How to define L-factors? complete L-fc of autom forms

2 methods

1. via LLC

2. via zeta integral

1. is uniform but a charm of LLC is

analytic L = Artin L

$$\text{motives} \underset{\text{etale coh}}{\longrightarrow} \text{Galois rep} \underset{\text{LLC}}{\longrightarrow} \text{autom rep}$$

merom conti
funct eq
poles
algebraicity of special values
p-adic behavior

2. is case by case

Ex 1. Godement-Jacquet L (generalization of Tate's thesis)

$L^{GJ}(s,\pi)$ is a (normalized) generator of the ideal gen by $Z^{GJ}(s-\frac{1}{2}, \xi\otimes\xi', \phi)$

$$= \int_{GL_n(F)} \langle \pi(g)\xi, \xi'\rangle \phi(g) |\det g|^{s+\frac{n-1}{2}} dg$$

std L-factor of irr rep of $GL_n(F)$, F: loc

$\pi \ni \xi$

$\xi' \in \pi^\vee$ dual to π

$\phi \in \mathcal{S}(M_n(F))$

$\langle\ ,\ \rangle : \pi \times \pi^\vee \to \mathbb{C}$ perfect pairing

Ex2. JPSS

π: irr generic rep of $GL_n(F)$

π' : $GL_m(F)$, $m>n$

$W \in W(\pi)$ Whittaker model

$W' \in W(\pi')$

$$\int_{GL_n(F)} W(g) W'\begin{pmatrix} g & \\ & 1_{m-n}\end{pmatrix} |\det g|^{s+\frac{m-n}{2}} dg \sim \text{product } L \text{ of } \pi \,\&\, \pi'$$

formulation of LLC

strongly mult one

Ex 3 triple L

π: generic rep of $GL_2(F)$

$W_i \in W(\pi_i)$

$\delta \in GSp_6(F)$

$\iota: GL_2\times GL_2\times GL_2 \hookrightarrow GSp_6$

$$\int W_1(g_1)W_2(g_2)W_3(g_3) f^{(s)}(\delta\,\iota(g_1,g_2,g_3))\, dg_1 dg_2 dg_3 \qquad f^{(s)} \in I_3(s,\chi)$$

families of integral rep's

G-J JPSS

triple L ← research talk

doubling method

std L for all irr rep of $Sp_{2n}, Mp_{2n}, O(V), U(W)$

Pitale's lecture my lecture

Complete g.c.d thy

theta corresp

sym & ext sq L for GL_n

Sporadic cases ...

thy of γ-factor

product L for $GL_n \times$ classical gp's

Langlands-Shahidi

monoid ...

Lec 1. PS-Rallis thy

2. Lapid-Rallis thy

3. g.c.d thy & loc theta corr

4. loc & global theta

F: number or loc field

E/F : quad ext

$W, \langle , \rangle$: skew herm sp$/E$, $\dim_E W = n$

$G = U(W)$

$-W = (W, -\langle , \rangle)$

doubled sp

$$W^\square = W \oplus (-W) \quad \text{split}$$
$$\cup$$
$$W^\triangle = \{(w,w) \mid w \in W\} \quad \text{max tot iso}.$$

For tot iso sp $Y \subset W$

$$P(Y) = M(Y)N(Y) \quad \text{stabilizer of } Y \text{ in } G.$$

$$M(Y) \simeq GL_E(Y) \times U(Y^\perp/Y)$$

$P := P(W^\triangle) = MN < G^\square = U(W^\square)$

$M \simeq GL_n(E)$

$N \simeq Herm_n := \{x \in M_n(E) \mid {}^t\bar{x} = x\}$

$G \times G \xhookrightarrow{i} G^\square$

$X = P\backslash G^{\square} = \{\text{tot iso subsp of } W^{\square} \text{ of } \dim n\} \curvearrowleft G\times G.$

$g \mapsto W^{\Delta} g$

Def. $G\times G$- orbit $O(x) = x\, i(G\times G)$ is negligible

$\Leftrightarrow \mathrm{Stab}(x) = \{(g,g')\in G\times G \mid x\, i(g,g') = x\}$

contains nip radical of proper parab of $G\times G$ as a normal subgp

$W^{\square} = W^+ \oplus W^-$

$W^+ = \{(w,0) \mid w\in W\}$

$W^- = \{(0,w)\}$

$K^{\pm}(x) := \dim_E(\underbrace{x\cap W_\pm}_{= x_\pm})$

Prop 1 (orbit str)

(1) $K_+(x) = K_-(x) =: K(x)$

(2) $K(x) = K(y) \Leftrightarrow O(x) = O(y)$

(3) $O(x)$ is negligible $\Leftrightarrow K(x) > 0$

proof.
$$\begin{tikzcd} & W^{\square} \arrow[dl, "p_+"'] \arrow[dr, "p_-"] & \\ W_+ & & W_- \end{tikzcd}$$
$I_\pm := p_\pm(x)$

Claim $I_\pm = x_\pm^{\perp}$

Since x is tot iso $\Rightarrow$ $(v,0)\in x_+ \subset x$

$(v_1, v_2) \in x$

$\langle (v,0), (v_1,v_2)\rangle = \langle v, v_1\rangle = 0$

i.e. $x_+ \subset I_+^{\perp}$

$K_+(x) + \dim I_+ \le \dim I_+^{\perp} + \dim I_+ = n$

$K_-(x) + \dim I_- \le n$

$x_\mp = \ker(p_\pm|_x) \Rightarrow$ $n = \dim x = \dim\ker(p_\pm|_x) + \dim \mathrm{Im}(p_\pm|_x)$

$= K_\mp(x) + \dim I_\pm$

equality hold

$x_\pm = I_\pm^{\perp}$ as claimed.

$K_+(x) = K_-(x) \Rightarrow (1)$

$\langle\, ,\rangle$ is non deg on $x_\pm^{\perp}/x_\pm$

Define
$$\begin{tikzcd} x_+^{\perp}/x_+ \arrow[rr, "g_x"] & & x_-^{\perp}/x_- \\ & x \arrow[ul, "p_+"] \arrow[ur, "p_-"'] & \end{tikzcd}$$

by $(v_1 \bmod x_+)\, g_x = v_2 \bmod x_-$ for $(v_1, v_2)\in x$

$v_1 \in x_+ \Leftrightarrow v_2 \in x_- \rightsquigarrow g_x$ welldef

g_x is isometry $\Leftrightarrow$ x is tot iso.

$x \underset{=}{\subset} \{(v_1, v_2) \in W^0 \mid v_1 \in x_+^\perp,\ v_2 \in x_-^\perp,\ (v_1 \bmod x_+) g_x = v_2 \bmod x_- \}$

has dim $\dim x_+^\perp + \dim x_+ = n$

$(\clubsuit)$ x is determined by $x_\pm, x_\pm^\perp, g_x$

Assume $K(x) = K(y)$

x_+ & y_+ are tot iso of same dim

$\Rightarrow \exists\, g \in G$ s.t. $x_+ g = y_+$

We may assume $x_+ = y_+$
$x_- = y_-$

$\Rightarrow x_\pm^\perp = y_\pm^\perp$

$g_x, g_y : x_+^\perp / x_+ \longrightarrow x_-^\perp / x_-$

$g_x = g_y \gamma$ with $\gamma \in U(x_-^\perp / x_-)$

Extend γ to G

$x = y\, i(1, g)$ by $(\clubsuit) \Rightarrow (2)$

$P_\pm = M_\pm N_\pm = P(x_\pm)$

Let $K(x) = j$

$M_\pm \simeq GL_j(E) \times G_j$

$\mathrm{Stab}(x) = \{(g, g') \in P^+ \times P^- \mid (v_1 g \bmod x_+) g_x = v_2 g' \bmod x_- \}$

$\cong (GL_j(E) \times GL_j(E) \times G_j^0) \ltimes \underbrace{N^+ \times N^-}_{\text{trivial only if } j=0} \Rightarrow (3)$ //

F: # field

$\mathbb{A}$: adeles of F

$\mathbb{E}$: " E

$\chi: E^\times \backslash \mathbb{E}^\times \to \mathbb{C}^\times$ unitary Hecke char

$I_n(s,\chi) = \{ f: G^0(\mathbb{A}) \to \mathbb{C}$ $\underset{\substack{\uparrow \\ \text{fixed goot max cpt subgp}}}{K\text{-fin}}$ $\mid f\left(\begin{pmatrix} a & b \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} g\right) = \chi(\det a) |\det a|^{s+\frac{n}{2}} f(g) \}$

Def A hol section $f^{(s)}$ of $I_n(s,\chi)$ is a ft on $\mathbb{C}\times G^0(\mathbb{A})$, $(s,g)\mapsto f^{(s)}(g)$ s.t.

- hol in s
- $\forall s$, $f^{(s)}\in I_n(s,\chi)$
- right K-fin uniformly in s

Eisenstein series

$$E(g,f^{(s)})=\sum_{\gamma\in P(F)\backslash G^0(F)} f^{(s)}(\gamma g)$$ conv for $\mathrm{Re}\, s>\frac{n}{2}$

mero conti to $\mathbb{C}$

funct eq

(π, V_π) : irr cusp autom rep of $G(\mathbb{A})$

$(\pi^\vee, \bar{V}_\pi)$

For $\xi\in V_\pi$, $\xi'\in\bar{V}_\pi$

$$Z(\xi\otimes\xi', f^{(s)})=\int_{G(F)\times G(F)\backslash G(\mathbb{A})\times G(\mathbb{A})}\xi(g)\xi'(g')E(i(g,g'),f^{(s)})\,dg\,dg'$$

Thm (Basic id)

Petersson pairing

$$Z(\xi\otimes\xi', f^{(s)})=\int_{G(\mathbb{A})}\langle\pi(g)\xi,\xi'\rangle f^{(s)}(i(g,1))\,dg$$

pf. $$G^0(F)=\bigsqcup_{j=0}^{r}P(F)x_j\, i(G(F)\times G(F))$$

$r:=$ Witt index of W

$\kappa(x_j)=j$

$$E(i(g,g'),f^{(s)})=\sum_{j=0}^{r}\sum_{\gamma\in \mathrm{Stab}(x_j)\backslash G(F)\times G(F)} f^{(s)}(x_j\, i(\gamma(g,g'))) \quad \text{for } \mathrm{Re}\, s>\frac{n}{2}$$

$$Z(\xi\otimes\xi', f^{(s)})=\sum\int_{\mathrm{Stab}(x_j)\backslash G(\mathbb{A})\times G(\mathbb{A})}\xi(g)\xi'(g')f^{(s)}(x_j\, i(g,g'))\,dg\,dg'$$

$$=\sum\int_{\mathrm{Stab}(x_j,\mathbb{A})\backslash G(\mathbb{A})\times G(\mathbb{A})}f^{(s)}(x_j\, i(g,g'))\chi(\det g')^{-1}$$

$$\times\underbrace{\int_{\mathrm{Stab}(x_j)\backslash \mathrm{Stab}(x_j,\mathbb{A})}(\xi\otimes\xi')(r\, i(g,g'))\,dr\,dg\,dg'}_{\parallel}$$

$$\begin{cases}\langle\pi(g)\xi,\pi(g')\xi'\rangle & \text{if } j=0\\ 0 & j>0\end{cases}$$

by Prop 1(3) & Cusp Condition //

F : (non arch) loc field of char 0

E/F : quad ext

$(W, \langle , \rangle)$: skew herm sp

$\uparrow$

non deg or 0

$G = U(W)$ or $GL_E(W)$

Rmk. • arch case is similar (explicit computation of $Z(\xi \otimes \xi', f^{(s)})$ is difficult)

• $E = F \oplus F \Rightarrow G = GL_n(E)$ or $GL_n(F) \times GL_n(F)$

reduced to the case $\langle , \rangle \equiv 0$

Basic rep thy

π : adm rep of G.

$Q = LU$ parab of G

δ_Q : modulus of Q

Jacquet module

natural action

$\pi_Q = \pi / \{\pi(u)v - v \mid u \in U, v \in \pi\} \curvearrowleft L$

adm rep of L normalized by $\otimes \delta_Q^{-\frac{1}{2}}$

σ : adm rep of $L = Q/U$

$\mathrm{Ind}_Q^G \sigma = \{ f : G \to \sigma \text{ smooth} \mid f(qg) = \delta_Q(q)^{\frac{1}{2}} \sigma(q) f(g) \ \forall q \in Q \}$

adm rep of G

Frob reci

$\mathrm{Hom}_G(\pi, \mathrm{Ind}_Q^G \sigma) = \mathrm{Hom}_L(\pi_Q, \sigma)$

$(\mathrm{Ind}_Q^G \sigma)^\vee \simeq \mathrm{Ind}_Q^G \sigma^\vee$

More generally, $(\underbrace{\text{c-ind}}_{\text{cpt-ind}}{}_H^G \rho)^\vee \simeq \mathrm{ind}_H^G \rho^\vee \otimes \delta_H$

$H \subset G$ closed subgp

ρ smooth rep

Def π is s.c. $\Leftrightarrow$ $\pi_Q = 0$ for $\forall$ proper Q

$\Leftrightarrow$ $g \mapsto \langle \pi(g) \xi, \xi' \rangle \in C_c(Z \backslash G)$

sq-int $\Leftrightarrow$ $\qquad$ " $\qquad L^2(Z \backslash G)$

tempered $\Leftrightarrow$ $\qquad$ " $\qquad L^{2+\varepsilon}(Z \backslash G)$

$\Leftrightarrow$ sub quot of $\mathrm{Ind}_Q^G \sigma$ with sq-int σ

Given π, $\exists\, Q$ & s.c. σ s.t. $\pi \hookrightarrow \mathrm{Ind}_Q^G \sigma$

$k: G\times G \to E^\times$

$$k(g,g') = \begin{cases} (\det g')^{-1} & \text{if } \langle\, ,\rangle \text{ is nondeg} \\ \dfrac{\overline{\det g'}}{\det g'} & \text{otherwise} \end{cases}$$

Prop 2 (generic uniqueness)

$\dim \mathrm{Hom}_{G\times G}(I_n(s,\chi), (\pi^\vee \boxtimes \pi)\otimes \chi\circ k) = 1$ outside fin values of q^{-s}

Rmk Kudla-Rallis conj = holds for all s $\qquad q := \#$ of residue field

pf. Assume $\langle\, ,\rangle$ is nondeg

$$P\backslash G^\square = \bigsqcup_{j=0}^{n} \Omega_j$$

$$\Omega_j = P\backslash P x_j i(G\times G)$$

$$\overline{\Omega_j} = \bigsqcup_{k\geq j} \Omega_k$$

$P_j = M_j N_j$ parab of G

$M_j \simeq GL_j(E)\times G_j$

$\mathrm{Stab}(x_j) \simeq (GL_j(E)\times GL_j(E)\times G_j^\Delta) \ltimes N_j\times N_j$

$$I_n(s,\chi) = \mathrm{Ind}_P^{G^\square} \chi|\,|^s = I_n(s,\chi)^{(0)} \supset \cdots \supset I_n(s,\chi)^{(j)} \supset \cdots \supset I_n(s,\chi)^{(n)}$$

$$I_n(s,\chi)^{(j)} = \{ f\in I_n(s,\chi) \mid f|_{\Omega_{j+1}} = 0\}$$

$$Q_j(s,\chi) := I_n(s,\chi)^{(j)} / I_n(s,\chi)^{(j+1)}$$

$\xi_j : g \mapsto \chi(\det a)|\det a|^{s+\frac{n}{2}}$

$x_j g x_j^{-1} = \begin{pmatrix} a & * \\ & {}^t\bar{a}^{-1} \end{pmatrix} \in P$

geometric lem $\qquad \simeq \text{c-ind}_{\mathrm{Stab}(x_j)}^{G\times G} \xi_j$

$$\simeq \mathrm{Ind}_{P_j\times P_j}^{G\times G} \chi|\,|^{s+\frac{j}{2}} \boxtimes \chi|\,|^{s+\frac{j}{2}} \boxtimes \text{c-ind}_{G_j^\Delta}^{G_j\times G_j} 1$$

$\mathrm{Hom}_{G\times G}(Q_j(s,\chi), (\pi^\vee\otimes\pi)\otimes\chi\circ k)$

$\simeq \mathrm{Hom}_{M_j\times M_j}((\pi_{P_j}\boxtimes\pi^\vee_{P_j})\otimes\chi^{-1}\circ k,\ \chi^{-1}|\,|^{-s-\frac{j}{2}} \boxtimes \chi^{-1}|\,|^{-s-\frac{j}{2}} \boxtimes \mathrm{ind}_{G_j^\Delta}^{G_j\times G_j} 1)$

$$= \begin{cases} 0 & \text{if } j>0,\ \chi^{-1}|\,|^{-s-\frac{j}{2}} \notin \text{exponents of } \pi_{P_j} \\ \mathbb{C} & \text{if } j=0 \end{cases} //$$

For $\xi\in\pi$, $\xi'\in\pi^\vee$ & hol section $f^{(s)}$ of $I_n(s,\chi)$

$$Z(\xi\otimes\xi', f^{(s)}) = \int_G \langle\pi(g)\xi,\xi'\rangle f^{(s)}(i(g,1))\,dg \quad \text{conv for } \mathrm{Re}\, s \gg 0 \text{ (proved in Lec 2)}$$

For $\mathrm{Re}\, s\gg 0$ $\quad Z(f^{(s)}, s, \chi) : \xi\otimes\xi' \mapsto Z(\xi\otimes\xi', f^{(s)})$

$$Z(s,\chi) \in \mathrm{Hom}_{G\times G}(I_n(s,\chi), (\pi^\vee\boxtimes\pi)\otimes\chi\circ k)$$

$M(s,\alpha): I_n(s,\chi) \to I_n(-s, \bar{\chi}^{-1})$ $\bar{\chi} = \chi \circ \theta$ $Gal(E/F) = \langle \theta \rangle$

$[M(s,\alpha) f^{(s)}](g) = \int_N f^{(s)}(w_0 u g)\, du$ conv for $\mathrm{Re}\, s > \frac{n}{2}$

$w_0 = (1,-1) \in G \times G$ mero conti

$W^{\Delta} w_0 = W^{\nabla} := \{(v,-v) \mid v \in W\}$ long Weyl

For $\mathrm{Re}\, s \ll 0$ $Z(-s) \circ M(s,\alpha) \in \mathrm{Hom}_{G \times G}(I_n(s,\chi), (\pi^\vee \boxtimes \pi) \otimes \alpha \circ \kappa)$

If $Z(s,\alpha)$ has mero conti to $\mathbb{C}$

$\Rightarrow {}^{\exists}\Gamma(s,\pi,\alpha)$ s.t. $Z(-s, \bar{\chi}^{-1}) \circ M(s,\chi) = \Gamma(s,\pi,\alpha) Z(s,\chi)$ by gen uniqueness

Lem 1. If $\langle \xi_1, \xi_1' \rangle \neq 0 \Rightarrow {}^{\exists} f_1^{(s)}$ s.t. $Z(\xi_1 \otimes \xi_1', f_1^{(s)}) = 1$

pf. $G \simeq i(G,1) \simeq \Omega_0 \subset P \backslash G^0$ open

U : small neighbor of 1 in G

$$f_1^{(s)}(g) = \begin{cases} \frac{1}{\langle \xi_1, \xi_1' \rangle} & \text{if } g \in i(U,1) \\ 0 & \text{if } g \notin P\, i(U,1) \end{cases}$$ //

Rmk To find a good test vector, one needs a good coodinate of an open subset of $P \backslash G^0$

e.g. $\langle\ ,\ \rangle = 0$

$G^0 = GL_{2n}(E) \supset P = MN$

$\bar{P} = M\bar{N}$ an opposite

For $\phi \in \mathcal{S}(\bar{N})$ $f_\phi^{(s)} \in I_n(s,\chi)$ defined by $f_\phi^{(s)}(u) = \phi(u)$

$Z(\xi \otimes \xi', f_\phi^{(s)}) = Z^{(1)}(s, \xi' \otimes \xi, \phi)$

This type of section plays important role

e.g. p-adic L for unitary gp

Eischen - Harris - Li - Skinner

Bernstein's principle

general setting Y : $\mathbb{C}$-vec sp, countable dim

$Y^* := \mathrm{Hom}(Y, \mathbb{C})$

$D/\mathbb{C}$: cm alg var

For $d \in D$, $\Xi_d = \{(y_v(d), \lambda_v(d))\}_{v \in R}$

$y_v(d) \in Y \otimes \mathbb{C}[D]$

$\lambda_v(d) \in \mathbb{C}[D]$

If ${}^{\exists}\phi \neq {}^{\exists}\Omega \subset D$ open s.t. $\forall d \in \Omega$ ${}^{\exists !} Z_d \in Y^*$ s.t. $Z_d(y_v(d)) = \lambda_v(d)$ $\forall v \in R$

$\Rightarrow {}^{\exists} Z(d) \in Y^* \otimes \mathbb{C}(D)$ s.t. $Z(d)(y_v(d)) = \lambda_v(d)$

Prop 3. $Z(\xi \otimes \xi', f^{(s)})$ has mero conti to $\mathbb{C}$

$f^{(s)}|_K$ is indep of $s \Rightarrow Z(\xi \otimes \xi', f^{(s)}) \in \mathbb{C}(q^{-s})$

pf. Let $\mathbb{C}[D] = \mathbb{C}[q^s, q^{-s}]$

$Y = (\pi \boxtimes \pi^\vee) \otimes \alpha \circ \kappa \otimes I_n(s,\chi) \curvearrowleft^{\Pi_s} G \times G$

$I_n(s,\chi) = \{ f : P \cap K \backslash K \to \mathbb{C} \mid \dots \}$ $\curvearrowleft$ action depends on s G^0

the sp indep of s

$v = (g, g', y) \in R = G \times G \times Y \cup \{0\}$

$y_v(d) = \Pi_s(g,g')y - y \in Y \otimes \mathbb{C}[D]$ $y_0(d) = \xi_1 \otimes \xi_1' \otimes f_1^{(s)}$

$\lambda_v(d) = 0$ $\lambda_0(d) = 1$

$\Omega = \{\mathrm{Re}\, s \gg 0\}$ $Z(s,\alpha)$ is unique solution of Ξ_d for $\mathrm{Re}\, s \gg 0$ by generic uniqueness//

~~Prop 2.7~~ Model thy 1.

π : irr rep of $GL_n(F)$

$\xi \in \pi$, $\xi' \in \pi^\vee$, $\phi \in \mathcal{S}(M_n(F))$

$Z^{GJ}(-s, \xi'\otimes\xi, \hat{\phi}) = \gamma^{GJ}(s+\frac{1}{2}, \pi, \psi)\, Z^{GJ}(s, \xi\otimes\xi', \phi)$

$L^{GJ}(s,\pi) :=$ "g.c.d" of $Z^{GJ}(s-\frac{1}{2}, \xi\otimes\xi', \phi)$

$$\frac{Z^{GJ}(-s, \xi'\otimes\xi, \hat{\phi})}{L^{GJ}(\frac{1}{2}-s, \pi^\vee)} = \varepsilon^{GJ}(s+\tfrac{1}{2}, \pi, \psi)\frac{Z^{GJ}(s, \xi\otimes\xi', \phi)}{L^{GJ}(s+\frac{1}{2}, \pi)}$$

Multiplicativity
$\left\{ \gamma^{GJ}(s,\pi,\psi) \right.$ depends only on cuspidal support of π

π is the Langlands quot of a std module $\mathrm{Ind}_Q^G(\sigma_1 \boxtimes \cdots \boxtimes \sigma_k)$

$\Rightarrow L^{GJ}(s,\pi) = \prod_{i=1}^{k} L^{GJ}(s, \sigma_i)$

$\varepsilon^{GJ}(\quad, \psi) = \prod \varepsilon^{GJ}(\quad, \psi)$

Specialty for GL_n

$\left\{ \begin{array}{l} \text{If } \pi \text{ is s.c. \& } n>1 \Rightarrow L^{GJ}(s,\pi) = 1 \\ \sigma \text{ is s.c} \Rightarrow L^{GJ}(s, [\sigma, |\,|\otimes\sigma, \cdots, |\,|^{k-1}\otimes\sigma]) = L^{GJ}(\underbrace{s}_{+k-1}, \sigma) \end{array} \right.$

+ BZ classification $L^{GJ}(s,\pi)$ easily computable

How to normalize $Z(\xi\otimes\xi', \underbrace{M(s,\pi)}\, f^{(s)}) = \Gamma(s,\pi,\pi)\, Z(\xi\otimes\xi', f^{(s)})$?

Model thy 2 (Langlands - Shahidi)

$M(w,\sigma) : \mathrm{Ind}_Q^G \sigma \longrightarrow \mathrm{Ind}_{Q^w}^G \sigma^w$

σ : generic

G: quasi-split

V: unip radical of Borel subgp

ψ : generic char of V

$0 \neq l_\psi \in \mathrm{Hom}_V(\mathrm{Ind}_Q^G \sigma, \psi)$

Jacquet integral

$l_\psi \circ M(w,\sigma) = \underbrace{C(w,\sigma,\psi)}\, l_\psi$

local coefficient giving Shahidi's γ-factor

important to study $\mathrm{Ind}_Q^G \sigma$

normalization of $M(w,\sigma)$

$MN = P \subset G^\circ \supset G\times G \ni w_0 = (1,-1)$

For $A \in Her_n$

define $\psi_A : \underset{\underset{Her_n}{\cong}}{N} \to \mathbb{C}^\times$ by $\psi_A(z) = \psi(\mathrm{tr}(Az))$

If $\det A \neq 0 \Rightarrow \mathrm{Hom}_N(I_n(s,\chi),\psi_A) = \mathbb{C}\, l_{\psi A}$

"genericity"

$$l_{\psi A}(f^{(s)}) = \int_N^{reg} f^{(s)}(w_0 u)\overline{\psi_A(u)}\,du := \int_{\mathcal{L}} f^{(s)}(w_0 u)\overline{\psi_A(u)}\,du$$

$\uparrow$ sufficiently big open cpt subgp of N

By uniqueness

explicitly computed by Sweet

$l_{\psi A}\circ M(s,\chi) = c(s,\chi,A,\psi)\, l_{\psi A}$ need to study $I_n(s,\chi)$

$$M^*(s,\chi) = \underbrace{c(s,\chi,A,\psi)^{-1}\chi(\det A)|\det A|^s\, \epsilon_{E/F}\left((-1)^{\frac{n(n-1)}{2}}\det A\right)^{n-1}}_{\text{indep of } A} M(s,\chi)$$

$$\epsilon_{E/F} \underset{CFT}{\longleftrightarrow} E/F$$

quad char

$$Z(\xi\otimes\xi', M^*(s,\chi)f^{(s)}) = \pi(-1)\,\delta(W)\,\gamma(s+\tfrac{1}{2},\pi\times\chi,\psi)\, Z(\xi\otimes\xi', f^{(s)})$$

$$\delta(W) = \begin{cases} -1 & \text{if } n \text{ is even, } W \text{ nonsplit} \\ 1 & \text{otherwise}\end{cases}$$

Thm 2 (10 commandments)

1. $\gamma(s,\pi\times\chi|\ |^{s_0},\psi) = \gamma(s+s_0,\pi\times\chi,\psi)$
2. π is subquot of $\mathrm{Ind}_Q^G \sigma \Rightarrow \gamma(s,\pi\times\chi,\psi) = \gamma(s,\sigma\times\chi,\psi) = \prod_{j=0}^{t}\gamma(s,\sigma_j\times\chi,\psi)$ (defined similarly)
3. W,π,χ,ψ unram $\Rightarrow \gamma(s,\pi\times\chi,\psi) = \dfrac{L^{GJ}(1-s, BC(\pi^\vee)\otimes\chi^{-1})}{L^{GJ}(s, BC(\pi)\otimes\chi)}$
4. $\gamma(s,\pi\times\chi,\psi)\gamma(1-s,\pi^\vee\times\chi^{-1},\psi^{-1}) = 1$
5. $\gamma(s,\pi^\vee\times\chi,\psi) = \gamma(s,\pi\times\chi\circ\theta,\psi)$
6. $\gamma(s,\pi\times\chi,\psi_a) = |a|^{n(s-\frac{1}{2})}\gamma(s,\pi\times\chi,\psi)$
7. W is aniso, absolute s.s. rank $r \le 1$
   $\pi = 1$ if $r = 1$ $\Rightarrow \gamma(s,\pi\times\chi,\psi) = \gamma^{GJ}(s, BC(\pi)\otimes\chi,\psi)$ (defined by accidental)
8. $\langle\, ,\, \rangle = 0 \Rightarrow \gamma(s,\pi\times\chi,\psi) = \gamma^{GJ}(s,\pi\otimes\chi,\psi)\,\gamma^{GJ}(s,\bar{\pi}^\vee\otimes\chi,\psi)$
9. $F = \mathbb{R}$ or $\mathbb{C} \Rightarrow \gamma(s,\pi\times\chi,\psi) =$ Langlands γ-factor $\bar{\pi} = \pi\circ\theta$
10. π: irr cusp autom rep of $G(\mathbb{A})$
    χ: Hecke char of $\mathbb{E}^\times/E^\times$ $\Rightarrow L^S(s,\pi\times\chi) = \prod_{v\in S}\gamma(s,\pi_v\times\chi_v,\psi_v)\cdot L^S(1-s,\pi^\vee\times\chi^{-1})$
    $1 \neq \psi: F\backslash\mathbb{A} \to \mathbb{C}^\times$

2, 6, 7, 10 determines $\gamma(s,\pi\times\chi,\psi)$ uniquely

proof. 1, 4, 5, 6, 10 are easy

7, 9 technical calculation

For some choice $f^{(s)}_{\phi,\phi'} \in I_n(s,\chi)$ $\quad Z(s,\chi,f^{(s)}_{\phi,\phi'}) = Z^{GJ}(s,\phi)\otimes Z^{GJ}(s,\phi') \Rightarrow 8$

We will see 2.

new notation (compound case)

$Q = P(Y) = LU \qquad Y \subset W$

tot iso dim j

$\mathcal{Y} = Y \boxplus Y^{\perp}/Y$

formal sum

$\mathcal{Y}^{\square} = Y^{\square} \boxplus (Y^{\perp}/Y)^{\square} \supset \mathcal{Y}^{\Delta}$ tot iso

$L \simeq GL_j(E) \times G(Y^{\perp}/Y)$

$L^{\square} \simeq GL_{2j}(E) \times G(Y^{\perp}/Y)^{\square} \supset L \times L$

$P(\mathcal{Y}^{\Delta}) \simeq P(Y^{\Delta}) \times P((Y^{\perp}/Y)^{\Delta})$

$f^{(s)} \in I^{\mathcal{Y}}(s,\chi) = I_j(s,\chi) \boxtimes I_{n-j}(s,\chi)$

$\wr\!\!\wr$

$\mathrm{Ind}_{P(Y^{\Delta})}^{GL_{2j}(E)} (\chi|\,|^{s} \boxtimes \bar{\chi}^{-1}|\,|^{-s})$

σ : adm rep of L

$\zeta \in \sigma,\ \zeta' \in \sigma^{\vee}$

$\xi \in \mathrm{Ind}_Q^G \sigma,\ \xi' \in \mathrm{Ind}_Q^G \sigma^{\vee}$

$\langle \xi, \xi' \rangle = \int_{Q\backslash G} \langle \xi(g), \xi'(g) \rangle dg$

$Q\backslash G = (Q\backslash G)^{\square}$

$Z(\xi \otimes \xi', f^{(s)}) = \int_{G^{\Delta}\backslash G\times G} \langle \pi(g)\xi, \pi^{\vee}(g')\xi' \rangle \chi(k(g,g'))\, f^{(s)}(i(g,g'))\, dg\, dg'$

$= \int_{Q^{\Delta}\backslash G\times G} \langle \xi(g), \xi'(g') \rangle \chi(k(g,g'))\, f^{(s)}(i(g,g'))\, dg\, dg'$

$= \int_{Q\times Q\backslash G\times G} \int_{L^{\Delta}\backslash L\times L} \boxed{\int_{U^{\Delta}\backslash U\times U} f^{(s)}(i(ulg))\, du\ \delta_{Q\times Q}(l)^{-\frac{1}{2}}}$

$U^{\Delta}\backslash U\times U = N(Y^{\Delta}) \cap P \backslash N(Y^{\Delta})$

$\chi(k(lg)) \langle (\zeta\otimes\zeta')(lg) \rangle\, dl\, dg$

intertwining

$I_n(s,\chi) \to \mathrm{Ind}_{P(Y^{\Delta})}^{G^{\square}} \cdots$

$\Delta_{\mathcal{Y}} : P(Y^{\Delta}) \to \mathbb{R}_+^{\times}$

$\quad \| \quad L^{\square} N(Y^{\Delta})$

$\Delta_{\mathcal{Y}}(p) = \begin{cases} |\det(p|_{Y^{\Delta}})|^{\frac{j-n}{2}} & \text{if } \langle\,,\rangle \text{ is nondeg} \\ \dfrac{|\det p|^{\frac{1}{2}}}{|\det(p|_{Y^{\Delta}})|^{\frac{n}{2}}} & \text{otherwise} \end{cases}$

Prop 4. (Multiplicativity)

extension of $\delta_{Q\times Q}^{-\frac{1}{2}}$ to $P(Y^{\Delta})$

$f^{(s)} \to [\Psi(s) f^{(s)}](g) : p \to \Delta_{\mathcal{Y}}(p)^{-1} \delta_{P(Y^{\Delta})}(p)^{-\frac{1}{2}} \int_{N(Y^{\Delta})\cap P\backslash N(Y^{\Delta})} f^{(s)}(upg)\, du$

$\Psi(s) : I_n(s,\chi) \to \mathrm{Ind}_{P(Y^{\Delta})}^{G^{\square}} (I^{\mathcal{Y}}(s,\chi) \otimes \Delta_{\mathcal{Y}})$

conv for $\mathrm{Re}\, s > -\frac{1}{2}$

merom conti

$$Z(\xi\otimes\xi', f^{(s)}) = \int_{\underbrace{Q\times Q\backslash G\times G}_{cpt}} \chi(K(g,g'))\, Z^{\gamma}(\xi(g)\otimes\xi'(g'), [\underline{\Psi(s)} f^{(s)}](i(g,g'))\, dg\,dg'$$

proof. $\begin{array}{ccc} S := P\cap P(Y^0) & \hookrightarrow & P(Y^0) \\ \downarrow & & \downarrow \text{proj} \\ P(y^{\Delta}) & \hookrightarrow & L^0 \end{array}$ $\quad S \ni m$

$\det(m|_{W^{\Delta}}) = \det(\mathrm{proj}(m)|_{y^{\Delta}})$

$\delta_{P(y^{\Delta})}(m)^{\frac{1}{2}} \cdot \Delta_{y}(m)\cdot \delta_{P(Y^0)}(m)^{\frac{1}{2}}\cdot \delta_P(m)^{-\frac{1}{2}}$ = modulus of m on $N(Y^0)\cap P\backslash N(Y^0)$ //

Corollaries (1) Convergence

(2) $E/F, W, \pi, \chi$ unram

ξ_0, ξ_0' K-fixed, $\langle \xi_0, \xi_0'\rangle = 1 \Rightarrow Z(\xi_0\otimes\xi_0', f_0^{(s)}) = \dfrac{L^{G}(s+\frac{1}{2}, BC(\pi)\otimes\chi)}{b(s,\chi)}$

$f_0^{(s)}$: normalized spherical

$b(s,\chi) = \prod_{j=1}^{n} L(2s+j, \chi|_{F^\times}\epsilon_{E/F}^{n-j})$

proof. Cor 1 is true for σ

$\Rightarrow$ 〃 $\mathrm{Ind}_Q^G \sigma$.

general conv $\Leftarrow$ s.c. conv $\Leftarrow$ unram char $\Leftarrow$ unram $n=1$.

majorized by unram char

Computation for $U(1)\times U(1) \hookrightarrow U(1,1)$ trivial

$GL(1)\times GL(1) \hookrightarrow GL(2)$ easy //

Study of $\Psi(s)$

$M_{y}(s,\chi) : I^{y}(s,\chi) \to I^{y}(-s, \bar\chi^{-1})$ similarly defined.

$M^*_{y}(s,\chi)$ normalization

Lem2. $\Psi(-s)\circ M^*(s,\chi) = \mathrm{Ind}_{P(Y^0)}^{G^0} M^*_{y}(s,\chi)\circ\Psi(s)$

Lem2 $\Rightarrow$ 2.

Recall $\underline{Z(-s)}\circ M^*(s,\chi) = \pi(-1)\, \varsigma(W)\, \gamma(s+\frac{1}{2}, \mathrm{Ind}_Q^G \sigma\times\chi, \psi)\, Z(s)$

$$LHS = \underbrace{\int_{Q\times Q} Z^{\gamma}(-s)\circ\Psi(-s)}\circ M^*(s,\chi)$$

$\qquad\qquad \|$ by Lem2

$$= \int Z^{\gamma}(-s)\cdot M^*_{y}(s,\chi)\circ\Psi(s)$$

$$= \sigma(-1)\,\varsigma(W)\,\gamma(s+\tfrac{1}{2}, \sigma\times\chi, \psi) \underbrace{\int Z^{\gamma}(s)\circ\Psi(s)}_{\substack{\| \\ Z(s)}} \quad //$$

proof of Lem2

(easy) generic uniqueness

$\dim \mathrm{Hom}_{G^0}(I_n(s,\chi),\ \mathrm{Ind}_{P(Y^0)}^{G^0}(I^{y}(-s,\bar{\chi}^{-1})\otimes \mathcal{O}_y)) = 1$ for almost all s

$\Rightarrow\ \Psi(-s)\circ M^*(s,\chi) = \alpha(s,\chi)\ \mathrm{Ind}_{P(Y^0)}^{G^0} M^*_y(s,\chi)\circ \Psi(s)$

"split generic" $A \mapsto$ "split generic" $\psi_B : N(y^0) \to \mathbb{C}^\times$

$(-1)^{\frac{n(n-1)}{2}}\det A = 1$

$$\ell_{\psi_A}(f^{(s)}) = \int_{P(Y^0)\cap N \backslash N} \ell^{y}_{\psi_B}([\Psi(s) f^{(s)}](w_0 u))\ \overline{\psi_A(u)}\, du$$

Apply $\int \ell^{y}_{\psi_B}$ to $\sim$

LHS $= \ell_{\psi_A}\circ M^*(s,\chi) = \ell_{\psi_A}$

RHS $= \alpha(s,\chi)\ \ell_{\psi_A}$ $\qquad \alpha(s,\chi)\equiv 1$ //

Thy of L-factors

How to define "g.c.d"?

$M^*(s,\chi) f^{(s)}$ may not be holomorphic

$$Z(\xi\otimes\xi', M^*(s,\chi) f^{(s)}) = \pi(-1)\,\zeta(W)\,\gamma(s+\tfrac{1}{2}, \pi\times\chi, \psi)\ Z(\xi\otimes\xi', f^{(s)})$$

symmetry necessary

Def. mero sections are $\alpha(s) f^{(s)}$ with hol section $f^{(s)}$ & mero ft $\alpha(s)$ on $\mathbb{C}$.

mero section $f^{(s)}$ is good $\Leftrightarrow$ $f^{(s)}$ is hol for $\mathrm{Re}\, s \geq 0$
$M^*(s,\chi) f^{(s)}$ is hol for $\mathrm{Re}\, s < 0$

Lem3 (1) hol sections are good sections

(2) $b(s,\chi)^{-1}\cdot$ good sections are hol sections

(3) good sections $=$ hol sections $+$ $M^*(-s,\bar{\chi}^{-1})$ (hol sections)

(4) $f^{(s)}$ is good $\Rightarrow$ $[\Psi(s) f^{(s)}](g)$ is a good section of $I^{y}(s,\chi)$ $\forall g\in G^0$

We prove (4) let $f^{(s)}$ be good

$\Psi(s)$ is conv for $\mathrm{Re}\, s > -\frac{1}{2}$ $\Rightarrow$ $[\Psi(s) f^{(s)}](g)$ is hol for $\mathrm{Re}\, s > -\frac{1}{2}$

$$M^*_y(s,\chi)[\Psi(s) f^{(s)}](g) = [\underset{\substack{\uparrow\\ \text{conv for } \mathrm{Re}\, s < \frac{1}{2}}}{\Psi(-s)}\ \underbrace{M^*(s,\chi) f^{(s)}}_{\text{hol for } \mathrm{Re}\, s<0}](g) \quad \text{hol for } \mathrm{Re}\, s < 0 \ //$$

By multiplicativity

$-\infty < \min\{\mathrm{ord}_{s=s_0} Z(\xi\otimes\xi', f^{(s)}) \mid \xi\in\pi, \xi'\in\pi^\vee, \text{ good } f^{(s)}\} \le 0$

$\exists!$ Euler factor $L(s,\pi\times\chi) = \dfrac{1}{P(q^{-s})}$ s.t.

$\forall \xi, \xi'$, good $f^{(s)}$ $\quad \dfrac{Z(\xi\otimes\xi', f^{(s)})}{L(s+\frac{1}{2}, \pi\times\chi)}$ is entire

$\forall s_0\in\mathbb{C}$ $\exists$ 〃 s.t. 〃 $\Big|_{s=s_0} \neq 0$.

Rmk · $\varepsilon(s,\pi\times\chi,\psi) = \gamma(s,\pi\times\chi,\psi)\dfrac{L(s,\pi\times\chi)}{L(1-s,\pi^\vee\times\chi^{-1})}$ monoid.

$L(1-s,\pi^\vee\times\chi^{-1}) = L(1-s,\pi\times\bar{\chi}^{-1})$

$\pi^\vee \simeq \pi\circ\theta$

$$(\maltese)\quad \frac{Z(\xi\otimes\xi', M^*(s,\chi)f^{(s)})}{L(\frac{1}{2}-s,\pi\times\bar{\chi}^{-1})} = \pi(-1)\chi(w)\,\varepsilon(s+\tfrac{1}{2},\pi\times\chi,\psi)\frac{Z(\xi\otimes\xi', f^{(s)})}{L(s+\frac{1}{2},\pi\times\chi)}$$

Analytic properties of $\{Z(\xi\otimes\xi', f^{(s)})\}$ = those of $L(s+\frac{1}{2},\pi\times\chi)$

Cor 2 (1) If σ is subquot of $\mathrm{Ind}_Q^G\,6$ $\Rightarrow$ $\dfrac{L(s,\pi\times\chi)}{L(s,6\times\chi)}$ entire

(2) Moreover, if $\forall\,\mathrm{Re}\,s_0\ge 0$, $\exists\,\xi,\xi'$, hol section $f^{(s)}$ of $I_n(s,\bar{\chi}^{-1})$

s.t. $\dfrac{Z(\xi\otimes\xi', f^{(s)})}{L(s+\frac{1}{2}, 6\times\bar{\chi}^{-1})}\Big|_{s=s_0} \neq 0 \Rightarrow L(s,\pi\times\chi) = L(s,6\times\chi)$.

(3) If π,χ unram $\Rightarrow$ $L(s,\pi\times\chi) = L^{(G)}(s, BC(\pi)\otimes\chi)$

proof. (1) matrix coef of π is that of $\mathrm{Ind}_Q^G\,6$ $\Rightarrow$ (1)

(2). follows from $(\maltese)$

(3) 〃 (1) & $Z(\xi_0\otimes\xi_0', \underbrace{b(s,\chi)f_0^{(s)}}_{\text{good section}}) = L^{(G)}(s+\frac{1}{2}, BC(\pi)\otimes\chi)$ //

Lem 4. π & χ tempered $\Rightarrow$ $L(s,\pi\times\chi)$ is hol for $\mathrm{Re}\,s>0$

proof sq-int case $\Rightarrow$ tempered case
multiplicativity

Assume π sq-int

$\langle\,,\,\rangle$ unitary

$$|Z(\xi\otimes\xi', f^{(s)})| \le \underbrace{\int_G |\langle\pi(g)\xi,\xi'\rangle|^2 dg \int_G |f^{(s)}(\iota(g,1))|^2 dg}_{\text{Cauchy-Schwartz}}$$

conv for $\mathrm{Re}\,s\ge 0$

π: tempered $\Rightarrow \gamma$ determines L & ε

$$\gamma = \varepsilon \cdot \frac{L(1-s, \pi^\vee \times \chi^{-1})}{L(s, \pi\times\chi)} \quad \text{no cancelation}$$

Lapid-Rallis L-factor

$L \simeq GL_{n_1}(E) \times \cdots \times GL_{n_k}(E) \times U(W_0)$

$\sigma = \sigma_1 \boxtimes \cdots \boxtimes \sigma_k \boxtimes \sigma_0$

$e(\sigma_i) \in \mathbb{R}$ s.t. $\sigma_i \otimes |\det|^{-e(\sigma_i)}$ has unitary cent char

If σ_0 : tempered

σ_i : ess tempered $\Rightarrow \exists!$ irr quot $\iota^G_Q(\sigma)$ of $\mathrm{Ind}^G_Q \sigma$

$e(\sigma_1) > \cdots > e(\sigma_k) > 0$

irr adm rep of $G \overset{1:1}{\longleftrightarrow} \{(Q,\sigma)\}/\sim$

$$L^{LR}(s, \iota^G_Q(\sigma)\times\chi) = \underbrace{L(s, \sigma_0\times\chi) \prod_{j=1}^{k} L^{GJ}(s, \sigma_j\otimes\chi)}\, L^{GJ}(s, \sigma_j^\vee\otimes\chi)$$

$\varepsilon^{LR}(\quad,\psi) = \varepsilon(\quad,\psi)\prod \varepsilon^{GJ}(\quad,\psi)\,\varepsilon^{GJ}(\quad,\psi)$

hol in $\mathrm{Re}\, s > 0$.

Thm 3

$L(s, \pi\times\chi) = L^{LR}(s, \pi\times\chi)$

$\varepsilon(\quad,\psi) = \varepsilon^{LR}(\quad,\psi)$.

proof Find ξ & ξ' & $f^{(s)}$ hol

s.t. $\dfrac{Z(\xi\otimes\xi', f^{(s)})}{\prod_{j=1}^{k} L^{GJ}(s, \sigma_j^\vee\otimes\chi)}$ has no zero in $\mathrm{Re}\, s \geq 0$ //

$+\frac{1}{2}$

$Q = LU$

$L^\square \simeq GL_{2j}(E) \times G_{n-j}^\square \quad \iota(\sigma, 1)$

$P(Y^\triangle) \times G_{n-j} \ni g$

$\bar{P}(Y^\triangle) = M(Y^\triangle)\bar{N}(Y^\triangle)$ an opposite

$\bar{Q} = L\bar{U}$

$U \times \bar{N}(Y^\triangle) \times G_{n-j} \times \bar{U} \hookrightarrow P\backslash G^\square$

good coordinate!

$W^\square = W^\triangle \oplus W^\nabla$

Weil rep $\omega_\psi^\square$

§ 4. Application to loc theta

$(V, (\,,\,))$: herm non-deg, $\dim V = m$

$(W, \langle\,,\,\rangle)$: skew herm $\dim W = n$

(χ_V, χ_W) : a pair of char's of $E^\times$ s.t.

$\chi_V|_{F^\times} = \epsilon_{E/F}^m$

$\chi_W|_{F^\times} = \epsilon_{E/F}^n$

$\mathbb{W} = V \otimes_E W$

$\langle\!\langle\,,\,\rangle\!\rangle = \mathrm{Tr}_{E/F}((\,,\,)\otimes\langle\,,\,\rangle)$ symplectic form

$W^\square = W \oplus (-W)$

$\mathbb{W}^\square = \mathbb{W} \oplus (-\mathbb{W}) = W^\square \otimes V$

ω_ψ : Weil rep of $Mp(\mathbb{W}) \longrightarrow Sp(\mathbb{W})$ (metaplectic)

$\omega_{V,W} := \omega_\psi \circ \iota_+$

$$\begin{tikzcd} & Mp(\mathbb{W}) \arrow[r] & Sp(\mathbb{W}) \\ & & U(W)\times U(V) \arrow[u] \arrow[ul, dashed, "\iota"'] \end{tikzcd}$$

Harris-Kudla-Sweet fix ι

$Mp(\mathbb{W}^\square) \curvearrowright \mathcal{S}(V\otimes W^\nabla) \cong V^n$

$$\begin{tikzcd} & Mp(\mathbb{W}^\square) \arrow[d] & \\ U(W^\square)\times U(V) \arrow[r, "\iota^0"] \arrow[ur, dashed, "\tilde{\iota}_0"] & Sp(\mathbb{W}^\square) & Mp(\mathbb{W})\times Mp(\mathbb{W}) \arrow[ul] \arrow[dl] \\ U(W)\times U(-W)\times U(V) \arrow[r] \arrow[u] \arrow[urr, dashed] & Sp(\mathbb{W})\times Sp(\mathbb{W}) \arrow[u] & \end{tikzcd}$$

$\iota_+ \times \iota_- \times \iota_0$

For an irr rep π of $G = U(W)$

$$\omega_{V,W} / \cap \ker f \simeq \pi \boxtimes \Theta_{W,V}(\pi)$$

$f : \omega_{V,W} \to \pi$ G-hom

adm rep of $H = U(V)$

Howe conjecture (proved by Waldspurger (odd res char) Gan-Takeda)

If $\Theta_{W,V}(\pi) \neq 0 \Rightarrow \theta_{W,V}(\pi)$ unique irr quot of $\Theta_{W,V}(\pi)$

$$\theta_{W,V}(\pi) \simeq \theta_{W,V}(\pi') \Rightarrow \pi \simeq \pi'$$

For irr rep σ of H

$\Theta_{V,W}(\sigma)$ & $\theta_{V,W}(\sigma)$ are similarly defined

Q. When $\Theta_{W,V}(\pi) \neq 0$?

Persistence $\Theta_{W,V}(\pi) \neq 0 \Rightarrow \Theta_{W,V \oplus \mathcal{H}_r}(\pi) \neq 0$

$\mathcal{H}_r$: split herm sp of dim $2r$

bottom sp's

$V_0^{\pm}$: even dim, aniso, $\epsilon(V_0^{\pm}) = \pm 1$

$V_1^{\pm}$: odd " $V_1^{\pm}$ "

$$\epsilon(V) = \epsilon_{E/F}(\operatorname{disc} V)$$

$$\operatorname{disc} V = (-1)^{\frac{m(m-1)}{2}} \det V \in F^\times / N_F^E(E^\times)$$

It suffices to determine

$$r_\epsilon^{\pm}(\pi) := \min \{ r \mid \theta_{W, V_\epsilon^{\pm} \oplus \mathcal{H}_r}(\pi) \neq 0 \} \quad \text{for } \epsilon = 0, 1$$

Tower property

Rmk if π is s.c. $\Rightarrow$ $\Theta_{W, V_\epsilon^{\pm} \oplus \mathcal{H}_{r_0}}(\pi) = \theta_{W, V_\epsilon^{\pm} \oplus \mathcal{H}_{r_0}}(\pi)$ is s.c.

$r_0 = r_\epsilon^{\pm}(\pi)$

Cuspidal supp of $\theta_{W, V_\epsilon^{\pm} \oplus \mathcal{H}_r}(\pi)$ described in terms of $\theta_{W, V_\epsilon^{\pm} \oplus \mathcal{H}_{r_0}}(\pi)$

$r > r_0$

Conservation relation

$$r_\epsilon^{+}(\pi) + r_\epsilon^{-}(\pi) = n$$

Example

For $\Phi \in \mathcal{S}(V^n)$, $a \in GL_n(E)$, $b \in \mathrm{Her}_n$, $h \in U(V)$

$$\left(\omega_{V,W}^{P}\left(\begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix}\right)\Phi\right)(x) = \chi_V(\det a) |\det a|^{\frac{m}{2}} \Phi(xa)$$

$$\left(\omega_{V,W}^{P}\left(\begin{pmatrix} 1_n & b \\ & 1_n \end{pmatrix}\right)\Phi\right)(x) = \psi(\operatorname{tr}(b\, Q(x)))\, \Phi(x) \qquad Q(x) = (\langle x_i, x_j \rangle) \in \mathrm{Her}_n$$

$$\left(\omega_{V,W}^{P}(h)\Phi\right)(x) = \underbrace{}^{\chi_W(\det h)} \Phi(h^{-1}x)$$

Notation $\chi_V := \chi_V \circ \det$ char of $G = U(W)$

$\chi_W := \chi_W \circ \det$ " $H = U(V)$

~~Rmk $\omega_{V,W} = \omega_{V,W}^{P} \otimes \chi_W$~~

$(V,(\,,\,))$: Herm sp, $\dim_E V = m$

$(W,\langle\,,\,\rangle)$: skew $\quad \dim_E W = n$

$\chi_V, \chi_W : E^\times \to \mathbb{C}^\times$

$\chi_V|_{F^\times} = \epsilon_{E/F}^m$ (viewed as a char of $G = U(W)$)

$\chi_W|_{F^\times} = \epsilon_{E/F}^n$ ($\chi_W \circ \det$ 〃 $H = U(V)$)

$V_0^\pm$: even dim aniso $\epsilon(V_0^\pm) = \pm 1$

$V_1^\pm$: odd 〃 $V_1^\pm$

$r_\epsilon^\pm(\pi) := \min\{r \mid \Theta_{W, V_\epsilon^\pm \oplus \mathcal{H}_r}(\pi) \neq 0\}$

$r(V)$: Witt index of V

if $\epsilon(V) = -\epsilon(V') \Rightarrow \dim V + \dim V' = 2n+2$

$\Leftrightarrow r(V) + r(V') = n$

$\Theta_{V,W^0}(\chi_W) \subset I_n(\frac{m-n}{2}, \chi_V)$

big theta

reducibility & compos series of $I_n(\frac{m-n}{2},\chi_V)$ are explained by $\Theta_{V,W^0}(\chi_W)$.

Herm sp's are classified by $\epsilon(V) = \epsilon_{E/F}\left((-1)^{\frac{m(m-1)}{2}} \det V\right) \in \{\pm 1\}$

$\dim V = \dim V' = m$

$\epsilon(V) = -\epsilon(V')$

$m \le n \Rightarrow \Theta_{V,W^0}(\chi_W) = \theta_{V,W^0}(\chi_W)$

$$\underbrace{\Theta_{V,W^0}(\chi_W) \oplus \Theta_{V',W^0}(\chi_W)}_{\text{max s.s. submodule}} \subset I_n\left(\tfrac{m-n}{2}, \chi_V\right)$$

When $m=n$, $I_n(0,\chi_V) = \Theta_{V,W^0}(\chi_W) \oplus \Theta_{V',W^0}(\chi_W)$

$m > n \Rightarrow$

$I_n(\frac{m-n}{2},\chi_V) = \Theta_{V,W^0}(\chi_W) + \Theta_{V',W^0}(\chi_W)$.

If $m > n$

(☆) $\exists V_0$ s.t. $\dim V_0 = 2n-m$ $\Rightarrow$ $I_n(\frac{m-n}{2},\chi_0) / \Theta_{V,W^0}(\chi_W) \simeq \Theta_{V_0,W^0}(\chi_W)$

$\epsilon(V_0) = -\epsilon(V)$

Rmk. $n=1 \Rightarrow G^0 \doteq SL_2(F)$, deduce compos series of p.s. of $SL_2(F)$

Seesaw pairs

$$\begin{tikzcd} H \arrow[dr, dash] & G' \\ H' \arrow[u, hook] \arrow[ur, dash] & G \arrow[u, hook] \end{tikzcd}$$

irr rep σ of H' $\quad$ irr rep π of G

(G,H), (G',H') are dual pairs $\Rightarrow \mathrm{Hom}_G(\Theta_{V',W'}(\sigma), \pi) \simeq \mathrm{Hom}_{H'}(\Theta_{W,V}(\pi), \sigma)$

proof. $\mathrm{Hom}_{G\times H'}(\omega_{V,W}, \pi\otimes\sigma) \simeq \mathrm{Hom}_{G\times H'}(\pi \boxtimes \Theta_{W,V}(\pi), \pi\otimes\sigma) \simeq$ RHS //

Lem ※. $\mathrm{Hom}_{G\times G}(\Theta_{V,W^0}(\chi_W), \pi \boxtimes (\pi^\vee \otimes \chi_V)) \neq 0 \Leftrightarrow \Theta_{V,W}(\pi) \neq 0$.

proof.

$$\begin{tikzcd} G^0 & H\times H \\ G\times G \arrow[u, hook] \arrow[ur, dash] & H^0 \arrow[u, hook] \arrow[ul, dash] \end{tikzcd}$$

$\pi \boxtimes (\pi^\vee \otimes \chi_V) \qquad \chi_W$

$\mathrm{Hom}_{G\times G}(\Theta_{V,W^0}(\chi_W), (\pi\boxtimes\pi^\vee) \otimes k\circ\chi_V)$

$\simeq \mathrm{Hom}_H(\Theta_{V,W}(\pi) \otimes \Theta_{V,-W}(\pi^\vee\otimes\chi_V), \chi_V)$

Kudla's choice of $\iota_{V,W} : U(V)\times U(W) \to Mp(W)$

$\Theta_{V,-W}(\pi^\vee \otimes \chi_V) \simeq \Theta_{V,W}(\pi)^{MVW} \otimes \chi_W \to \pi^{MVW} \otimes \chi_W \simeq \pi^\vee \otimes \chi_W$

if $\Theta_{V,W}(\pi) \neq 0$.

$\rho^{MVW} := \rho \circ$ (outer autom of H)

$\simeq \rho^\vee$ if ρ is irreducible //

$$0 \neq Z^*\left(\tfrac{m-n}{2},\chi_V\right) = \lim_{s\to\frac{m-n}{2}} \frac{Z(s,\chi_V)}{L(s+\frac{1}{2},\pi^\vee\times\chi_V)} \in \mathrm{Hom}_{G\times G}\left(I_n(s,\chi_V), (\pi\boxtimes\pi^\vee)\otimes\chi_V\circ\kappa\right)$$

$$Z^*\left(\tfrac{m-n}{2},\chi_V\right)\Big|_{\Theta_{V,W^0}(\chi_W)} \neq 0 \Rightarrow \Theta_{V,W}(\pi)\neq 0.$$

$\Leftarrow$

$\boxed{\text{Can be deduced from conservation relation } r^{+}_{\epsilon}(\pi)+r^{-}_{\epsilon}(\pi)=n}$

$\vdots$

$$=0 \Rightarrow Z^*\left(\tfrac{m-n}{2},\chi_V\right): I_n\left(\tfrac{m-n}{2},\chi_V\right)\Big/\Theta_{V,W^0(\chi_W)} \overset{(*)}{\simeq} \Theta_{V_0,W^0}(\chi_W) \to (\pi\boxtimes\pi^\vee)\otimes\chi_V\circ\kappa$$

$$\Rightarrow \Theta_{W,V_0}(\pi)\neq 0 \ \&\ r(V)+r(V_0)=n-1 \Rightarrow \Theta_{W,V}(\pi)=0.$$

proof of $\boxed{}$ Assume $\dim V\equiv\dim V'\equiv\epsilon \bmod 2$

$\epsilon(V)=-\epsilon(V')$

$\sigma := \Theta_{W,V}(\pi)\neq 0$

$\sigma' := \Theta_{W,V'}(\pi)\neq 0$.

$$\begin{array}{ccc} U(V\oplus(-V')) & & G\times G \\ \uparrow & \times & \uparrow \\ U(V)\times U(-V') & & G^\Delta \\ \sigma\boxtimes(\sigma'^\vee\otimes\chi_W) & & \chi_V \end{array}$$

$\Theta_{V,W}(\sigma)\to\pi$

$\Theta_{V',W}(\sigma')\to\pi$

$\Theta_{V',W}(\sigma')^{MVW}\to\pi^{MVW}\simeq\pi^\vee$

is

$\Theta_{-V',W}(\sigma'^\vee\otimes\chi_W)\otimes\chi_V^{-1}$

$$\mathrm{Hom}_{U(V)\times U(-V')}\left(\Theta_{W,V\oplus(-V')}(\chi_V), \sigma\boxtimes(\sigma'^\vee\otimes\chi_W)\right) \simeq \mathrm{Hom}_G\left(\Theta_{V,W}(\sigma)\otimes\Theta_{V',W}(\sigma')^{MVW}\otimes\chi_V,\chi_V\right)\neq 0.$$

$\Rightarrow \Theta_{W,V\oplus(-V')}(\chi_V)\neq 0$.

Lem 4

$\epsilon(V\oplus(-V'))=-1$ & $r_0^-(\chi_V)=n \Rightarrow \underline{\dim(V\oplus(-V'))\geq 2n+2} \Rightarrow r(V)+r(V')\geq n$

even dim, non split; known fact

i.e. $r^+_\epsilon(\pi)+r^-_\epsilon(\pi)\geq n$

Assume V & V' are 1st occurrence

$\dim V\geq\dim V'$

known fact $r^\pm_\epsilon(\pi)\leq n$

$\underline{n+1}\leq\dim V\leq 2n+2$

Take V_1 s.t. $V\simeq V_1\oplus\mathcal{H}_1$

$n-1\leq\dim V_1\leq 2n$

Take V_0 s.t. $\underbrace{\dim V_0}_{m_0}+\underbrace{\dim V_1}_{m_1}=2n$, $\epsilon(V_0)=-\epsilon(V_1)$

$$m_0\leq m_1 \Rightarrow I_n\left(\tfrac{m_1-n}{2},\chi_V\right)\Big/\Theta_{V_1,W^0}(\chi_W)\simeq\Theta_{V_0,W^0}(\chi_W) \text{ by } (*)$$

$$0\to\mathrm{Hom}_{G\times G}\left(\Theta_{V_1,W^0}(\chi_W),(\pi\boxtimes\pi^\vee)\otimes\chi_V\circ\kappa\right)=0 \Leftarrow \Theta_{W,V_1}(\pi)=0.$$

$$\to\mathrm{Hom}_{G\times G}\left(I_n\left(\tfrac{m_1-n}{2},\chi_V\right),\pi\boxtimes(\pi^\vee\otimes\chi_V)\right)\ni Z^*\left(\tfrac{m_1-n}{2},\chi_V\right)\neq 0.$$

$$\to\mathrm{Hom}_{G\times G}\left(\Theta_{V_0,W^0}(\chi_W),\pi\otimes(\pi^\vee\otimes\chi_V)\right)\neq 0 \Rightarrow \Theta_{\cancel{V_0,W}}^{W,V_0}(\pi)\neq 0$$

$\theta_{W,V}(\pi) \neq 0 \Rightarrow \theta_{W, V\oplus\mathcal{H}_r}(\pi) \neq 0$

$\left\{ \text{ If } r(V) \geq \dim W \Rightarrow \theta_{W,V}(\pi) \neq 0. \right.$

Assume $\theta_{W,V'}(\pi) = 0$ if $V = V' \oplus \mathcal{H}$

Then if $\theta_{W,V}(\pi) \neq 0 \Rightarrow \theta_{W,V}(\pi) \simeq \bigotimes_v{}' \theta_{W,V}(\pi_v)$ is cuspidal

When $\theta_{W,V}(\pi) \neq 0$?

global see saw

$$\begin{array}{ccc} H & & G' \\ \uparrow & \times & \\ H' & & G \end{array}$$

irr cusp rep of $H'(\mathbb{A})$ $\quad$ irr cusp rep of $G(\mathbb{A})$

$\sigma \in 6 \qquad \xi \in \pi$

$$\langle \theta_\phi(\xi), \sigma \rangle_{H'} = \langle \theta_\phi(\sigma), \xi \rangle_G$$

change of order of integration

$$\begin{array}{ccc} U(V) \times U(-V) & & G^0 \\ \uparrow & \times & \uparrow \\ U(V^\diamond) & & G \times G \end{array}$$

χ_W

$$\begin{aligned} \langle \theta_\phi(\xi), \theta_{\phi'}(\xi') \rangle_H &= \langle \theta_\phi(\xi)\, \overline{\theta_{\phi'}(\xi')}\chi_W, \chi_W \rangle_H \\ &= \langle \theta_{\phi\otimes\bar\phi'}(\chi_W), \xi \otimes \bar\xi' \rangle_{G\times G} \\ &= \langle E(f^{(s_0)}_{\phi\otimes\bar\phi'}), \xi \otimes \bar\xi' \rangle_{G\times G}. \\ &= Z(\xi \otimes \xi', f^{(s_0)}_{\phi\otimes\bar\phi'}) \\ &= \lim_{s\to s_0} \frac{L(\frac{1}{2}+s, \pi\times\chi_V)}{b(s,\chi_V)} \prod_v Z^*(\xi_v \otimes \xi'_v, f^{(s)}_{\phi_v\otimes\bar\phi'_v}) \end{aligned}$$

$\Theta_{V,W^\diamond}(\chi_W) \subset I_n(\frac{m-n}{2}, \chi_V)$:

$\{ f^{(s_0)}_\Phi \mid \Phi \in \omega_{V,W^\diamond} \} \qquad s_0 = \frac{m-n}{2}$

$\theta_\Phi(\chi_W) = E(f^{(s_0)}_\Phi)$

Siegel-Weil

Conclusion

$$\theta_{W,V}(\pi) \neq 0 \Leftrightarrow \lim_{s\to s_0} \frac{L(s+\frac{1}{2}, \check\pi\times\chi_V)}{b(s,\chi_V)} \neq 0.$$

~~$\exists\, \xi_v \in \pi_v^\vee$~~ $\qquad Z^* \big|_{\Theta_{V,W^\diamond}(\chi_{W,v})} \neq 0 \ \forall v.$

~~$\xi'_v \in \pi_v$ s.t. $Z^*(\xi_v$~~

$\Leftrightarrow$ ~~$\Phi_v \in \omega_{V,W}$~~ $\qquad$ } conservation relation

$$\lim_{s\to s_0} \frac{L(s+\frac{1}{2}, \pi^\vee\times\chi_V)}{b(s,\chi_V)} \neq 0 \quad \& \quad \Theta_{V,W}(\pi_v) \neq 0 \ \forall v$$

Thm Assume $\theta_{V',W}(\pi) = 0$ if $V = V' \oplus \mathcal{H}$

(1) When $m < n$,

$\theta_{W,V}(\pi) \neq 0 \Leftrightarrow L(s, \pi^\vee\times\chi_V)$ has a pole at $s = \frac{m-n+1}{2}$

$\qquad \forall v,\ \Theta_{W,V}(\pi_v) \neq 0.$

If $L(s, \pi^\vee\times\chi_V)$ has a pole at $s = \frac{m-n+1}{2} \Rightarrow \exists V,\ \dim V = m$ s.t. $\theta_{W,V}(\pi) \neq 0.$

(2) When $m \geq n$,

$\qquad$ hol & non zero.

$\theta_{W,V}(\pi) \neq 0 \Leftrightarrow L(\frac{m-n+1}{2}, \pi^\vee\times\chi_V) \neq 0$

$\qquad \forall v,\ \Theta_{W,V}(\pi_v) \neq 0.$

$\dim V' \le \dim V_0 = 2n+2-\dim V \Rightarrow r_\epsilon^+(\pi) + r_\epsilon^-(\pi) \le n$

$\uparrow$

first occurence

$m_0 \ge m$ similar

exchange the role of V_0 & V_1 in the exact seq //

Corollaries (1) (theta dichotomy)

If $\dim W = \dim V_+ = \dim V_-$ $\Rightarrow$ precisely one of $\Theta_{W,V^\pm}(\pi)$ is nonzero.

$\epsilon(V_\pm) = \pm 1$

$\uparrow$

(2) (ϵ-dichotomy)

determined by $\epsilon(\frac{1}{2}, \pi^\vee \times \chi_V^{\#}, \psi) \in \{\pm 1\}$

proof. $r(V_+) + r(V_-) = n-1$ + conservation $\Rightarrow$ (1)

Recall ~~Since~~ $I_n(0,\chi_v) = \Theta_{V^+,W^0}(\chi_W) \oplus \Theta_{V^-,W^0}(\chi_W)$

$\Theta_{W,V^\pm}(\pi) \ne 0 \Leftrightarrow Z^*(0,\chi_v)\big|_{\Theta_{V^\pm,W^0}(\chi_W)} \ne 0$

$Z^*(0,\chi_v) \circ M^*(0,\chi_v) = \underbrace{\epsilon}_{\pi(-1)\gamma(w)}(\frac{1}{2}, \pi^\vee \times \chi_V^{\#}, \psi)\, Z^*(0,\chi_v)$

$\pi(-1)\gamma(w)\,\epsilon(\frac{1}{2}, \pi^\vee \times \chi_v, \psi)$ = eigenvalue of $M^*(0,\chi_v)$ on $\Theta_{V^\pm,W^0}(\chi_W)$ //

Rmk. When π is s.c. & $m<n$

Prop (H-K-S) $L(s, \pi^\vee \times \chi_V)$ has a pole at $s = \frac{m-n+1}{2}$ $\Leftrightarrow$ $\exists V$, $\dim V = m$

$\Theta_{W,V}(\pi) \ne 0$.

Global theta corr

F : # field

$\pi \simeq \otimes_v' \pi_v$ irr cusp autom rep of $G(\mathbb{A})$

$Z(\xi \otimes \xi', f^{(s)}) = \dfrac{L(s+\frac{1}{2}, \pi \times \chi)}{b(s,\chi)} \prod_v Z^*(\xi_v \otimes \xi_v', f_v^{(s)})$

$\otimes_v \xi_v = \xi \in \pi$

$\xi' \in \pi^\vee$

$f^{(s)} = \otimes_v f_v^{(s)} \in I_n(s,\chi)$

$L(s, \pi \times \chi) = \epsilon(s, \pi \times \chi)\, L(1-s, \pi^\vee \times \chi^{-1})$

$L(s, \pi \times \chi)$ has a pole $\Rightarrow$ $\chi \circ \theta = \chi^{-1}$ ($\Leftrightarrow \chi = {}^\exists\chi_V$)

at most simple at $\frac{m-n+1}{2}$ for $m = 0,1,2,\dots,2n$, $m \ne n$, $m \equiv \epsilon \bmod 2$ $\chi|_{\mathbb{A}^\times} = \epsilon_{E/F}$

Q Is $\otimes_v' \Theta_{W,V}(\pi_v)$ autom?

$\mathbb{W} = V \otimes W$

$\omega_{V,W}$: Weil rep of $Mp(\mathbb{W})$

$\Theta : \omega_{V,W} \to \mathcal{A}(Sp(\mathbb{W}, F) \backslash Mp(\mathbb{W}))$

$\Theta_\phi(h, \xi) = \displaystyle\int_{G(F)\backslash G(\mathbb{A})} \overline{\xi(g)}\, \Theta(\phi, g, h)\, dg$ theta lift

$\Theta_{W,V}(\pi) = \langle \Theta_\phi(\xi) \mid \phi \in \omega_{V,W}, \xi \in \pi \rangle$

References.

Godement-Jacquet, LNM 260

Jacquet, "Principle L-ft of the linear gp", PSPUM 33-2

PS-Rallis, LNM 1254

Lapid-Rallis, "On the local factors of representations of classical gp's", Rallis volume

g.c.d thy & poles of std L,

Yamana, "L-ft & theta corr for classical gp's", Invent. 2014

Loc theta corr

Gan-Ichino, "Formal degree & loc theta corr", Invent. 2014. { Spliting / theta dichotomy for unitary gp's
H-K-S, JAMS 1996

Advanced one

" "The Gross-Prasad conjecture & loc theta corr", Invent. 2016

Siegel-Weil formula & Rallis inner prod.

Kudla-Rallis "A regularized SW formula", Annals 1994

Gan-Qiu-Takeda "~~On~~ The regularized SW formula & ~~non vanishing~~ Rallis inner prod formula", Invent 2014