E) := f(\Pi_E, \psi_E) - Nc(\psi_E)$ is called the *conductor* of $\Pi_E$.

We have

$$\mathrm{dep}(r \circ \phi) = \mathrm{dep}(\phi), \text{ for any } L\text{-parameter } \phi\colon W_F' \to {}^L M.$$

Let $\pi \in \mathrm{Irr}(\mathbf{M}(F))$ with $L$-parameter $\phi$. We have $\mathrm{dep}(\phi) = \mathrm{dep}(\lambda_M(\pi))$. Then we get

$$\mathrm{dep}(r \circ \phi) = \varphi_{E/F}(\mathrm{dep}(\lambda_G(\iota(\pi)))).$$

As before we put $\pi_E := \iota(\pi) \in \mathrm{Irr}(\mathrm{GL}_n(E))$. Recall that $N = n^2$.

Let $\mathrm{As}(\pi_E) \in \mathrm{Irr}(\mathrm{GL}_N(F))$ be the representation with $L$-parameter $r \circ \phi$. It is the *Asai lift* of $\pi_E$.

## Theorem (A-Plymen)

We assume (for simplicity of the exposition) that $n \geq 2$. Then we have

$$f(\mathrm{As}(\pi_E)) = \varphi_{E/F}\left(\frac{f(\pi_E) - n}{n}\right) + n^2.$$

## Proof

We have

$$f(\mathrm{As}(\pi_E)) = \begin{cases} N-1 & \text{if } N=1 \text{ and } \mathrm{As}(\pi_E) \text{ is unramified,} \\ N\mathrm{dep}(\mathrm{As}(\pi_E)) + N & \text{otherwise.} \end{cases}$$

We consider the case when $N \geq 2$. Since LLC for $\mathrm{GL}_N(E)$ preserves the depth, we get

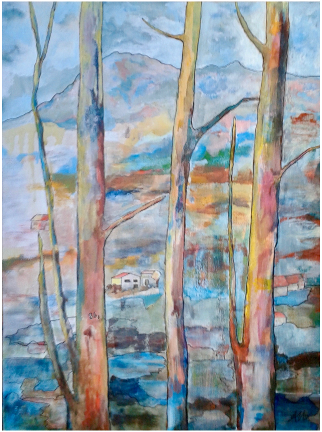
$$f(\mathrm{As}(\pi_E)) = N\mathrm{dep}(\mathrm{As}(\pi_E)) + N = N\mathrm{dep}(r \circ \phi) + N.$$

Then we obtain

$$f(\mathrm{As}(\pi_E)) = \varphi_{E/F}(\mathrm{dep}(\lambda_G(\pi_E))) + N.$$

We check that

$$f(\mathrm{As}(\pi_E) = \varphi_{E/F}(\mathrm{dep}(\phi_E)) + N = \varphi_{E/F}(\frac{f(\pi_E) - n}{n}) + N.$$



Thank you very much for your attention