From Coxeter Higher-Spin Theories

to

Strings and Tensor Models

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MV 1804.06520

Gelfond, MV 1805.11941, Didenko, Gelfond, Koribut, MV 1806....

AdS/CFT at 20 and Beyond

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Plan

- Introduction: HS theory versus Strings
- HS modules and difficulty of the naive extension of HS theory
- Framed oscillator algebra
- Nonlinear HS equations
- Perturbative expansion and locality
- Coxeter groups and Cherednik algebras
- Framed Cherednik systems
- Coxeter HS equations
- Relation with strings and tensor models
- Idempotent extension
- B_2–HS model and $\mathcal{N} = 4$ SUSY
- HS higgsing
- Conclusion

Challenge: Quantum Gravity and String Theory

Conjecture: trans-Planckian regime of exhibits high symmetries

D. Gross 1988, MV 1987...

Key idea of HS gauge theory: to understand what higher symmetries are possible

Important feature: $(A)dS$ background with $\Lambda \neq 0$ Fradkin, MV, 1987

HS theories: $\Lambda \neq 0$, $m = 0$, symmetric fields $s = 0, 1, 2, \dots \infty$

First Regge trajectory

String Theory: $\Lambda = 0$, $m \neq 0$ except for a few zero modes

Infinite set of Regge trajectories

What is a HS symmetry of a string-like extension of HS theory?

MV 2012, Gaberdiel and Gopakumar 2014-2018

String Theory as spontaneously broken HS theory?! $(s > 2, m > 0)$

Fronsdal Fields Versus Riemann Geometry

All $m=0$ HS fields are gauge fields Fronsdal 1978

$\varphi_{n_1\dots n_s}$ is a rank$-s$ symmetric tensor obeying $\varphi^{k}{}_{k}{}^{m}{}_{mn_5\dots n_s}=0$

Gauge transformation

$$\delta\varphi_{n_1\dots n_s}=\partial_{(n_1}\varepsilon_{n_2\dots n_s)}\,,\qquad \varepsilon^{m}{}_{mn_3\dots n_{s-1}}=0$$

Since HS symmetries do not commute with space-time symmetries

$$[T^n\,,T^{HS}]=T^{HS}\,,\qquad [T^{nm}\,,T^{HS}]=T^{HS}$$

Riemann geometry is not appropriate for HS theory

$$\delta_{HS}\varphi_{nm}\sim\varphi_{HS}$$

How (non)local is HS gauge theory?

Prokushkin, MV (1998), MV (2015),

Boulanger, Kessel, Skvortsov, Sleight, Taronna (2015,2016)...

Constructive progress and the homotopy formalism MV (2016,2017),

Gelfond, MV (2017), Didenko, Gelfond, Koributt, MV: to appear soon

Unfolded Dynamics

Cartan formalism: coordinate independence without metric.

Unfolded dynamics: multidimensional generalization of ODE

$$\frac{\partial}{\partial t} \to \mathsf{d}\,, \qquad q^i(t) \to W^\Omega(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W^\Omega_{n_1 \ldots n_p}(x)$$

$$\mathsf{d}\mathbf{W}^\Omega(\mathbf{x}) = \mathbf{G}^\Omega(\mathbf{W}(\mathbf{x}))\,, \qquad \mathsf{d} = dx^n \partial_n$$

$G^\Omega(W)$: **function of "supercoordinates"** W^Ω

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Phi_1 \ldots \Phi_n} W^{\Phi_1} \wedge \ldots \wedge W^{\Phi_n}$$

Covariant first-order differential equations

$d > 1$: **Compatibility conditions**

$$G^\Phi(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Phi} \equiv 0$$

- **General applicability**
- **Manifest (HS) gauge and diffeomorphism invariance**
- **Independence of ambient space-time:** geometry is in $G^\Omega(W)$

Vacuum and Fluctuations

h: a Lie algebra. $\omega = \omega^\alpha T_\alpha$: a h-valued 1-form h.

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^\alpha \wedge \omega^\beta [T_\alpha, T_\beta]$$

The unfolded equation with $W = \omega$ has the zero-curvature form

$$d\omega + \omega \wedge \omega = 0\,.$$

Background geometry in a coordinate independent way.

Minkowski or AdS_d space-time: h is Poincare or AdS algebra Let W^α contain p-forms $\mathcal{C}^i$ (e.g. 0-forms) and G^i be linear in ω and $\mathcal{C}$

$$G^i = -\omega^\alpha (T_\alpha)^i{}_j \wedge \mathcal{C}^j\,.$$

The compatibility condition: $\mathcal{C}^i$ form some h-module.

The unfolded equation

$$D_\omega \mathcal{C} = 0$$

$D_\omega \equiv d + \omega$: covariant derivative in the h-module

Central On-Shell Theorem

Infinite set of integer spins

$$\omega(y,\bar y\mid x)\,,\quad C(y,\bar y\mid x)\quad f(y,\bar y)=\sum_{n,m=0}^{\infty}\frac{1}{n!m!}f_{\alpha_1\ldots\alpha_n,\dot\alpha_1\ldots\dot\alpha_m}y^{\alpha_1}\ldots y^{\alpha_n}\bar y^{\dot\alpha_1}\ldots\bar y^{\dot\alpha_m}$$

The full unfolded system for free bosonic fields is **1989**

$$\star\qquad R_1(y,\bar y\mid x)=\frac{i}{4}\left(\eta\overline{H}^{\dot\alpha\dot\beta}\frac{\partial^2}{\partial\bar y^{\dot\alpha}\partial\bar y^{\dot\beta}}\,C(0,\bar y\mid x)+\bar\eta H^{\alpha\beta}\frac{\partial^2}{\partial y^{\alpha}\partial y^{\beta}}\,C(y,0\mid x)\right)$$

$$\star\star\qquad \tilde D_0 C(y,\bar y\mid x)=0$$

Vacuum: $sp(4)\sim o(3,2)$

$$\mathbf{R}_{\alpha\beta}:=\mathbf{d}\omega_{\alpha\beta}+\omega_{\alpha\gamma}\omega_{\beta}{}^{\gamma}-\mathbf{H}_{\alpha\beta}=0\,,\qquad \mathbf{R}_{\alpha\dot\beta}:=\mathbf{d}+\omega_{\alpha\gamma}\mathbf{h}^{\gamma}{}_{\dot\beta}+\overline{\omega}_{\dot\beta\dot\delta}\mathbf{h}_{\alpha}{}^{\dot\delta}=0$$

$$\mathbf{H}^{\alpha\beta}:=\mathbf{h}^{\alpha\dot\alpha}\wedge\mathbf{h}^{\beta}{}_{\dot\alpha}\,,\qquad \overline{\mathbf{H}}^{\dot\alpha\dot\beta}:=\mathbf{h}^{\alpha\dot\alpha}\wedge\mathbf{h}_{\alpha}{}^{\dot\beta}$$

$$R_1(y,\bar y\mid x)=D_0^{ad}\omega(y,\bar y\mid x)\qquad D_0^{ad}=D^L-h^{\alpha\dot\beta}\left(y_\alpha\frac{\partial}{\partial\bar y^{\dot\beta}}+\frac{\partial}{\partial y^{\alpha}}\bar y_{\dot\beta}\right)$$

$$\tilde D_0=D^L+h^{\alpha\dot\beta}\left(y_\alpha\bar y_{\dot\beta}+\frac{\partial^2}{\partial \mathbf{y}^{\alpha}\partial\bar{\mathbf{y}}^{\dot\beta}}\right)\qquad D^L=\mathrm{d}_x-\left(\omega^{\alpha\beta}y_\alpha\frac{\partial}{\partial y^{\beta}}+\bar\omega^{\dot\alpha\dot\beta}\bar y_{\dot\alpha}\frac{\partial}{\partial\bar y^{\dot\beta}}\right)$$

$\star\star$ **implies that higher-order terms in y and $\bar y$ describe higher-derivative descendants of the primary HS fields**

HS Algebra and Modules

Free field analysis: realization of the HS algebra hs_1 as Weyl algebra

$$[y_\alpha\,, y_\beta]_* = 2i\varepsilon_{\alpha\beta}\,, \qquad [\bar{y}_{\dot\alpha}\,, \bar{y}_{\dot\beta}]_* = 2i\varepsilon_{\dot\alpha\dot\beta}$$

Fradkin, MV 1987

AdS_4 algebra $sp(4) \sim o(3,2)$

Naive way to extend the spectrum of fields $y_\alpha \to y^n_\alpha$

does not lead to physically acceptable HS theories

The Fock hs_1-module F_1 describes free boundary conformal fields

$$D|0\rangle = h_1|0\rangle$$

Lowest weight representations of the naively extended algebras hs_p

built from p copies of oscillators have too high weights

$$h_p = ph_1$$

$F_1 \otimes F_1$ = massless fields in the bulk   Flato, Fronsdal (1978)

For $p > 1$ the lowest weights in $F_p \otimes F_p$ have no room for gravity

(massless spin-two)

Framed Oscillator Algebras

The problem is resolved in the framed oscillator algebras replacing usual oscillator algebra

$$[y^n_\alpha\,, y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta}I\,,$$

where I is the unit element by

$$[y^n_\alpha\,, y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta}I_n$$

"Units" I_n are assigned to each specie of the oscillators forming a set of commutative central idempotents

$$I_i I_j = I_j I_i\,, \qquad I_i I_i = I_i$$

This allows us to consider Fock modules F_i obeying

$$I_j F_i = \delta_{ij} F_i$$

equivalent to those of the single-oscillator case

Fields of the Nonlinear System

Nonlinear HS equations demand doubling of spinors and Klein operator

$$\omega(Y|x) \longrightarrow W(Z;Y;k|x)\,, \qquad C(Y|x) \longrightarrow B(Z;Y;k|x)$$

Some of the nonlinear HS equations determine the dependence on Z_A in terms of "initial data"

$W(0;Y;k|x) = \omega^{dyn}(Y|x) + \omega^{top}(Y|x)k$

$B(0;Y;k|x) = C^{dyn}(Y|x)k + C^{top}(Y|x)$

$S(Z;Y;k|x) = dZ^A S_A(Z;Y;k|x)$ **is a connection along** Z^A

Topological fields: finite # d.o.f.: tensors

Klein operator k generates chirality automorphisms

$$kf(A) = f(\tilde{A})k\,, \quad A = (a_\alpha\,, \bar{a}_{\dot\alpha}) : \quad \tilde{A} = (-a_\alpha\,, \bar{a}_{\dot\alpha})$$

$$P(Y) = P^{\alpha\dot\alpha} y_\alpha \bar{y}_{\dot\alpha} \quad \longrightarrow \quad \tilde{P}(Y) = -P(Y)\,, \qquad \tilde{M}(Y) = M(Y)$$

Nonlinear HS Equations

HS star product

$$(f*g)(Z,Y) = \int dSdT \exp iS_A T^A f(Z+S, Y+S) g(Z-T, Y+T)$$

$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2iC_{AB}\,,$ $\qquad Z-Y \, : \, Z+Y$ **normal ordering**

Inner Klein operators:

$\kappa = \exp iz_\alpha y^\alpha\,, \qquad \bar{\kappa} = \exp i\bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}\,, \qquad \kappa * f = \tilde{f} * \kappa\,, \qquad \kappa * \kappa = 1$

$$\begin{cases} \mathrm{d}_x W + W * W = 0 \\ \mathrm{d}_x B + W * B - B * W = 0 \\ \mathrm{d}_x S + W * S + S * W = 0 \\ \mathbf{S * B - B * S = 0} \\ \mathbf{S * S = i(dZ^A dZ_A} + \eta \mathbf{dz}^\alpha \mathbf{dz}_\alpha \mathbf{B * k} * \kappa + \bar{\eta} \mathbf{d\bar{z}}^{\dot{\alpha}} \mathbf{d\bar{z}}_{\dot{\alpha}} \mathbf{B * k} * \bar{\kappa}) \end{cases}$$

1992

Dynamical content is located in the x-independent twistor sector

The non-zero curvature has the form of Z_2-Cherednik algebra

Perturbative Analysis

Vacuum solution $B_0 = 0\,, \qquad S_0 = Z_A dZ^A\,, \qquad W_0 = \frac{1}{2}\Omega^{AB}(x) Y_A Y_B$

$$\mathsf{d}W_0 + W_0 \star W_0 = 0 \quad \Longrightarrow \quad \Omega^{AB}(x) \text{ **describes** } AdS_4$$

S_0 **induces** d_Z**-complex:** $[S_0\,, f]_\star = -2i\mathsf{d}_Z f\,, \qquad \mathsf{d}_Z := dZ^A \frac{\partial}{\partial Z^A}$

Nontrivial space-time equations on $\omega(Y|x)$ **and** $C(Y|x)$ **in** d_Z **-cohomology**

via reconstruction of Z Variables

order–n d_Z**-dependent equations**

$$\mathsf{d}_Z f_n(Z;Y|dZ) = g_n(Z;Y|dZ)\,, \qquad \mathsf{d}_Z g_n(Z;Y|dZ) = 0$$

Solution

$$f_n(Z;Y|dZ) = \mathsf{d}^*_{\mathbf{Z}} V_n(Z;Y|dZ) + h_n(Y) + \mathsf{d}_Z \epsilon_n(Z;Y|dZ)$$

Shifted homotopy

Conventional homotopy $\partial = Z^A \frac{\partial}{\partial \mathrm{d} Z^A}$ $\rightarrow$ **resolution**

$$\mathrm{d}^*_{\mathbf{Z}} V(Z;Y|dZ) = Z^A \frac{\partial}{\partial dZ^A} \int_0^1 \frac{dt}{t} V(tZ;Y|tdZ)$$

lead to nonlocalities beyond the free-field level

Giombi, Yin 2009; Boulanger, Kessel, Skvortsov, Taronna 2015; MV 2017

Shifted homotopy

$$Z_A \rightarrow Z_A + Q_A \,, \qquad \partial \rightarrow \partial = (Z_A + Q_A) \frac{\partial}{\partial Z_A} \,, \qquad \frac{\partial}{\partial Z_A}(Q_B) = 0$$

$Q_A C(Y) = \frac{\partial}{\partial Y^A} C(Y)$ **is admissible**

Homotopy operators ∂ can be independent for different sectors

$$Q_A = c_0 Y_A + \sum_j c_j \partial_{jA} \,,$$

Appropriate choice of Q_A leads to local results in the lowest order and decreases non-locality in the higher orders

Gelfond, MV 1805.11941

Didenko, Gelfond, Korybut, MV 1806....

Coxeter Groups and Cherednik Algebras

A rank-p Coxeter group $\mathcal{C}$ is generated by reflections with respect to a system of root vectors $\{v_a\}$ in a p-dimensional Euclidean vector space V. An elementary reflection associated with the root vector v_a

$$R_{v_a} x^i = x^i - 2 v_a^i \frac{(v_a, x)}{(v_a, v_a)}\,, \qquad R_{v_a}^2 = I$$

Cherednik deformation of the semidirect product of the oscillator algebra with the group algebra of $\mathcal{C}$ is

$$[q_\alpha^n\,, q_\beta^m] = -i\epsilon_{\alpha\beta} \left(2\delta^{nm} + \sum_{v\in\mathcal{R}} \nu(v) \frac{v^n v^m}{(v,v)} k_v \right)\,, \qquad k_v q_\alpha^n = R_v{}^n{}_m q_\alpha^m k_v$$

q_α^n $(\alpha = 1, 2,\ n = 1; \ldots, p)$

Coupling constants $\nu(v)$ are invariants of $\mathcal{C}$ being constant on the conjugacy classes of root vectors under the action of $\mathcal{C}$.

Double commutator of q_α^n respects Jacobi identities.

B_p–Coxeter System

Important case of the Coxeter root system is B_p with the roots

$$R_1 = \{\pm e^n \qquad 1 \leq n \leq p\}\,, \qquad R_2 = \{\pm e^n \pm e^m \quad 1 \leq n < m \leq p\}\,.$$

Apart from permutations B_p contains reflections of basis axes $v^n_\pm = e^n$.

R_1 and R_2 form two conjugacy classes of B_p.

The Coxeter group of $3d$ HS theory is $A_1 \sim B_1$.

B_2 underlies the string-like HS models.

The fact of fundamental importance for HS theories is that for any Coxeter root system the generators

$$t_{\alpha\beta} := \frac{i}{4} \sum_{n=1}^{p} \{q^n_\alpha\,, q^n_\beta\}$$

obey the $sp(2)$ commutation relations properly rotating all indices α

$$[t_{\alpha\beta}\,, q^n_\gamma] = \epsilon_{\beta\gamma} q^n_\alpha + \epsilon_{\alpha\gamma} q^n_\beta$$

Framed Cherednik Systems

A_{p-1} **system.** **In addition to** $q_{\alpha n}$ **and** k_{nm}, $n,m=1,\dots p$ **introduce** I_n

$$I_nI_m = I_mI_n\,,\qquad I_nI_n = I_n\,,\qquad I_nq_{\alpha n} = q_{\alpha n}I_n = q_{\alpha n}\,,\qquad I_nq_{\alpha m} = q_{\alpha m}I_n\,.$$

In presence of I_n **the deformed oscillator relations respecting Jacobi**

$$[q_{\alpha\, n}\,,q_{\beta\, m}] = -i\epsilon_{\alpha\beta}\Big(\delta_{nm}\Big(2I_n+\nu\sum_{l=1}^{p}\widehat{k}_{ln}\Big)-\nu\widehat{k}_{nm}\Big)\,,\qquad \widehat{k}_{nm}=I_nI_mk_{nm}\,.$$

$\widehat{k}_{nm}$ **obey all relations of** S_p **except for involutivity replaced by**

$$\widehat{k}_{nm}\widehat{k}_{nm} = I_nI_m\,.$$

$$I_l\widehat{k}_{nm} = \widehat{k}_{nm}I_l\quad \forall\; l,n,m\,,\qquad I_n\widehat{k}_{nm} = I_m\widehat{k}_{nm} = \widehat{k}_{nm}\,.$$

General Framed Cherednik Algebra

$$[q^n_\alpha\,,q^m_\beta] = -i\epsilon_{\alpha\beta}\left(2\delta^{nm}I_n+\sum_{v\in\mathcal{R}}\nu(v)\frac{v^nv^m}{(v,v)}\widehat{k}_v\right)\,,\qquad \widehat{k}_v := k_v\prod I_{i_1(v)}\dots I_{i_k(v)}$$

Framed Cherednik algebra still possesses inner $sp(2)$ **automorphisms**

$$t_{\alpha\beta} := \frac{i}{4}\sum_{n=1}^{p}\{q^n_\alpha\,,q^n_\beta\}I_n$$

Framed Star Product

x-dependent fields W, S and B depend on p sets of variables Y^n_A, Z^n_A $(A = 1, \dots M)$, I_n, anticommuting differentials dZ^A_n $(n = 1, \dots p)$ and Klein-like operators $\hat{k}_v$ associated with all roots of $\mathcal{C}$. Coxeter HS field equations are formulated in terms of the star product

$$(f{*}g)(Z;Y;I) = \frac{1}{(2\pi)^{pM}} \int d^{pM}S\, d^{pM}T \exp\,[iS^A_n T^B_m \delta^{nm} C_{AB}]\, f(Z_i{+}I_iS_i;Y_i{+}I_iS_i;I)$$

$$I_n * Y^n_A = Y^n_A * I_n = Y^n_A\,, \qquad I_n * Z^n_A = Z^n_A * I_n = Z^n_A\,, \qquad I_n * I_n = I_n$$

Implying

$$[Y^n_A, Y^m_B]_* = -[Z^n_A, Z^m_B]_* = 2iC_{AB}\delta^{nm} I_n\,, \qquad [Y^n_A, Z^m_B]_* = 0\,.$$

This star product admits inner Coxeter-Klein operators

$$\exp i \frac{v^n v^m Z_{\alpha n} Y^{\alpha}{}_m}{(v,v)}$$

Coxeter HS Equations

Unfolded equations for $\mathcal{C}$-HS theories remain the same except for

$$iS * S = dZ^{An} dZ_{An} + \sum_i \sum_{v \in \mathcal{R}_i} F_{i*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v$$

κ_v are generators of $\mathcal{C}$ acting trivially on all elements except for $dZ_{\alpha n}$

$$\kappa_v * dZ_\alpha^n = R_v{}^n{}_m dZ_\alpha^m * \kappa_v$$

$F_{i*}(B)$ is any star-product function of the zero-form B on the conjugacy classes R_i of $\mathcal{C}$. In the important case of the Coxeter group B_p

$$iS * S = dZ_{An} dZ^{An} + \sum_{v \in \mathcal{R}_1} F_{1*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v + \sum_{v \in \mathcal{R}_2} F_{2*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v$$

with arbitrary $F_{1*}(B)$ and $F_{2*}(B)$ responsible for the HS and stringy/tensorial features, respectively

$F_{2*}(B) \neq 0$ for $p \geq 2$.

The framed construction leads to a proper massless spectrum.

Color and Multi-Particle Extensions

W, S and B are allowed to be valued in any associative algebra A.

To make contact with the tensorial boundary theory $A = (Mat_N)^p$ with elements represented by $a^{u_1 \ldots u_p}{}_{v_1 \ldots v_p}$, $u_i, v_i = 1 \ldots N$.

p is the tensor degree of the boundary model

Multi-particle extensions are associated with the semi-simple Coxeter groups. The simplest option with $\mathcal{C} = B_p^{\mathcal{N}}$ is the product of $\mathcal{N}$ of B_p systems

$$B_p^{\mathcal{N}} := \underbrace{B_p \times B_p \times \ldots}_{\mathcal{N}} .$$

The limit $\mathcal{N} \to \infty$ along with the graded symmetrization of the product factors expressing the spin-statistics gives the (graded symmetric) multi-particle algebra $M(h(\mathcal{C}))$ of the HS algebra $h(\mathcal{C})$

$M(h(\mathcal{C})) = U(h(\mathcal{C}))$: Hopf algebra.

Klein Operators and Single-Trace Operators

Enlargement of the field spectra of the rank-$p>1$ Coxeter HS models:

$C(Y^n_\alpha;k_v)$ **depend on p copies of oscillators Y^n_α and Klein operators k_v**

Qualitative agreement with enlargement of the boundary operators in tensorial boundary models.

Klein operators of Coxeter reflections permute master field arguments

At $p=2$ the star product of two master fields $C(Y_1,Y_2|x)k_{12}$ gives

$$(C(Y_1,Y_2|x)k_{12})*(C(Y_1,Y_2|x)k_{12}) = C(Y_1,Y_2|x)*C(Y_2,Y_1|x)\,.$$

$p=2$ **system: strings of fields with repeatedly permuted arguments**

$$C^n_{string} := \underbrace{C(Y_1,Y_2|x)*C(Y_2,Y_1|x)*C(Y_1,Y_2|x)\ldots}_{n}\,.$$

are analogous of the single-trace operators in AdS/CFT.

$C(Y_1,Y_2|x)$ **and** $C(Y_1,Y_2|x)*C(Y_2,Y_1|x)$: **single-trace-like**

$C(Y_1,Y_2|x)*C(Y_1,Y_2|x)$: **double-trace-like.**

From Coxeter HS Theory to Strings and Tensor Models

The spectrum of the B_2 HS model is analogous to that of String Theory with the infinite set of Regge trajectories.

B_2- HS theory has parallels with the stringy Gaberdiel-Gopakumar HS models: dependence on $Y^A_{1,2}$ is like having two HS symmetry algebras

B_p–HS models with $p \geq 2$ have two coupling constants.

F_{1*} is analogous to that of the B_1-HS theory.

F_{2*} first appears in the rank-two stringy model and, containing the Klein operators that permute different Y-variables, generates single-trace-like strings of operators and their tensor generalizations.

To establish relation with usual string theory in flat space the limit $F_{2*}/F_{1*} \to \infty$ is most interesting.

Idempotent Extension

Let A be an associative algebra with the star product and a set of idempotents

$$\pi_i * \pi_i = \pi_i \,, \qquad \pi_i \in A \,.$$

$$a_i{}^j \in A_i{}^j : \quad a_i{}^j = \pi_i * a * \pi_j \,, \quad a \in A \,.$$

The matrix-like composition law in A_π

$$(a * b)_i{}^j = \sum_k a_i{}^k * b_k{}^j$$

A is the algebra of functions of dx, dZ, Z, Y, k_v, x

The set of idempotents π_i has to be $\mathcal{C}$-invariant

The idempotent-extended $\mathcal{C}$-HS equations have the same form with the replacement of $A \to A_{\{\pi\}}$.

Idempotent extensions of the Coxeter HS systems describe lower-dimensional brane-like objects.

Vector-Like and Supersymmetric Models

Fock idempotent in the B_1 $4d$ HS theory

$$\pi_i^{star} = 4 I_i \exp y_{i\alpha} \bar{y}_i^{\alpha}$$

${A_0}^i$**-module describes $3d$ conformal fields= $4d$ singletons:**

Idempotent realization of Klebanov-Polyakov AdS_4/CFT_3

vector model HS holography checked by Giombi and Yin in 2009

B_2 HS model

$4d$ **conformal massless fields are valued in the Fock module π** 2002

$$a_\alpha * \pi = 0\,, \qquad \bar{b}^{\dot\beta} * \pi = 0\,, \qquad \phi_i * \pi = 0\,, \qquad \pi * \bar{a}^{\dot\alpha} = 0\,, \ldots$$

$$[a_\alpha, b^\beta]_* = \delta_\alpha^\beta\,, \qquad [\bar{a}_{\dot\gamma}, \bar{b}^{\dot\beta}]_* = \delta_{\dot\gamma}^{\dot\beta}\,, \qquad \{\phi_i, \bar{\phi}^j\}_* = \delta_i{}^j\,,$$

$i, j = 1, \ldots N$. **Bilinears:** $su(2,2;N)$. **Clifford oscillators: color** $Mat_{2^{2N}}$.

The system is consistent at $N \geq 4$ when $\#_B \leq \#_F$.

$N = 4$ **SYM: the only $N = 4$ massless system with spins $s \leq 1$.**

Higher-Spin Higgsing

To make connection with Strings the most fundamental question is breaking of HS symmetries

Spontaneous breaking of HS symmetries resulting in the massive HS fields is only possible in string-like models with the infinite number of Regge trajectories like B_2 multi-particle theory

The simplest option is to give VEV to a topological field $B(Y_1; Y_2)$

$$B_0 = Y_{iA} \cdot Y_j^A(\alpha k^{ij} + \beta\sigma_1^{ij} + \gamma\delta^{ij})$$

that preserves the AdS symmetry but breaks down the HS one.

Spontaneous symmetry breaking: mixing between the massless rank-one particle module and rank-two current module.

Conclusion

Coxeter HS theories

main principle: formal consistency & massless fields in the spectrum

Tensor-like models are natural duals of the rank-p Coxeter HS models

B_2–HS model is conjectured to be string-like

$\mathcal{N} = 4$ SYM is argued to be a natural dual of the B_2–HS model

B_p–Coxeter HS theories have two coupling constants and are formulated in AdS: the stringy B_2-HS models are different from the genuine String Theory in flat space

Multi-particle states of a lower-dimensional model = elementary states in a higher-dimensional (particularly, $10d$ model)

The original $3d$ and $4d$ spinorial theories: branes in the $10d$ theory with the $3d$ HS model as a brick from which the others are composed.

Vector Coxeter HS models in any d can also be introduced