Problem Set

Lecture 1

(1) Let $g = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{GL}_{2n}(R)$, with $A, B, C, D \in M_n(R)$. Then the following are equivalent.

i) $g \in \mathrm{GSp}_{2n}(R)$ with multiplier $\mu(g) = \mu$.

ii) ${}^{\mathrm{t}}g \in \mathrm{GSp}_{2n}(R)$ with multiplier $\mu({}^{\mathrm{t}}g) = \mu$.

iii) $\mu g^{-1} = \begin{bmatrix} {}^{\mathrm{t}}D & -{}^{\mathrm{t}}B \\ -{}^{\mathrm{t}}C & {}^{\mathrm{t}}A \end{bmatrix}$.

iv) The blocks A, B, C, D satisfy the conditions

$$ {}^{\mathrm{t}}AC = {}^{\mathrm{t}}CA,\ {}^{\mathrm{t}}BD = {}^{\mathrm{t}}DB \text{ and } {}^{\mathrm{t}}AD - {}^{\mathrm{t}}CB = \mu 1_n. \tag{1} $$

v) The blocks A, B, C, D satisfy the conditions

$$ A\,{}^{\mathrm{t}}B = B\,{}^{\mathrm{t}}A,\ C\,{}^{\mathrm{t}}D = D\,{}^{\mathrm{t}}C \text{ and } A\,{}^{\mathrm{t}}D - B\,{}^{\mathrm{t}}C = \mu 1_n. \tag{2} $$

(2) (Koecher principle) Let $F \in M_k(\Gamma_n)$ be such that

$$ F(Z) = \sum_{T = {}^{\mathrm{t}}T \text{ half-integral}} A(T) e^{2\pi i \mathrm{Tr}(TZ)}. $$

i) Let us denote by $\mathcal{S} := \{T = {}^{\mathrm{t}}T \text{ half-integral}\}$. Define an equivalence relation $\sim$ on $\mathcal{S}$ as follows. For $T_1, T_2 \in \mathcal{S}$, let $T_1 \sim T_2$ if there exists a $g \in \mathrm{SL}_n(\mathbb{Z})$ such that $T_1 = {}^{\mathrm{t}}gT_2 g$. Denote by $\{T\}$ the equivalence class of T under $\sim$. Show that

$$ F(Z) = \sum_{\mathcal{S}/\sim} A(T) \sum_{T' \in \{T\}} e^{2\pi i \mathrm{Tr}(T'Z)}. $$

Conclude that, if $A(T) \neq 0$ then the series $\sum_{T' \in \{T\}} e^{-2\pi \mathrm{Tr}(T')}$ converges absolutely.

ii) Suppose T is not positive semi-definite. Show that there is a matrix $g \in \mathrm{SL}_n(\mathbb{Z})$ such that the $(1,1)$ entry of the matrix ${}^{\mathrm{t}}gTg$ is negative. In particular, we can assume without loss of generality, that the matrix entry $T_{11} < 0$.

iii) Let T be as in part ii) above. For any positive integer m, consider the matrix

$$ g_m = \begin{bmatrix} 1 & m & \\ & 1 & \\ & & 1_{n-2} \end{bmatrix} \in \mathrm{SL}_n(\mathbb{Z}). $$

Use the subsequence $\{{}^{\mathrm{t}}g_m T g_m : m \in \mathbb{N}\}$ in $\{T\}$ to show that $\sum_{T' \in \{T\}} e^{-2\pi \mathrm{Tr}(T')}$ diverges. Hence, conclude that $A(T) = 0$.

(3) Let $n = 2$ and consider the Hecke operator $T(p)$. We have the following coset decomposition

$$ T(p) = \Gamma_2 \begin{bmatrix} 1_2 & \\ & p1_2 \end{bmatrix} \Gamma_2 = \Gamma_2 \begin{bmatrix} p1_2 & \\ & 1_2 \end{bmatrix} \sqcup \bigsqcup_{a \in \mathbb{Z}/p\mathbb{Z}} \Gamma_2 \begin{bmatrix} 1 & & a & \\ & p & & \\ & & p & \\ & & & 1 \end{bmatrix} $$

$$ \sqcup \bigsqcup_{\alpha, d \in \mathbb{Z}/p\mathbb{Z}} \Gamma_2 \begin{bmatrix} p & & & \\ -\alpha & 1 & & d \\ & & 1 & \alpha \\ & & & p \end{bmatrix} \sqcup \bigsqcup_{a,b,d \in \mathbb{Z}/p\mathbb{Z}} \Gamma_2 \begin{bmatrix} 1 & & a & b \\ & 1 & b & d \\ & & p & \\ & & & p \end{bmatrix} $$

Suppose $F \in S_k(\Gamma_2)$ with Fourier coefficients $\{A(T)\}$. Applying the definition of the action of $T(p)$ on F, show that the T^{th} Fourier coefficient of $T(p)F$ is given by

$$A(pT) + p^{2k-3}A(\frac{1}{p}T) + p^{k-2}\big(\sum_{\alpha\in\mathbb{Z}/p\mathbb{Z}} A(\frac{1}{p}\begin{bmatrix}1 & \alpha\\ & p\end{bmatrix}T\begin{bmatrix}1 & \\ \alpha & p\end{bmatrix}) + A(\frac{1}{p}\begin{bmatrix}p & \\ & 1\end{bmatrix}T\begin{bmatrix}p & \\ & 1\end{bmatrix})\big).$$

Here, we assume that $A(S) = 0$, if S is not half-integral.

(4) If $F \in S_k(\Gamma_2)$ is a Hecke, with Hecke eigenvalues $\lambda(m)$ and Satake p-parameters $\alpha_{0,p}, \alpha_{1,p}, \cdots, \alpha_{n,p}$. It is known that

$$L_p(s, F, \text{spin})^{-1} = 1 - \lambda(p)p^{-s} + (\lambda(p)^2 - \lambda(p^2) - p^{2k-4})p^{-2s} - \lambda(p)p^{2k-3}p^{-3s} + p^{4k-6}p^{-4s}.$$

(a) Show that

$$\alpha_{0,p}^2\alpha_{1,p}\alpha_{2,p} = p^{2k-3}.$$

(b) Set $\alpha_p = p^{3/2-k}\alpha_{0,p}$ and $\beta_p = \alpha_p\alpha_{1,p}$. Show that

$$\lambda(p) = p^{k-3/2}(\alpha_p + \alpha_p^{-1} + \beta_p + \beta_p^{-1}),$$

$$\lambda(p^2) = p^{2k-3}\big((\alpha_p + \alpha_p^{-1})^2 + (\alpha_p + \alpha_p^{-1})(\beta_p + \beta_p^{-1}) + (\beta_p + \beta_p^{-1})^2 - 2 - 1/p\big).$$

Lecture 2

(1) Write $A(T) = A(n, r, m)$ for $T = \begin{bmatrix} n & r/2 \\ r/2 & m\end{bmatrix}$. Show that, if F_f is the Saito-Kurokawa lift of f, then the Fourier coefficients satisfy the recurrence relations

$$A(n, r, m) = \sum_{d|\gcd(n,r,m)} d^{k-1}A(\frac{nm}{d^2}, \frac{r}{d}, 1).$$

(2) We know that $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfies the functional equation $\xi(s) = \xi(1-s)$. Also, if $g \in S_{k'}(\Gamma_1)$, then $\Lambda(s, g) := (2\pi)^{-s}\Gamma(s)L(s, g)$ satisfies the functional equation $\Lambda(k' - s, g) = (-1)^{k'/2}\Lambda(s, g)$. Use these functional equations to derive the functional equation for $\Lambda(s, F_f)$, where F_f is the Saito Kurokawa lift of $f \in S_{2k-2}(\Gamma_1)$.

(3) Show that

$$L(s, F_f, \text{std}) = \zeta(s)L(s + k - 1, f)L(s + k - 2, f).$$

(4) There is an open conjecture that any $F \in S_k(\Gamma_2)$ has a non-zero Fourier coefficient $A(T)$ with T of the form $\begin{bmatrix} a & b/2 \\ b/2 & 1\end{bmatrix}$. Show that this is true if F is a Saito-Kurokawa lift.

(5) Let $F \in M_k(\Gamma_n)$ be such that all its Fourier coefficients $A(T)$, for $T > 0$, satisfy $|A(T)| \ll_F \det(T)^{k/2}$. Then there is a Hecke eigenform $G \in M_k(\Gamma_n)$ such that all its Fourier coefficients $A(T)$, for $T > 0$, satisfy $|A(T)| \ll_F \det(T)^{k/2}$. In addition, if F is non-cuspidal, then we can choose G to be non-cuspidal.

(6) Suppose $F \in M_k(\Gamma_n)$ is a non-cuspidal Hecke eigenform such that all its Fourier coefficients $A(T)$, for $T > 0$, satisfy $|A(T)| \ll_F \det(T)^{k/2}$. Show that $\Phi^n F = 0$.