UNITARITY AND BOUNDS IN

CONFORMAL FIELD THEORIES

Based on

1707.03007: with Subham Dutta Chowdhury and Shiroman Prakash

and

1707.07166 and on going work with Surbhi Khetrapal, S. Prem Kumar

- Consider $d = 2$ CFT.

Using the Virasoro algebra and positivity of the norm.
Conformal dimension of any primary

$$\Delta \geq 0.$$

The central charge

$$c \geq 0.$$

For the $\mathcal{N} = 2$ super conformal algebra which admits a $U(1)$ current, we can conclude conclude any primary of weight Δ with charge q

$$\Delta \geq |q|$$

- Note that the constraints are of two types.

Constraints on the allowed states in the theory

$$\Delta \geq 0, \qquad \Delta \geq |j|.$$

Constraints on the theory itself

$$c \geq 0.$$

First discuss two constraints.

- Constraints in the space of all $d = 3$ conformal field theories.

- Constraint on states carrying spin-3 charge in large c CFT in $d = 2$ with $\mathcal{W}_3$ symmetry.

These constraints will be arrived by two seemingly different physical criteria. but we will see that they are ultimately related to the bounds due to unitarity.

- The chaos bound is also an example of such constraints.

We discuss its violation when states in 2d CFT's do not satisfy the unitarity bound, $\Delta > 0$

CONSTRAINTS ON $d = 3$ CFT'S

• Conformal field theories admit a stress tensor.

The three point function of the stress tensor in CFT's is constrained by conformal invariance. eg. in $d = 4$

$$\langle TTT \rangle = n_s^T \langle TTT \rangle_{\rm freeboson} + n_f^T \langle TTT \rangle_{\rm freefermion} + n_v^T \langle TTT \rangle_{\rm freevector}$$

$\langle TTT \rangle_{\rm freeboson}$ is the correlator obtained by performing Wick contraction on the stress tensor of a massless free scalar theory in $d = 4$. Similar definition for the other correlators. Osborn, Petkou (1993).

n_s, n_f, n_v need not be integers are theory dependent parameters.

Hofman and Maldacena (2008) devised a thought experiment:
The conformal collider experiment
to obtain constraints on theory dependent parameters.

Such constraints allow one to find the space of allowed conformal field theories.

- Similar constraints can be also placed on the correlator

$$\langle jTj\rangle = n^j_s\langle TTT\rangle_{\rm freeboson} + n^j_f\langle TTT\rangle_{\rm freefermion} + n^j_v\langle TTT\rangle_{\rm freevector}$$

• Such constraints have been explored both in field theory and in holographic theories for conformal field theories in $d \geq 4$.

• We will explore it for conformal field theories in $d = 3$.

What is special about $d = 3$?

• The 3 point functions admit a parity odd structure.

$$\langle TTT\rangle = n_s^T\langle TTT\rangle_{\rm freeboson} + n_f^T\langle TTT\rangle_{\rm freefermion} + p_T\langle TTT\rangle_{\rm parityodd}$$

$$\langle jTj\rangle = n_s^j\langle TTT\rangle_{\rm freeboson} + n_f^j\langle TTT\rangle_{\rm freefermion} + p_T^j\langle TTT\rangle_{\rm parityodd}$$

Giombi, Prakash, Yin (2011).

• Once one normalizes the three point function using the two point function, we will have 2 parameters.

The conformal collider constraints restrict the theories in this parameter space.

- The conformal collider:

One considers a localised excitation at say the origin with energy E.
The excitation propagates outward in a spherical wave.
Keep a detector in the direction $\hat{n}$, say y direction.
Measure the integrated energy over time. eg. $d = 4$

$$\hat{E}_{\hat{n}} = \lim_{r\to 0} r^2 \int_{-\infty}^{\infty} dt \eta^i T^t_i(t, r\hat{n})$$

$$\langle E_{\hat{n}} \rangle = \frac{\langle 0|O^\dagger E_{\hat{n}} O|0\rangle}{\langle 0|\langle O^\dagger O|0\rangle}$$

• O creates the state we are interested in.

$O \sim T_{\mu\nu}$ or $O \sim j$.

• The requirement that the integrated energy measured by the detector is positive results in the conformal collider bounds.

• Thus choosing O to be T or j results in constraints on the parameters of the three points functions $\langle TTT \rangle$ or $\langle jTj \rangle$.

• It can be shown that the energy flux measurement operator in $d = 3$ is equivalent to

$$\hat{E} = \lim_{x^+ \to \infty} \frac{x^+}{2} \int_{-\infty}^{\infty} \frac{dx^-}{2} T_{--}$$

$x^{\pm} = t \pm y$.

Therefore the condition on the positivity of the integrated energy flux is called the average null energy condition (ANC).

The excited states are defined by

$$O_E|0\rangle = \int dtdxdye^{-iEt}O(t,x,y)|0\rangle$$

We choose

$$O(\epsilon,T) = \epsilon_{ij}T^{ij} = \epsilon\cdot T \qquad O(\epsilon,j) = \epsilon_i j^i = \epsilon\cdot j$$

By conservation laws the polarizations are purely spatial.

In $d=4$, there are 3 independent polarizations for the stress tensor and 2 independent polarization for the current.

In $d=3$, there are 2 independent polarization for the stress tensor and 2 for the current.

Let us consider a linear combination of polarizations

$$\sum_a \alpha^a \epsilon^{(a)}$$

Then the expectation value of the energy flux operator looks like

$$\sum_{a,b} \alpha^a \alpha^b \langle \epsilon^{(a)} \cdot O | \hat{E} | \epsilon^{(b)} \cdot O \rangle = \sum_{a,b} \alpha^a \alpha^b M_{ab}$$

$$M_{ab} = \langle \epsilon^{(a)} \cdot O | \hat{E} | \epsilon^{(b)} \cdot O \rangle$$

Thus the requirement that the ANC is satisfied turns into a condition that the eigen values of this matrix is positive.

Let us recall the conditions obtained in $d = 4$ CFT's for the $\langle TTT \rangle$ correlator.

Since there are 3 independent polarizations, the matrix is a 3×3.

There is a simple choice of polarization for which the matrix is diagonal.

Define

$$t_2 = \frac{15(-4n_v + n_f)}{n_s + 12n_v + 3n_f}, \quad t_4 = \frac{15(n_s + 2n_v - 2n_f)}{2(n_s + 12n_v + 3n_f)}$$

.

Then the condition that the three diagonal elements are positive are

$$1-\frac{t_2}{3}-2\frac{t_4}{15}\geq 0, \qquad : \mathrm{I}$$
$$2(1-\frac{t_2}{3}-2\frac{t_4}{15})+t_2\geq 0, \qquad : \mathrm{II}$$
$$\frac{3}{2}(1-\frac{t_2}{3}-2\frac{t_4}{15})+t_2+t_4\geq 0 \qquad : \mathrm{III}$$

There are 2 parameters t_2, t_4. The region satisfied by the inequalities is a triangle.

All $d=4$ CFT's which satisfy the ANC lie in the triangle.

In fact theories which admit an Einstein gravity holgraphic dual lie at the origin.

• For $d = 3$ we have 2 independent polarisations for both charge and stress tensor excitations.

We have a 2×2 matrix for both the charge and stress tensor excitations.

- The energy matrix for stress tensor excitations.

$\epsilon_{xy} = 1; \;\; \epsilon'_{xx} = 1, \epsilon'_{yy} = -1$

$$\hat{E}(T) = \begin{pmatrix} \langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle \\ \langle 0|\mathcal{O}_E^\dagger(\epsilon'; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon'; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle \end{pmatrix},$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon; T)|0\rangle = \frac{1}{\langle \mathcal{O}_E(\epsilon; T)|\mathcal{O}_E(\epsilon, T)\rangle} \times$$

$$\int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot T(x) T_{--}(x_1) \epsilon \cdot T(0)\rangle,$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; T)\mathcal{E}\mathcal{O}_E(\epsilon'; T)|0\rangle =$$

$$(\langle \mathcal{O}_E(\epsilon'; T)|\mathcal{O}_E(\epsilon,' T)\rangle \langle \mathcal{O}_E(\epsilon; T)|\mathcal{O}_E(\epsilon, T)\rangle)^{-\frac{1}{2}}$$

$$\times \int d^3x e^{iEt} \lim_{x_1^+ \to \infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot T(x) T_{--}(x_1) \epsilon' \cdot T(0)\rangle.$$

- The energy matrix for charge excitations $\epsilon^x = 1, \epsilon'^y = 1$.

$$\hat{E}(j) = \begin{pmatrix} \langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle \\ \langle 0|\mathcal{O}_E^\dagger(\epsilon'; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle & \langle 0|\mathcal{O}_E^\dagger(\epsilon'; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle \end{pmatrix},$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon; j)|0\rangle = \frac{1}{\langle \mathcal{O}_E(\epsilon; j)|\mathcal{O}_E(\epsilon, j)\rangle} \times$$
$$\int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot j(x) T_{--}(x_1) \epsilon \cdot T(0)\rangle,$$

$$\langle 0|\mathcal{O}_E^\dagger(\epsilon; j)\mathcal{E}\mathcal{O}_E(\epsilon'; j)|0\rangle =$$
$$(\langle \mathcal{O}_E(\epsilon'; j)|\mathcal{O}_E(\epsilon,' j)\rangle\langle \mathcal{O}_E(\epsilon; j)|\mathcal{O}_E(\epsilon, j)\rangle)^{-\frac{1}{2}}$$
$$\times \int d^3x e^{iEt} \lim_{x_1^+\to\infty} \frac{x_1^+}{4} \int dx_1^- \langle \epsilon \cdot j(x) T_{--}(x_1) \epsilon' \cdot j(0)\rangle.$$

Evaluation of the energy matrix for charge excitations

$$\langle j(x)T(x_1)j(0)\rangle = \frac{\epsilon_2^{\sigma} I_{\sigma}^{\alpha}(x-x_1)\epsilon_3^{\rho} I_{\rho}^{\beta}(-x_1)\epsilon_1^{\mu\nu} t_{\mu\nu\alpha\beta}(X)}{|x_1-x|^3|x_1|^3|x|} + p_j \frac{Q_1^2 S_1 + 2P_2^2 S_3 + 2P_3^2 S_2}{|x_1-x||x||-x_1|},$$

$$
\begin{aligned}
t_{\mu\nu\alpha\beta}(X) &= \left(-\frac{2c}{3}+2e\right)h^1_{\mu\nu}(\hat{X})\eta_{\alpha\beta}+(3e)h^1_{\mu\nu}(\hat{X})h^1_{\alpha\beta} \\
&\quad +ch^2_{\mu\nu\alpha\beta}(\hat{X})+eh^3_{\mu\nu\alpha\beta}, \\
Q_1^2 &= \epsilon_1^\mu\epsilon_1^\nu\left(\frac{x_{1\mu}}{x_1^2}-\frac{x_{1\mu}-x_\mu}{(x_1-x)^2}\right)\left(\frac{x_{1\nu}}{x_1^2}-\frac{x_{1\nu}-x_\nu}{(x_1-x)^2}\right), \\
P_2^2 &= -\frac{\epsilon_1^\mu\epsilon_1^\nu I_{\mu\nu}(x_1)}{2x_1^2}, \\
P_3^2 &= -\frac{\epsilon_1^\mu\epsilon_2^\nu I_{\mu\nu}(x_1-x)}{2(x_1-x)^2},
\end{aligned}
$$

$$
\begin{aligned}
S_1 &= \frac{1}{4|x_1 - x||x^3| - x_1|} \left(\varepsilon^{\mu\nu}{}_{\rho} x_\mu (x_1 - x)_\nu \epsilon_2^\rho \epsilon_3^\alpha x_\alpha \right. \\
&\left. - \frac{\varepsilon^{\mu}{}_{\nu\rho}}{2} \left(|x_1 - x|^2 x_\mu + |x|^2 (x_1 - x)_\mu \right) \epsilon_2^\nu \epsilon_3^\rho \right), \\
S_2 &= \frac{1}{4|x_1 - x||x|| - x_1|^3} \left(\varepsilon^{\mu\nu}{}_{\rho} (x_{1\mu}) x_\nu \epsilon_3^\rho \epsilon_1^\alpha x_{1\alpha} \right. \\
&\left. - \frac{\varepsilon^{\mu}{}_{\nu\rho}}{2} \left(-|x|^2 x_{1\mu} + |x_1|^2 x_\mu \right) \epsilon_3^\nu \epsilon_1^\rho \right), \\
S_3 &= \frac{1}{4|x_1 - x|^3 |x|| - x_1|} \left(\varepsilon^{\mu\nu}{}_{\rho} (x_1 - x)_\mu (-x_{1\nu}) \epsilon_1^\rho \epsilon_2^\alpha (x_1 - x_\alpha) \right. \\
&\left. - \frac{\varepsilon^{\mu}{}_{\nu\rho}}{2} \left(|x|^2 (x - x_1)_\mu + |x - x_1|^2 (-x_\mu) \right) \epsilon_1^\nu \epsilon_2^\rho \right),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{X} &= \frac{x - x_1}{|x - x_1|^2} + \frac{x_1}{|x_1|^2}, \\
I_{\alpha\beta}(x) &= \eta_{\alpha\beta} - \frac{2x_\alpha x_\beta}{x^2}, \\
h^1_{\mu\nu}(\hat{x}) &= \frac{x_\mu x \nu}{x^2} - \frac{1}{3}\eta_{\mu\nu}, \\
h^2_{\mu\nu\sigma\rho}(\hat{x}) &= \frac{x_\mu x_\sigma}{x^2}\eta_{\nu\rho} + (\mu \leftrightarrow \nu, \rho \leftrightarrow \sigma) - \frac{4}{3}\frac{x_\mu x_\nu}{x^2}\eta_{\sigma\rho} - \frac{4}{3}\frac{x_\sigma x_\rho}{x^2}\eta_{\mu\nu} \\
&\quad + \frac{3}{16}\eta_{\mu\nu}\eta_{\sigma\rho}, \\
h^3{\mu\nu\sigma\rho} &= \eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{2}{3}\eta_{\mu\nu}\eta_{\sigma\rho}, \\
c &= \frac{3(2n^j_f + n^j_s)}{256\pi^3}, \qquad e = \frac{3n^j_s}{256\pi^3}.
\end{aligned}
$$

For the normalisation we need

$$\langle j_\mu(x) j_\nu \rangle = \; = \; \frac{C_V}{x^4} I_{\mu\nu}(x),$$

$$C_V = \frac{8}{3}\pi(c+e).$$

Carrying out all the steps we obtain the energy matrix for charge excitations

$$\hat{E}(j) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{a_2}{2}) & \frac{E}{8\pi}\alpha_j \\ \frac{E}{8\pi}\alpha_j & \frac{E}{4\pi}(1 + \frac{a_2}{2}) \end{pmatrix},$$

where

$$\begin{aligned} a_2 &= -\frac{2(n_f^j - n_s^j)}{(n_f^j + n_s^j)}, \\ \alpha_j &= \frac{4\pi^4 p_j}{(n_f^j + n_s^j)}. \end{aligned}$$

The trace of this matrix is positive. Therefore the condition that the eigen values are positive leads to

$$a_2^2 + \alpha_j^2 \leq 4.$$

This region is a disc of radius 2 centered at the origin in the a_2, α_j plane.

Evaluation of the energy matrix for the stress tensor excitations proceeds similarly.

The three point function is given by

$$\langle T(x)T(x_1)T(0)\rangle = \frac{\epsilon_1^{\mu\nu}\mathcal{I}^T_{\mu\nu,\mu'\nu'}(x)\epsilon_2^{\sigma\rho}\mathcal{I}^T_{\sigma\rho,\sigma'\rho'}(x_1)\epsilon_3^{\alpha\beta}t^{\mu'\nu'\sigma'\rho'}{}_{\alpha\beta}}{x^6x_1^6} +$$

$$p_T\frac{(P_1^2Q_1^2+5P_2^2P_3^2)S_1+(P_2^2Q_2^2+5P_3^2P_1^2)S_2+(P_3^2Q_3^2+5P_3^2P_1^2)S}{|x-x_1||x_1||-x|}$$

where each of the tensor structure is defined.

The calculation is more involved.

The final result for the energy matrix for stress tensor excitations

$$\hat{E}(T) = \begin{pmatrix} \frac{E}{4\pi}(1 - \frac{t_4}{4}) & \frac{E}{16\pi}\alpha_T \\ \frac{E}{16\pi}\alpha_T & \frac{E}{4\pi}(1 + \frac{t_4}{4}) \end{pmatrix},$$

where,

$$\begin{aligned} t_4 &= -\frac{4(n_f^T - n_s^T)}{n_f^T + n_s^T}, \\ \alpha_T &= \frac{8\pi^4 p_T}{3(n_f^T + n_s^T)}. \end{aligned}$$

The condition that the eigen values of this matrix is positive leads to

$$t_4^2 + \alpha_T^2 \le 16.$$

- Where does Large N Chern Simons theories lie ?

$$
\begin{aligned}
\langle jjT\rangle &= n_s^j\langle jjT\rangle_{\text{free boson}} + n_f^j\langle jjT\rangle_{\text{free fermion}} + p_j\langle jjT\rangle_{\text{parity odd}},\\
\langle TTT\rangle &= n_s^T\langle TTT\rangle_{\text{free boson}} + n_f^T\langle TTT\rangle_{\text{free fermion}} + p_T\langle TTT\rangle_{\text{parity odd}}
\end{aligned}
$$

$$
n_s^T(f) = n_s^j(f) = 2N\frac{\sin\theta}{\theta}\sin^2\frac{\theta}{2}, \qquad n_f^T(f) = n_s^j(f) = 2N\frac{\sin\theta}{\theta}\cos^2\frac{\theta}{2},
$$

$$
p_j(f) = \alpha' N\frac{\sin^2\theta}{\theta}, \qquad p_T(f) = \alpha N\frac{\sin^2\theta}{\theta},
$$

where the t 'Hooft coupling is related to θ by

$$
\theta = \frac{\pi N}{\kappa}.
$$

• The existence of the parity odd term was first confirmed in the fermionic theory using a one loop calculation by Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin (2011), Aharony, Gur-Ari, Yacoby (2011).

• The full dependence on the t'Hooft coupling was argued using weakly broken higher spin symmetry by Maldacena, Zhiboedov (2012).

• This was perturbatively checked to all orders in t'Hooft coupling confirmed for the bosonic theory by Aharony, Gur-Ari, Yacoby (2012).

• The Maldacena, Zhiboedov analysis can in principle determine the precise normalisation once one decides on the normalisation of the tensor structure of the parity odd term.

• We fix it by first taking the normalization of the parity odd term as given by GMPTW.

• We redo the one-loop perturbative analysis to obtain

$$\alpha' = \frac{1}{\pi^4}.$$

$$\alpha = \frac{3}{\pi^4}.$$

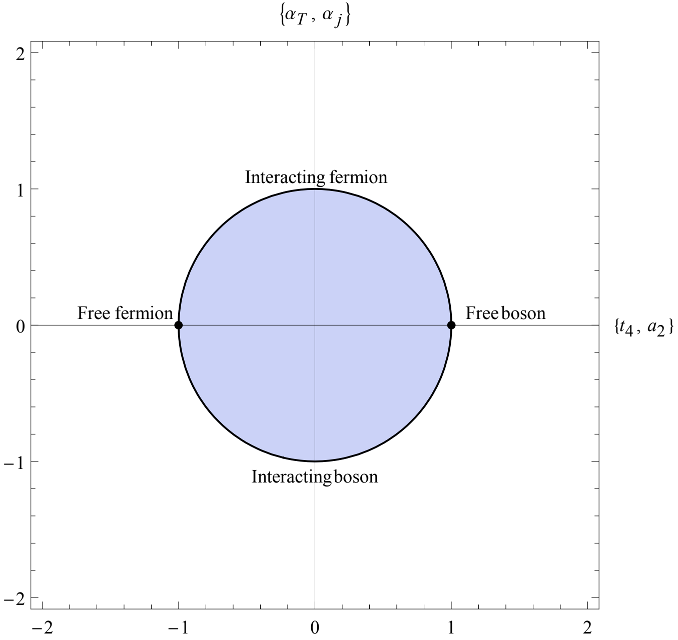
Substituting these values

$$a_2 = -2\cos\theta, \qquad \alpha_j = 2\sin\theta,$$
$$t_4 = -4\cos\theta, \qquad \alpha_T = 4\sin\theta.$$

Large N Chern-Simons theory with fundamental matter lie on the circle bounding the disc.

Their location on the bounding circle is parametrized by the 't Hooft coupling $\theta = \frac{\pi N}{\kappa}$.

To conclude



$\{\alpha_T, \alpha_j\}$
$\{t_4, a_2\}$
Interacting fermion
Free fermion
Free boson
Interacting boson
2
1
0
−1
−2
−2
−1
0
1
2

• Such conformal collider constraints on OPE coefficients were recently generalised by Córdova, Maldacena and Turiaci (1710.03199).

More importantly they also confirmed the saturation of the bounds by large N Chern-Simons theories

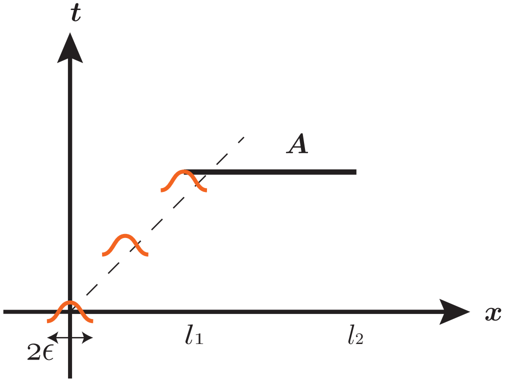
• The fact that Chern-Simons theories coupled to fundamental matter at large N saturate the ANC seems to be related to the saturation of the unitarity bound for spin s currents.

$$\Delta = 1 + s + O(1/N)$$

CONSTRAINT ON SPIIN-3 CHARGED STATES

• Consider a localized excitation of width ϵ in 2d CFT held at finite temperature β , at $x = 0, t = 0$ created by a conformal primary $\mathcal{O}$ of dimension $\Delta_{\mathcal{O}}$.

What is the change in entanglement entropy when the pulse enters the entanglement interval (l_1, l_2).


t
A
x
l1
l2
2ϵ

• The details:

The density matrix corresponding to the pulse is given by

$$\hat{\rho}_\epsilon = \mathcal{N}\, e^{-iHt} \left(e^{-\epsilon H}\mathcal{O}(0)e^{\epsilon H}\right) \rho_\beta \left(e^{\epsilon H}\mathcal{O}^\dagger(0)e^{-\epsilon H}\right) e^{iHt}\,,$$

$$\rho_\beta = e^{-\beta H}\,.$$

The evolution by imaginary time ϵ regulates the evaluation of subsequent correlators.

It also smears the pulse to a finite width ϵ.

• There is a jump in the entanglement entropy as the pulse enters the interval.

• The jump can be evaluate for large c CFT's using the large c limit for the heavy-heavy-light-light correlator.

• Equivalently it can be evaluated using the holographic dual of the local quench.

- The holographic dual to the local quench in the CFT is obtained by considering an in falling massive particle $m \sim \Delta_{\mathcal{O}}$, in the BTZ black hole
starting at distance ϵ from the boundary.

- The entangling interval A is located on the boundary.

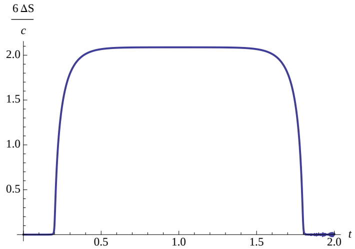
- The time dependent back reacted geometry of the in falling particle in the BTZ black hole.
Horowitz, Itzhaki (1999), Caputa, Simon, Stikonas, Takayanagi (2015).

Essentially it is conical defect moving along its in falling geodesic in the BTZ geometry

• The time dependent entanglement entropy can be obtained by evaluating the Wilson line/geodesic in the geometry.

One evaluates the Wilson line in the conical defect and substitutes the transformation that takes the co-ordinates to the in falling geodesic in the BTZ background.

The result agrees with that using the large c limit of the HHLL correlator.



• To obtain a simple expression for the jump, perform the following scaling.

We keep the energy of the pulse finite as $\epsilon \to \infty$. The energy density is given by

$$\langle \mathcal{O} | T_{00} | \mathcal{O} \rangle \sim \frac{\Delta}{\epsilon^2}$$

Therefore if

$$\Delta = \frac{\epsilon E}{\pi}$$

then total energy carried by the pulse is finite.

Then the jump in Entanglement entropy when the interval is infinite $l_2 \to \infty$ and at large times $t >> l_1$ is

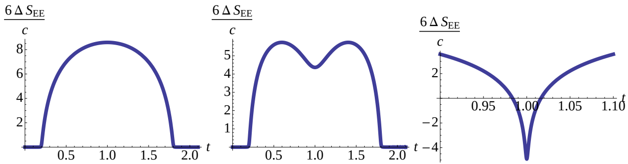
$$\Delta S_{EE}|_{l_2 \to \infty, t >> l_1} = \frac{c}{6} \ln \left(1 + \frac{6\beta}{\pi c} E \right)$$

• The local quench can be studied when the local operator carries spin-3 charge.

This can be done either using W_3 conformal blocks in the large c limit for the HHLL correlator.

Equivalently we can obtain it using the holographic description of the in falling spin-3 charged conical defect in the BTZ black hole embedded in the $SL(3)$ Chern-Simons theory.

The jump in entanglement entropy behaves as



To obtain the critical value of charge of the spin-3 excitation beyond which the jump in the EE ceases to be real.

Perform the scaling so that the spin-3 charge carried by the pulse is finite.

$$\langle \mathcal{O} | \mathcal{W} | \mathcal{O} \rangle \sim \frac{\mathcal{W}}{\epsilon^3}$$

We scale

$$\mathcal{W} = \frac{4\epsilon^2 q}{\pi^2}$$

and keep q finite when $\epsilon \to 0$.

The jump in entanglement entropy in the large interval large time limit

$$\Delta S_{EE}|_{l_2\to\infty, t>>l_1} = \frac{c}{24}\ln\left[\left(1+\frac{6\beta E}{\pi c}\right)^4 - \frac{q^2}{c^2}\left(\frac{6\beta}{\pi}\right)^4\right]$$

Thus to avoid the entanglement entropy becoming negative or complex, we must have

$$\sqrt{\frac{|q|}{c}} < \frac{E}{c} + \frac{\pi}{6\beta}$$

Rewriting in terms of the charge $\mathcal{W}$ and conformal dimension $\cdot$ we obtain

$$\sqrt{\frac{|\mathcal{W}|}{c}} < \frac{2\Delta}{c}$$

• This bound on the charge can also be obtained from the chaos bound.

• The OTO correlator can be obtained by an analytical continuation of the correlator used to study the jump in the entanglement entropy of the local quench.

• Recall for the quench:
(l_1, l_2) denotes the entangling interval.
ϵ, the width of the excitation or quench.
t the time after which the excitation is released from the origin.

To obtain the OTO correlator,

$$\mathcal{C}_{OTO} = \langle \frac{\langle H^\dagger(t - i\epsilon, 0)|_1 L(0, l_2)|_2 H(t + i\epsilon, 0)|_4 L(0, l_1)|_3\rangle_\beta}{\langle H^\dagger(t - i\epsilon, 0)H(t + i\epsilon, 0)\rangle_\beta \langle L(0, l_2)L(0, l_1)\rangle_\beta}$$

The cross ratio

$$z = \frac{(z_2 - z_3)(z_1 - z_4)}{(z_2 - z_1)(z_3 - z_4)} = \frac{i \sin(\frac{2\pi\epsilon}{\beta}) \sinh \frac{\pi}{\beta}(l_2 - l_1)}{\sinh \frac{\pi}{\beta}(l_1 - t + i\epsilon) \sinh \frac{\pi}{\beta}(l_2 - ti - i\epsilon)}$$

we need to $t > l_1$, excitation has crossed the interval, analytically continue

$$t \to t + i\epsilon_1, \qquad \epsilon_1 >> \epsilon$$

and then take $t >> l_2$ and for simplicity $l_2 >> l_1$

$$z = \frac{4\pi i\epsilon}{\beta} e^{-\frac{2\pi}{\beta}(t + i\epsilon_1 - l_2)}$$

The OTO is given by

$$\mathcal{C}_{OTO} = \left[\left(1 - \frac{6E}{\beta c\pi} e^{\frac{2\pi}{\beta}(t+i\epsilon_1-l_2)}\right)^4 - \left(\frac{36q\beta^2}{\pi^2 c} e^{\frac{4\pi}{\beta}(t+i\epsilon_1-l_2)}\right)^2\right]^{-2h_l}$$

It is clear from this that if

$$\sqrt{\frac{|q|}{c}} < \frac{E}{c}$$

then the departure from unity of the correlator is determined by the Lyapunov index

$$\lambda_{\text{spin } 2} = \frac{2\pi}{\beta}$$

else we have

$$\lambda_{\text{spin } 3} = \frac{4\pi}{\beta}$$

- We conclude that bound

$$\sqrt{\frac{|\mathcal{W}|}{c}} < \frac{2\Delta}{c}$$

obtained from entanglement entropy considerations also ensures that the chaos bound is satisfied.

• This conclusion can also be obtained by studying the scrambling time of mutual information of 2 entangling intervals caused by an infalling particle carrying higher spin charge in the eternal BTZ black hole.

The expression for scrambling time is

$$t_* = \frac{l_1 + l_2}{2} + \frac{\beta}{\pi} \ln \left[\sinh \tfrac{\pi}{\beta}(l_2 - l_1) \right] + \frac{\beta}{2\pi} \ln \left| \frac{E^4}{S_\beta^4} - \frac{9q^2\beta^2}{\pi^2 S_\beta^2} \right|^{-\frac{1}{4}},$$

$$S_\beta = \frac{c\pi}{3\beta}$$

• We see that scrambling time increases as the higher spin charge q is increased and diverges when the bound is saturated

$$\frac{E}{c} = \sqrt{\frac{q}{c}}$$

But only once it is crossed and E is negligible the Lyapunov index shifts to the larger spin-3 value.

Lets examine how this bound on charge is related to unitarity.

• In 1707. 0717, Afkhami-Jeddi,Colville, Hartman, Maloney, Perlmutter examined the Kac-determinant of these theories at level 1

$$\begin{pmatrix} \langle\Delta,\ \mathcal{W}|L_1 L_{-1}|\Delta,\ \mathcal{W}\rangle & \langle\Delta,\ \mathcal{W}|L_1 W_{-1}|\Delta,\ \mathcal{W}\rangle \\ \langle\Delta,\ \mathcal{W}|W_1 L_{-1}|\Delta,\ \mathcal{W}\rangle & \langle\Delta,\ \mathcal{W}|W_1 W_{-1}|\Delta,\ \mathcal{W}\rangle \end{pmatrix}$$

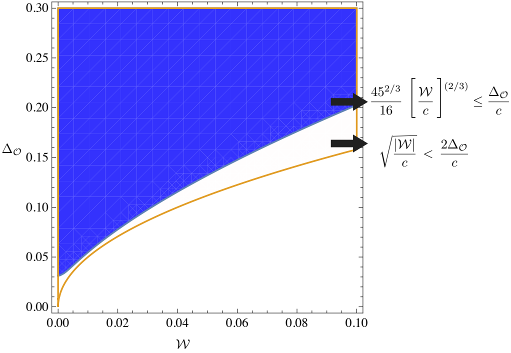
There is a constraint on the charge $\mathcal{W}$ by demanding the eigen values of the level 1 Kac-Determinant is positive.

There is no further constraint at higher levels.

On taking the large c limit of the constraint we obtain

$$9\left(\frac{\mathcal{W}}{c}\right)^2 \leq \frac{64}{5}\left(\frac{\Delta}{c}\right)^3 - \frac{2}{5}\left(\frac{\Delta}{c}\right)^2$$

Comparing the constraint obtained from considerations of entanglement entropy



UNITARITY AND CHAOS BOUND

• Can we see more directly that the presence of a state which does not satisfy the unitarity bound

$$\Delta > 0$$

violates the chaos bound ?

- Consider the density matrix

$$\rho_{\mathcal{O}} = \mathcal{O}(0)\rho_\beta \mathcal{O}^\dagger(\infty), \qquad \rho_\beta = e^{\beta H}$$

The operator $\mathcal{O}$ has dimension $h = \bar{h} = \Delta$.
We will take $\Delta \sim O(c)$.

Evaluating the expectation value of the stress tensor in this state we obtain

$$\begin{aligned}
\mathrm{Tr}(\rho_{\mathcal{O}} T_{ww}(w)) &= \frac{\langle \mathcal{O}(\infty) T_{ww}(w) \mathcal{O}(-\infty)\rangle_\beta}{\langle \mathcal{O}(\infty)\mathcal{O}(-\infty)\rangle_\beta} = \left[(\frac{2\pi}{\beta})^2 \Delta - \frac{c\pi^2}{6\beta^2} \right], \\
&= -\frac{c\pi^2}{6\beta^2}\left(1 - \frac{24\Delta}{c}\right)
\end{aligned}$$

The effective temperature of the state shifts to

$$\beta \to \beta' = \frac{\beta}{\sqrt{1 - \frac{24\Delta}{c}}}$$

What is the OTO of four operators in the state $\rho_{\mathcal{O}}$. ?

• As a preliminary step: let us probe this state by two light operators of dimension h_L.
The correlator reduces to

$$\begin{aligned}\mathcal{C}_{4pt} &= \mathrm{Tr}\left(\rho_{\mathcal{O}}\mathcal{O}_L(w_1,\bar{w}_1)\mathcal{O}_L(w_2,\bar{w}_2)\right), \\ &= \frac{\langle\mathcal{O}(\infty)\mathcal{O}_L(w_1,\bar{w}_1)\mathcal{O}_L(w_2,\bar{w}_2)\mathcal{O}(\infty)\rangle_\beta}{\langle\mathcal{O}(\infty)\mathcal{O}(-\infty)\rangle_\beta}\end{aligned}$$

This is the four point function of two light operators and two heavy operators at temperature β.

We can use the result for the vacuum conformal block at large c

$$
\begin{aligned}
&= \left.\frac{\langle \mathcal{O}(\infty)\mathcal{O}_L(z_1\bar{z}_1)\mathcal{O}_L(z_2\bar{z}_2)\mathcal{O}(0)\rangle}{\langle \mathcal{O}(\infty)\mathcal{O}(0)\rangle}\right|_{\text{holomorphic;plane}} \\
&= \exp\left\{-h_L\left((1-\alpha)\log(z_1 z_2) + 2\log\frac{(z_1^\alpha - z_2^\alpha)}{\alpha}\right)\right\}
\end{aligned}
$$

$$
\alpha = \sqrt{1 - \frac{24\Delta}{c}}
$$

Converting the result that on the cylinder of temperature β

$$\mathcal{C}_{4pt} = \left(\frac{\pi}{\beta'}\right)^{4h_L} \left(\frac{1}{\sinh \frac{\pi}{\beta'}(w_1 - w_2) \sinh \frac{\pi}{\beta'}(\bar{w}_1 - \bar{w}_2)}\right)^{2h_L}$$

Note that the result is equal to the two point function of the light operators as if the theory is at temperature β'.

Using this result for evaluating the entanglement entropy we obtain

$$S_{EE}(L) \quad = \quad \frac{c}{3} \log \left\{ \frac{\beta}{\pi} (1 - \frac{24\Delta}{c})^{-1/2} \sinh \left(\frac{\pi}{\beta} \sqrt{1 - \frac{24\Delta}{c}} L \right) \right\}$$

Examine the OTO in the state $\rho_{\mathcal{O}}$.
For this we need the 6 point function

$$\mathcal{C}_{6pt} = \mathrm{Tr}\left(\rho_{\mathcal{O}}\mathcal{O}_L(w_2,\bar{w}_2)\mathcal{O}_L(w_3,\bar{w}_3)\mathcal{O}'_L(w_4,\bar{w}_4)\mathcal{O}'_L(w_1,\bar{w}_1)\right)$$

$$= \frac{\langle\mathcal{O}(\infty)\mathcal{O}_L(w_2,\bar{w}_2)\mathcal{O}_L(w_3,\bar{w}_3)\mathcal{O}'_L(w_4,\bar{w}_4)\mathcal{O}'_L(w_1,\bar{w}_1)\mathcal{O}(-\infty)\rangle_\beta}{\langle\mathcal{O}(\infty)\mathcal{O}(-\infty)\rangle_\beta}$$

$\mathcal{O}'_L$ is an operator of dimension Δ_L.
The conformal dimensions we have satisfy $\Delta >> \Delta_L >> h_L$.

• The answer for the six point function can be obtained by considering the four point function HHLL correlator of operators $\mathcal{O}'_L$ and $\mathcal{O}_L$ at temperature β'.
The result can then be used to obtain the OTO.
The OTO correlator of these operators in the state $\rho_{\mathcal{O}}$ is given by

$$\mathcal{C}_{OTO} = \left(1 - \frac{6\beta' E}{\pi c} e^{\frac{2\pi}{\beta'}(t+i\epsilon_1 - l_2)}\right)^{-2h_L}$$

Therefore the Lyapunov index is given by

$$\lambda_L = \frac{2\pi}{\beta'} = \frac{2\pi}{\beta}\sqrt{1 - \frac{24\Delta}{c}}$$

When $\Delta > 0$; satisfies the unitarity constraint the Lyapunov exponent satisfies the bound.

when $\Delta < 0$, the Lyapunov exponent violates the bound.

Is there a system where such a state $\rho_{\mathcal{O}}$ is realized.

Consider the Chern-Simons theory based on the group $W_3^{(2)}$. The algebra of generators is given by

$$\begin{aligned}
\left[\hat{L}_i, \hat{L}_j\right] &= (i-j)L_{i+j} && i,j = 0, \pm, \\
\left[J_0, G^{\pm}_m\right] &= \pm G^{\pm}_m && m,n = \pm 1/2, \\
\left[\hat{L}_i, G^{\pm}_m\right] &= \left(\frac{i}{2} - m\right) G^{\pm}_{i+m}, \\
\left[G^{+}_n, G^{-}_m\right] &= -\frac{\hat{c}}{6}\left(n^2 - \frac{1}{4}\right)\delta_{n+m,0} - \hat{L}_{n+m} + \frac{3}{2}(n-m)J_{n+m}.
\end{aligned}$$

This is the Polyakov-Bershadsky algebra.

It consists of a stress tensor with two spin $3/2$ currrents together with a $U(1)$ current.

$W_3^{(2)}$ arises from a different $SL(2)$ embedding in $SL(3)$.

$$\hat{L}_0 = \frac{1}{2} L_0, \qquad \hat{L}_\pm = \pm \frac{1}{4} W_{\pm 2}, \qquad J_0 = \frac{1}{2} W_0,$$

$$G^\pm_{1/2} = \frac{1}{\sqrt{8}} \left(W_1 \mp L_1 \right), \qquad G^\pm_{-1/2} = \frac{1}{\sqrt{8}} \left(L_{-1} \pm W_{-1} \right).$$

Let us consider the BTZ like solution in the Chern-Simons theory based on $W_3^{(2)}$.

$$A = b^{-1}\, a\, b + b^{-1} db, \qquad \bar{A} = b\, \bar{a}\, b^{-1} + b\, db^{-1}, \qquad b = e^{\rho \hat{L}_0} = e^{\frac{\rho}{2} L_0}$$

$$a = \left(\hat{L}_1 - \frac{w_{-2}}{k} \hat{L}_{-1}\right) dz, \qquad \bar{a} = -\left(\hat{L}_{-1} - \frac{w_{-2}}{k} \hat{L}_1\right) d\bar{z},$$

This has the same form as the BTZ connection.

The difference lies in the use of $\hat{L}$ matrices instead of the L matrices in case of BTZ.

The level of the Chern-Simons theory is related to the central charge by

$$k = \frac{c}{12\mathrm{Tr}\left[\hat{L}_0\hat{L}_0\right]} = \frac{c}{6}$$

w_{-2} is a constant.

We impose that that holonomy of this connection is identical to that of the BTZ connection.

This requires the holonomy along the time direction with periodicity β to be identity and equal to

$$\exp\left(\int_0^\beta dta_t\right) \sim \begin{pmatrix} e^{2\pi i} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2\pi i} \end{pmatrix}$$

That is the eigen values of $\int_0^\beta dta_t$ are identical to that of the BTZ connection.

This imposes the following conditions on the flat connection,

$$\det(\beta a_t) = 0, \quad \mathrm{Tr}(\beta^2 a_t^2) = -8\pi^2,$$

This fixes w_{-2} and hence the expectation value of the stress tensor in the dual CFT.

$$\langle T_{zz}\rangle_{BTZ\ like} = -w_{-2} = -\frac{2}{3}\frac{c\pi^2}{\beta^2}.$$

Note that this differs from the expectation value of the stress tensor in the usual thermal circle of length β.

$$\langle T_{zz}\rangle_{\beta} = -\frac{c\pi^2}{6\beta^2}$$

• We can understand the difference in the expectation value of the stress tensor due to the presence of a nontrivial operator $\mathcal{O}$ and the assuming that the state represented by the BTZ-like connection corresponds to ρ_β.

$$\langle T_{zz}\rangle = \frac{\langle \mathcal{O}(\infty) T_{zz} \mathcal{O}(-\infty)\rangle_\beta}{\langle \mathcal{O}(\infty)\mathcal{O}(-\infty)\rangle_\beta} = \left(\frac{2\pi}{\beta}\right)^2 \Delta - \frac{c\pi^2}{6\beta^2}.$$

We obtain

$$\Delta = -\frac{c}{8}$$

• The presence of the non-trivial operator $\mathcal{O}$ can also be cross checked by evaluating the entanglement entropy.

Using the holographic prescription of De Boer and Jottar (2013) for the BTZ like connection we get

$$S_{\rm hol} = \frac{c}{3}\log\left(\frac{\beta}{2\pi}\sinh\left(\frac{2\pi L}{\beta}\right)\right).$$

From comparing the argument of the sinh with the expression derived from the CFT due to the presence of the state $\mathcal{O}$ we obtain

$$\Delta = -\frac{c}{8}$$

• We can proceed and evaluate the OTO holographically in the BTZ like background by considering the back reacted solution of an in-falling particle.

The results in the Lyapunov exponent

$$\lambda = \frac{2\pi}{\beta}\sqrt{1 - \frac{24\Delta}{c}} = \frac{4\pi}{\beta}$$

- We have also studied the solution carrying both $U(1)$ as well as spin $3/2$ charge in the $W_3^{(2)}$ Chern-Simons theory and obtained similar conclusions.

• We have discussed two constraints on conformal field theories.

Constrained the allowed CFT's in 3 dimensions.

Constrained the allowed higher spin charge for CFT's in $d = 2$ with $\mathcal{W}_3$ symmetry.

• We used different physical considerations to obtain these bounds. But we have seen that the constraints are connected to unitarity.

• The presence of a operator of negative conformal dimension whose dimensions $\Delta \sim O(c)$ results in the violation of the chaos bound.

This again demonstrates that this bound is tied to unitarity.