Problem set

Exercice 1. (A counterexample to Schur's lemma)
Let K/k be a field extension. Let G denote the multiplicative group $K^\times$ and V denote the field K considered as a k-vector space. Given $x \in G$, let $\pi(x) \in \mathrm{Aut}_k(V)$ be the k-linear automorphism defined by $y \mapsto xy$.

(1) Prove that the k-representation (π, V) of G is irreducible.
(2) Prove that $\mathrm{End}_{kG}(\pi)$ is isomorphic to K.

Exercice 2. (A smoothness criterion for finite-dimensional representations)
Let Λ be an algebraically closed field of characteristic different from p and (π, V) be a Λ-representation of a reductive p-adic group G such that V has finite dimension over Λ.

(1) Prove that π is smooth if and only if $\mathrm{Ker}(\pi)$ is open in G. (For the proof of π *smooth implies* $\mathrm{Ker}(\pi)$ *open*, one may consider a Λ-basis of V.)
(2) What if Λ is assumed to be any commutative ring and G to be any topological group?

Exercice 3. (Finite-dimensional irreducible smooth representations of $GL_n(F)$)
Let (π, V) be a smooth Λ-representation of $GL_n(F)$ of finite dimension, where $n \geqslant 1$ and F is a locally compact non-Archimedean field of residual characteristic p, and Λ is an algebraically closed field of characteristic different from p.

(1) Given integers $1 \leqslant i, j \leqslant n$ with $i \neq j$ and a subgroup $A \subseteq F$, write $U_{ij}(A)$ for the subgroup of $G = GL_n(F)$ made of all unipotent matrices of the form $1 + aE_{ij}$ for some $a \in A$, where E_{ij} is the unipotent matrix with (i, j)-entry 1 and any other non-diagonal entry are 0. Let $\mathfrak{p}$ be the maximal ideal of the ring of integers of F. Prove that the subgroups $U_{ij}(\mathfrak{p}^k)$, for $k \in \mathbb{Z}$, form a basis of open neighbourhoods of 1 in $U_{ij}(F)$.
(2) Prove that $\mathrm{Ker}(\pi)$ contains $U_{ij}(F)$.
(3) Prove that $\mathrm{Ker}(\pi)$ contains $SL_n(F)$.
(4) Deduce that, if π is irreducible, then V has dimension 1.

Exercice 4. (Principal series representations of $GL_2(F)$)
Write $G = GL_2(F)$. Let B be the Borel subgroup of upper triangular matrices and T be the torus made of diagonal matrices. Consider the induced Λ-representation $\boldsymbol{i}_{T,B}^{G}(\epsilon_B)$, denoted (π, V). It is made of all locally constant functions $f : G \to \Lambda$ such that

$$f(bg) = \delta_B(b)f(g)$$

for all $b \in B$ and $g \in G$, where δ_B is defined by

$$\delta_B\left(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}\right) = |\alpha\delta^{-1}|, \quad \alpha, \delta \in F^\times, \quad \beta \in F,$$

where $|x|$ is the absolute value of $x \in F^\times$ (normalized by taking any uniformizer to q^{-1}).

(1) Prove (or admit) that one has

$$\int_{\mathrm{B}} h(xb)\ \mathrm{d}b = \delta_{\mathrm{B}}(x)^{-1} \cdot \int_{\mathrm{B}} h(b)\ \mathrm{d}b$$

for all $h \in \mathcal{C}_c^\infty(\mathrm{B}, \Lambda)$ and any Λ-valued Haar measure $\mathrm{d}b$ on B.

(2) Prove that the projection map $f \mapsto f_{\mathrm{B}}$ from $\mathcal{C}_c^\infty(\mathrm{G}, \Lambda)$ to V defined for all $g \in \mathrm{G}$ by

$$f_{\mathrm{B}}(g) = \int_{\mathrm{B}} \delta_{\mathrm{B}}^{-1}(b) f(bg)\ \mathrm{d}b$$

is well-defined and surjective.

(3) Let $\mathrm{d}g$ be a Λ-valued Haar measure on G. Prove that there is a unique Λ-valued Haar measure $\mathrm{d}b$ on B such that $\mathrm{d}g$ factors through the map $f \mapsto f_{\mathrm{B}}$, that is, there is a linear form $\mathscr{L}$ on V such that

$$\mathscr{L}(f_{\mathrm{B}}) = \int_{\mathrm{G}} f(g)\ \mathrm{d}g$$

for any $f \in \mathcal{C}_c^\infty(\mathrm{G}, \Lambda)$.

(4) Prove that the Jacquet module $\boldsymbol{r}_{\mathrm{T,B}}^{\mathrm{G}}(\pi)$ has length 2, with irreducible components ϵ_{B} and $\epsilon_{\mathrm{B}}^{-1}$ as Λ-characters of T.

(5) Prove that one has the following exact sequence of smooth Λ-representations of B:

$$0 \to \mathrm{X} \to \pi|_{\mathrm{B}} \to \delta_{\mathrm{B}} \to 0$$

where $\pi|_{\mathrm{B}}$ denotes the restriction of π to B and X is the subspace of V made of all functions vanishing at 1.

(6) Prove that the restriction of X to the unipotent radical N of B is isomorphic to $\mathcal{C}_c^\infty(\mathrm{N}, \Lambda)$.

(7) Prove that the kernel Y of the natural map from X onto $\boldsymbol{r}_{\mathrm{T,B}}^{\mathrm{G}}(\mathrm{V})$ is irreducible.

(8) Deduce that $\pi|_{\mathrm{B}}$ has length 3.

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