VECTOR SPACE OF CFTS

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MOTIVATIONS

- Crossing symmetry imposes strong constraints on CFTs
- So strong that they are thought to be sufficient
- Conformal bootstrap deals with implications of a single or a few crossing equations
- When supplemented with unitarity even a single crossing equation is quite powerful
- It seems to fix the CFT uniquely i.e. a solution to a single or a few crossing equations seems to admit uplift to the solution of all crossing equations

MOTIVATIONS

- However it is quite easy to come up with a single or a few crossing symmetric four-point function that results in a CFT that is not necessarily unitary
- Because a single constraint is so weak, its solution doesn't lift to the solution of all crossing equations
- This means that in order to constrain or classify CFTS that are not necessarily unitary, one must solve all the crossing equations
- This seems a difficult task to construct such conformal data but as we will show it can be accomplished with the help of alternative but equivalent conformal data

DISCLAIMER

- Strictly speaking, logically, it is a stretch to term the solutions of the crossing equations to be a conformal field theory
- Apart from crossing equations, presumably CFTs satisfy other conditions coming from consistency of theory on manifolds with interesting topology
- For example, in two dimensions such are modular invariance
- Formulating such constraints in higher dimension is an interesting problem in itself and we will not address it
- We will take CFTs to mean solutions to the crossing equation

CONVENTIONAL CONFORMAL DATA

- A CFT is defined in terms of its operator spectrum and their correlation functions
- Conformal symmetry allows one to express all higher point function in terms of 2 and 3 point functions
- Hence conformal dimension and 3-point function coefficient form the whole conformal data
- This we refer to as the conventional conformal data

ALTERNATE CONFORMAL DATA

- We define conformal data in a different way but show its equivalence to the conventional data

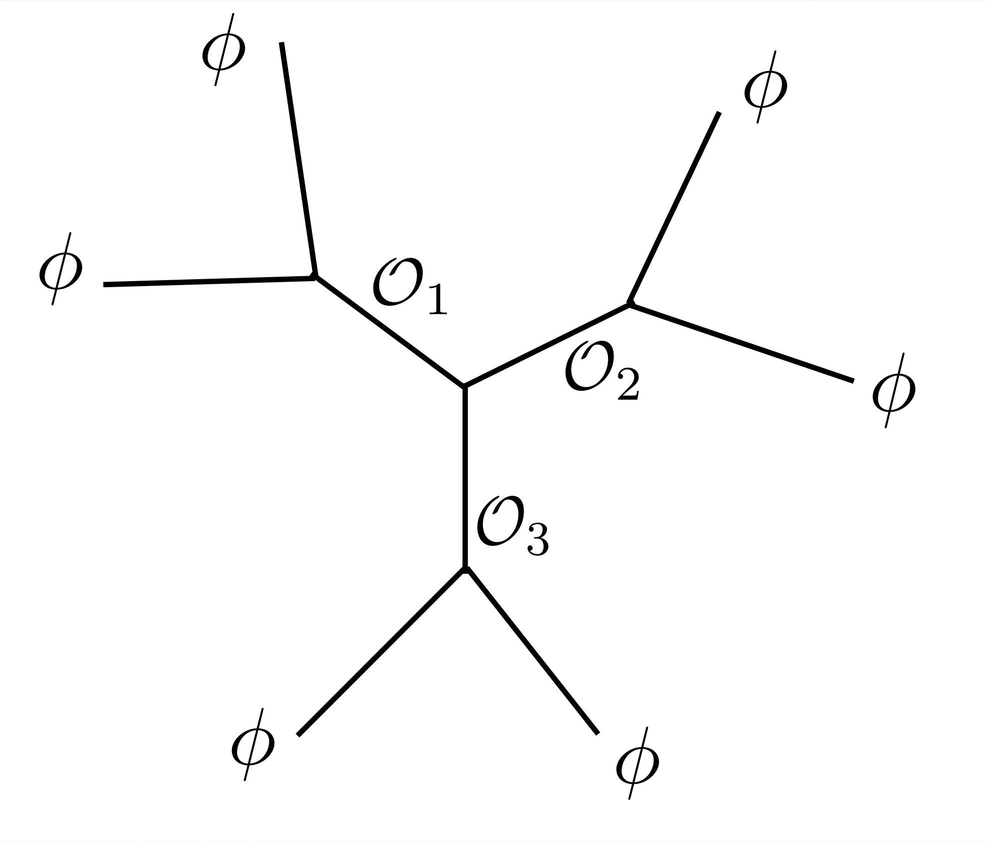
- Alternate data: set of n-point functions of a single chosen scalar operator $\langle\phi(x_1)\ldots\phi(x_n)\rangle$

- Clearly, these can be constructed from 2,3-point functions by sewing them together

- On the other hand, if one takes appropriate OPE limit then 2,3-point functions involving any operators

ALTERNATE CONFORMAL DATA

- For example if one wants to get 3-point functions of symmetric traceless operator then starting from 6-point function of



CONSTRAINTS

- The constraints on the conventional data follow from its consistency under sewing. This results in the crossing equation
- The constraints on the alternate data come from the fact that there are many OPE channels which can result in a 3-point function given operators
- All such channels should give the same 3-point function
- Constraint on the new data is that the 3-point functions that it yields should be well-defined

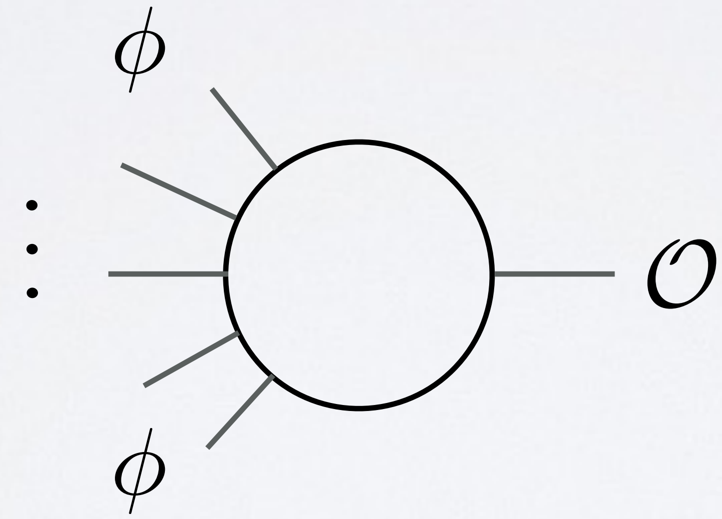
SIMPLIFYING CONSTRAINTS

- It turns out that the well-defined-ness of the 3-point function follows from a simpler constraint
- This is the constraint that n-point function when factorized in the channel with either Identity operator or the chosen operator should yield the correct lower point function
- i.e. the set of n-point functions of the chosen operator should be consistent under factorization
- In this formulation the constraint is easier to deal with

SIMPLIFYING CONSTRAINTS

- We will now prove this simplification claim

- In order to show that any 3-point function is uniquely defined we will first start with the correlators of the form $\langle \phi(x_1) \ldots \phi(x_n) \mathcal{O}(y) \rangle$



- Let us first define these correlators and then show that the definition is unique

SIMPLIFYING CONSTRAINTS

- Let $n_{\mathcal{O}}^{min}$ be the minimum number of ϕ's required in order to have a non-zero correlator $\langle\phi(x_1)\dots\phi(x_n)\mathcal{O}(y)\rangle$

- Consider $2n_{\mathcal{O}}^{min}$ point function of ϕ's. We take the two groups of $n_{\mathcal{O}}^{min}$ points separated by distance R and take $R\to\infty$

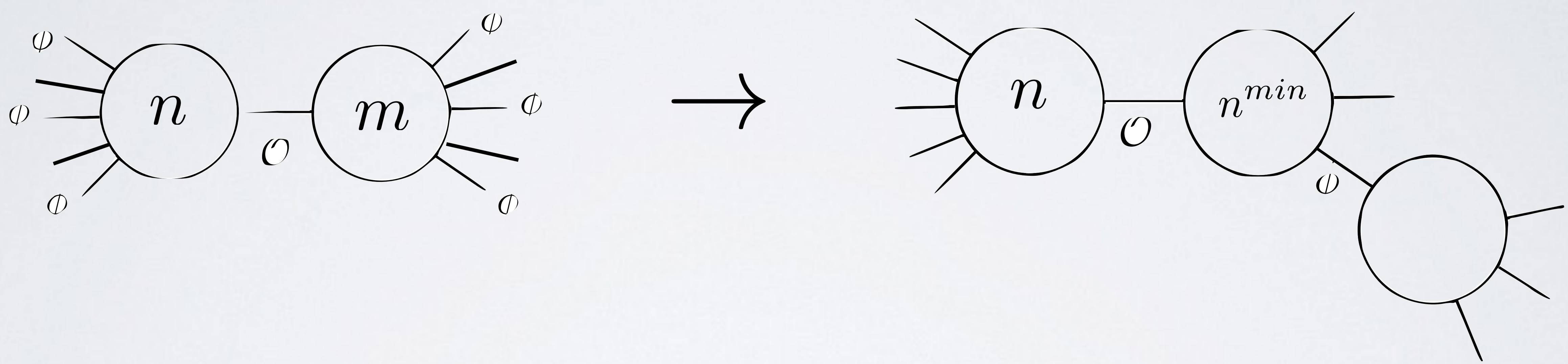
$$\langle\phi(x_1)\dots\phi(x_{n^{min}})\phi(x_1+R)\dots\phi(x_{n^{min}}+R)\rangle$$

$$\to \quad R^{-2\Delta_{\mathcal{O}}}\langle\phi^{n^{min}}\mathcal{O}\rangle\ \langle\phi^{n^{min}}\mathcal{O}\rangle$$

- Using this definition, we define $\langle\phi^n\mathcal{O}\rangle$ correlator for other $n\neq n^{min}$.

SIMPLIFYING CONSTRAINTS

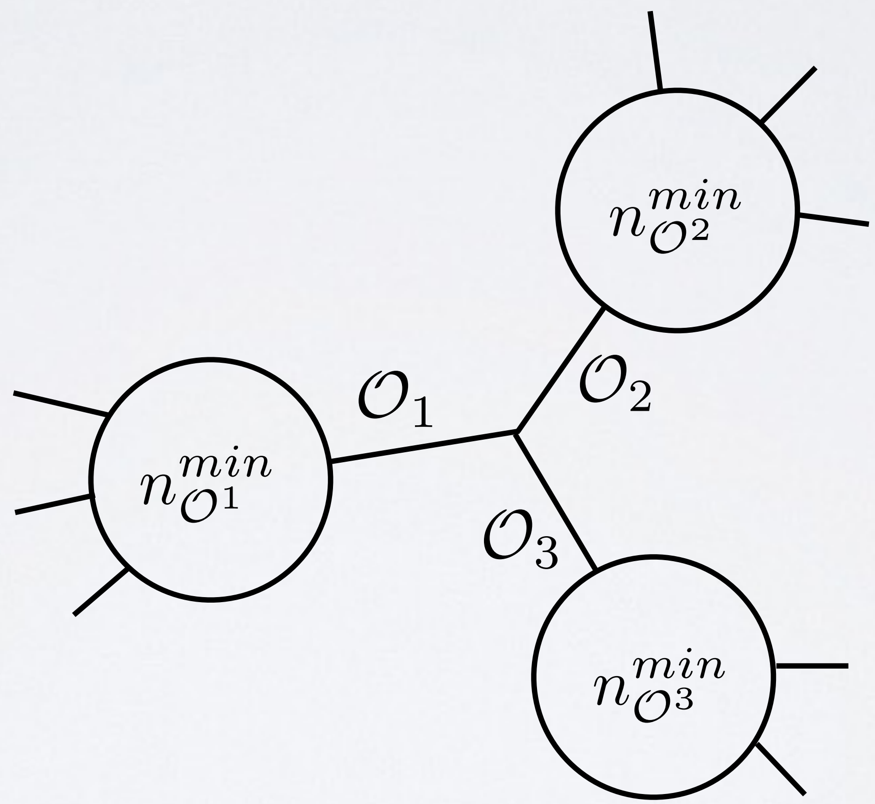
- Any other way of getting such a correlation function must match up with what we have defined



- We only assume that the factorization in ϕ channel is consistent to show that correlators involving $\mathcal{O}$ is well-defined

UNIQUENESS OF 3-POINT FUNCTION

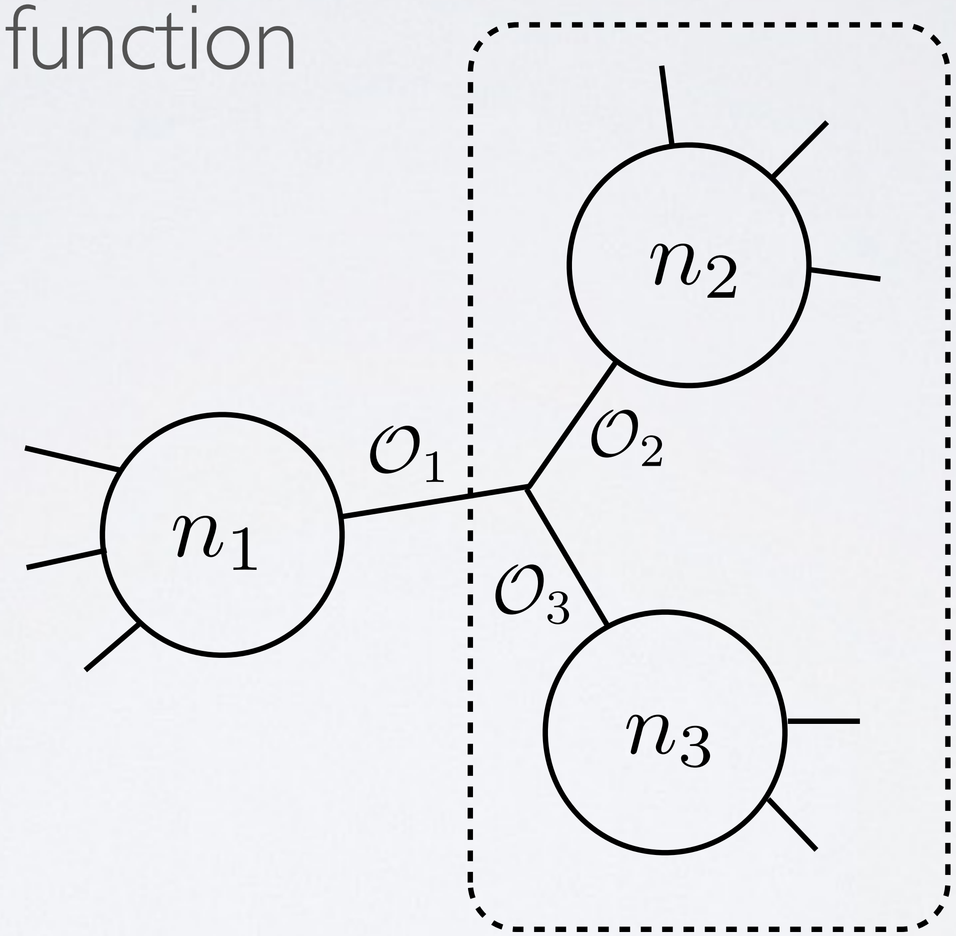
- 3-point function of $\mathcal{O}_i$ is defined starting from the $\sum_i n_{\mathcal{O}_i}^{min}$ point function of ϕ



- We need to show that the 3-point function obtained from any higher point function of ϕ is also the same.

UNIQUENESS OF 3-POINT FUNCTION

- Let is take a generic channel in higher point function that produces the required 3-point function



- At least for a single i, $n_i > n^{min}_{\mathcal{O}_i}$. Let that be the case for $i = 1$

BOTTOM LINE

- A solution to the conformal crossing equation is obtained by a set of n-point function of a single scalar operator that factorize consistently into lower point functions of the same scalar operator
- This result is true for any conformal field theory
- Now we assume that the CFT has abelian symmetry
- For such theories, we will construct new solutions to crossing starting from old ones. This operation is same as the vector space operation

CONSTRUCTION

- For a CFT with the abelian symmetry, we consider n-point function of scalar operator but only for very special scalar operator

- This operator is lowest dimensional charge 1 scalar Φ
- Let f_n be the n-point correlator of Φ s

- Given two CFTs with the same abelian symmetry $\{f_n^{(1)}\}$ and $\{f_n^{(2)}\}$

New solution: $\{g_n = (f_n^{(1)})^\alpha (f_n^{(2)})^\beta\}$

PROOF

- Lower point functions of Φ from higher point functions of Φ are obtained only in the factorization channel with either I or Φ .
- In the channel with identity, the identity exchange is the leading term in the large R expansion

$$f_{n+m} = f_n f_m + R^{-2\Delta_{\mathcal{O}}} \dots$$

- In the channel with Φ, the Φ exchange is the leading term in the large R expansion. This is due to the fact that Φ is lowest dimension operator of charge 1.

$$f_{n+m} = R^{-2\Delta_{\Phi}} f_{n+1} f_{m+1} + R^{-2\Delta_{\mathcal{O}'}} \dots$$

PROOF

- Now let us factorize g_n in the identity channel

$$g_{n+m} = (f^{(1)}_{n+m})^\alpha (f^{(2)}_{n+m})^\beta$$

- In the large R limit

$$g_{n+m} = (f^{(1)}_n f^{(1)}_m + R^{-2\Delta_{\mathcal{O}}})^\alpha (f^{(2)}_n f^{(2)}_m + R^{-2\Delta_{\mathcal{O}}})^\beta$$

$$g_{n+m} = g_n g_m + R^{-2\Delta_{\mathcal{O}}}$$

- Same argument works for the factorization in the Φ channel

- This concludes the proof of the claim

REMARKS

- The vector space operation preserves currents
- No assumption about the spectrum. Proliferation of operators.
- The OPE coefficients of the new operators goes to zero when the powers become integer
- For integer powers, the CFT obtained is the direct product CFT orbifolded by a discrete symmetry. The orbifold group is a subgroup of the "generalized permutation group".
- Unitary theories span a lattice in this space.

$$\text{Rank of the lattice} \stackrel{?}{=} \text{dimension of the vector space}$$