The SYK model and AdS_2

Juan Maldacena

Some references

SYK : Sachdev, Ye . Kitaev

AdS_2 : $NAdS_2$: Jackiw, Teitelboim, Almheiri and Polchinski,…

Nice review by Gabor Sarosi : https://arxiv.org/abs/1711.08482

My own work on this was done with:

Douglas Stanford

Zhenbin Yang

Some orientation first

Models of holography

Large N quantum system	Anomalous dimension	**Gravity/string dual**
• Free boundary theories.	$\gamma_{S>2} = 0$	• Bulk theories with massless higher spin fields.
• O(N) interacting theories.	$\gamma_{S>2} \sim 1/N$	• Very slightly massive higher spins
• Sachdev Ye Kitaev Model	$\gamma_{S>2} \sim 1$	• O(1) masses for the higher spin fields.
• Maximally supersymmetric Yang Mills at very strong t'Hooft coupling, g^2 N >> 1	$\gamma_{S>2} \gg 1$	• Einstein gravity theory. Higher spin particles are very massive.

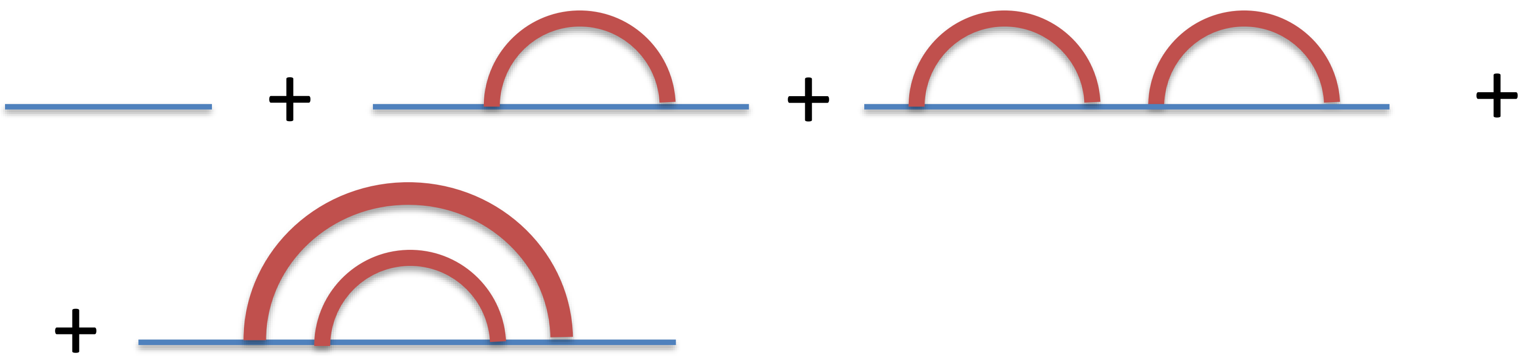
Harder

Easier

Solvable large N models

A simple solvable model

Rainbow diagrams.



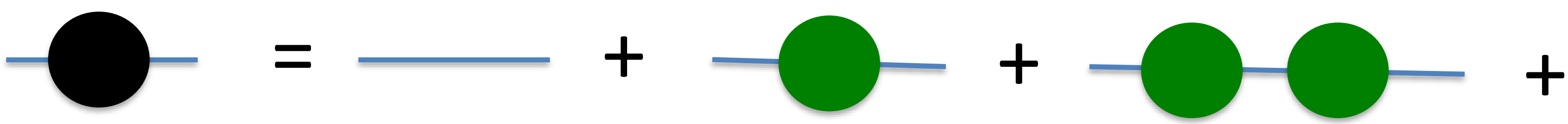
Eg : 2d QCD, O(N) models,
large N Chern Simons theories with fundamental matter in 2 +1 dimensions

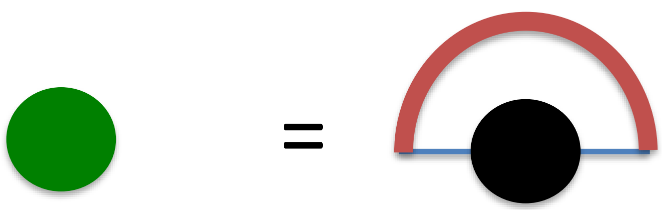
Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin,..

Izuka, Polchinski, Okuda

Summing rainbow diagrams

Rainbow diagrams.





$$\frac{1}{\mathbf{G}(\omega)} = \frac{1}{G_0} - \Sigma(\omega)$$

$$\Sigma(t,t') = P(t,t')\mathbf{G}(t,t')$$

Special Case

$$P = J^2 = \text{constant}$$

Izuka Polchinski Okuda

Similar to what we get for the following model

N Majorana fermions

$$\{\psi_i, \psi_j\} = \delta_{ij}$$

$$H = i \sum_{j,k} J_{jk} \psi_j \psi_k$$

$$\langle J_{ij}^2 \rangle = J^2/N$$

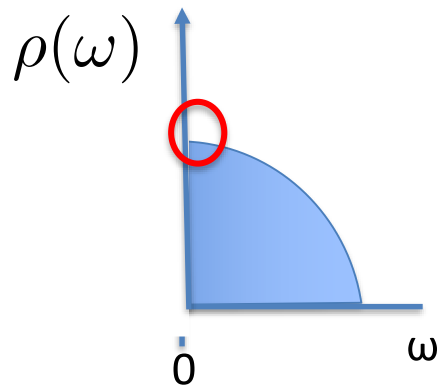
(no sum)

Random couplings, gaussian distribution.

To leading order → treat J as an additional field → same structure as before.

Solution of the special model

$$H = i \sum_{j,k} J_{jk} \psi_j \psi_k$$



Diagonalize J → semicircle distribution of energies.

Low energies → constant distributions → like a massless fermion on a circle of size N

Simple emergence of approximate scale invariance.

This model is too simple → no chaos, no black hole like-behavior.

The SYK model

N Majorana fermions $\{\psi_i, \psi_j\} = \delta_{ij}$

Sachdev Ye Kitaev
Georges, Parcollet

$$H = \sum_{i_1,\cdots,i_4} J_{i_1 i_2 i_3 i_4} \psi_{i_1} \psi_{i_2} \psi_{i_3} \psi_{i_4}$$

Random couplings, gaussian distribution. $\langle J^2_{i_1 i_2 i_3 i_4} \rangle = J^2/N^3$

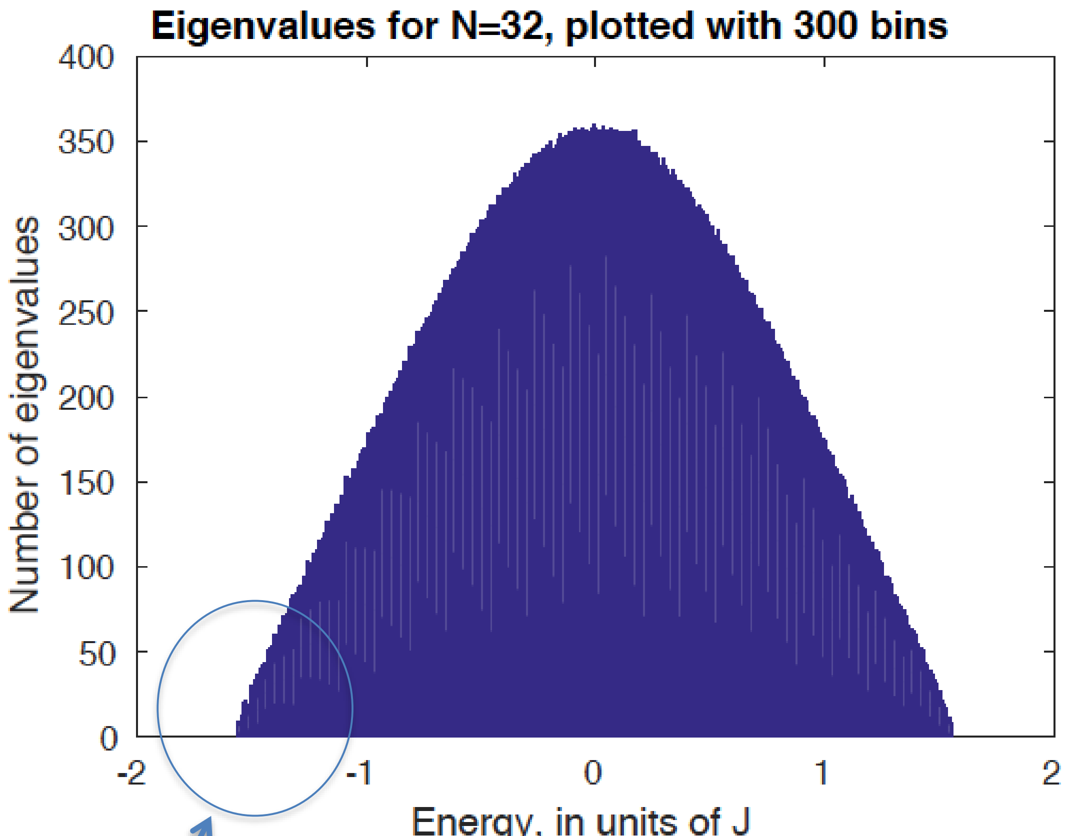
To leading order → treat J_{ijkl} as an additional field

J = dimensionful coupling. We will be interested in the strong coupling region

$$1 \ll \beta J, \ \tau J \ll N$$

Spectrum

D. Stanford



$$\dim_H = 2^{\frac{N}{2}}$$

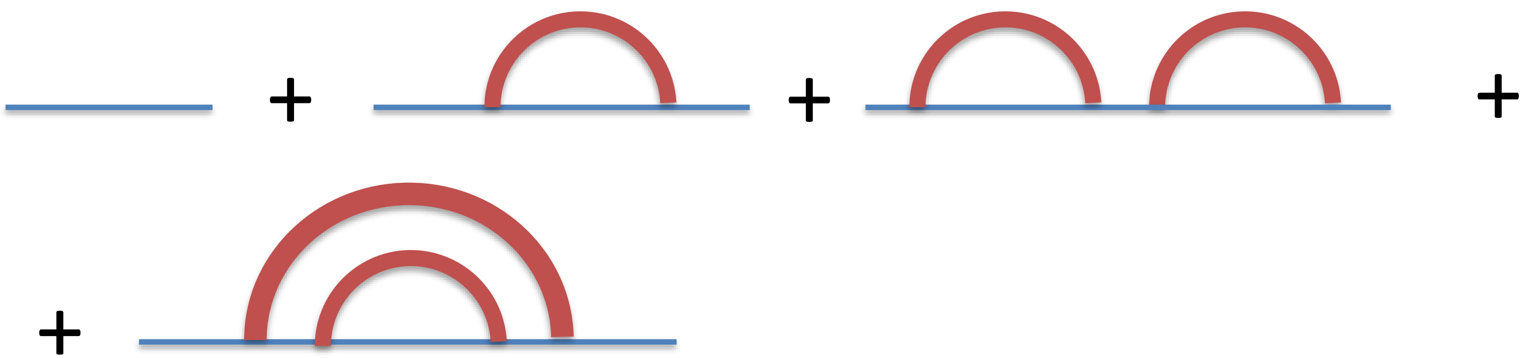
Number of random couplings $\propto N^4 \ll 2^N$

(specific, but random J's)

Exponentially large number of states contributes to the low energy region we consider

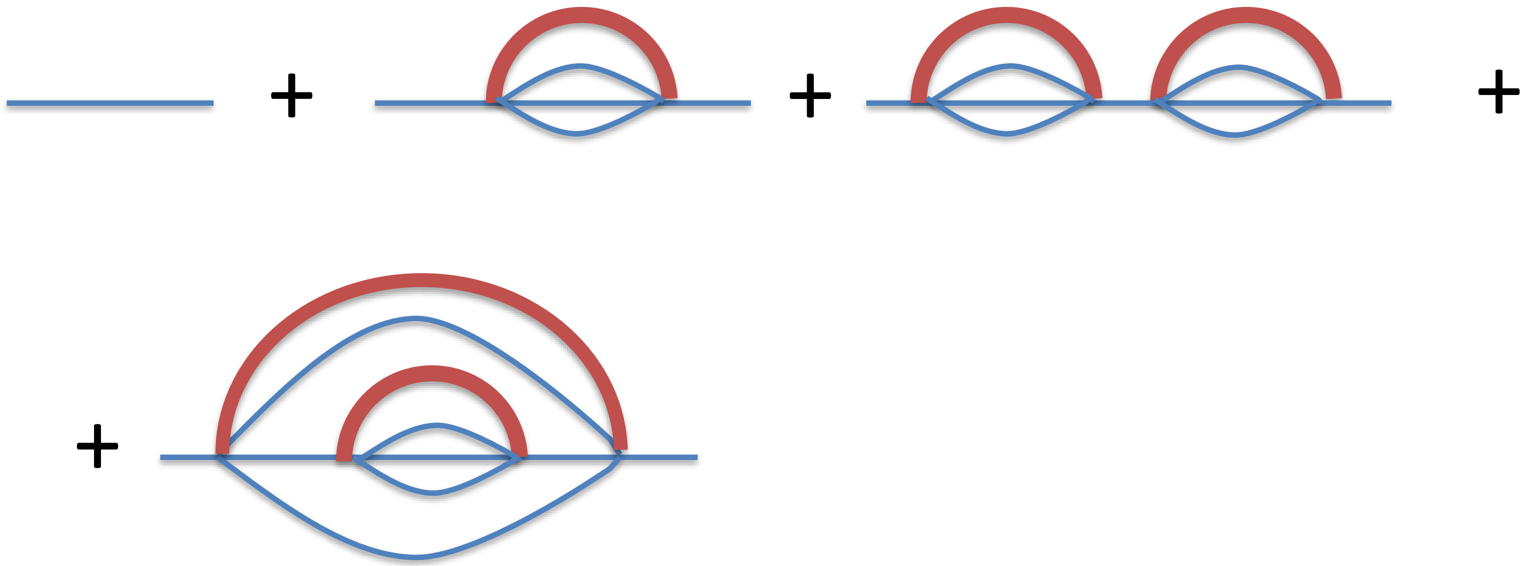
Large N limit

Before we had rainbows



Large N limit

Now:



$$\frac{1}{\mathbf{G}(\omega)} = \frac{1}{G_0} - \Sigma(\omega)$$

$$\Sigma(t,t') = J^2 \mathbf{G}(t,t')^3$$

Generalization:

$$H = i^{q/2} \sum_{i_1, i_2, \cdots, i_q} J_{i_1, i_2, \cdots, i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

$$\frac{1}{\mathbf{G}(\omega)} = \frac{1}{G_0} - \Sigma(\omega)$$

$$\Sigma(t,t') = J^2 \mathbf{G}(t,t')^{q-1}$$

q=2 → case we had before.

q=4 → SYK

q = Infinity → analytically solvable equations.

In the IR → Conformal symmetry

Make a scale invariant ansatz

$$\frac{1}{\mathbf{G}(\omega)} = \cancel{\frac{1}{G_0}} - \Sigma(\omega)$$

$$\Sigma(t,t') = J^2 \mathbf{G}(t,t')^{q-1}$$

$$G_c(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

is a solution if $\Delta = \frac{1}{q}$

If G is a solution, and we are given an arbitrary function f(τ), we can generate another solution:

$$G_c \longrightarrow G_{c,f}(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta} G_c(f(\tau), f(\tau'))$$

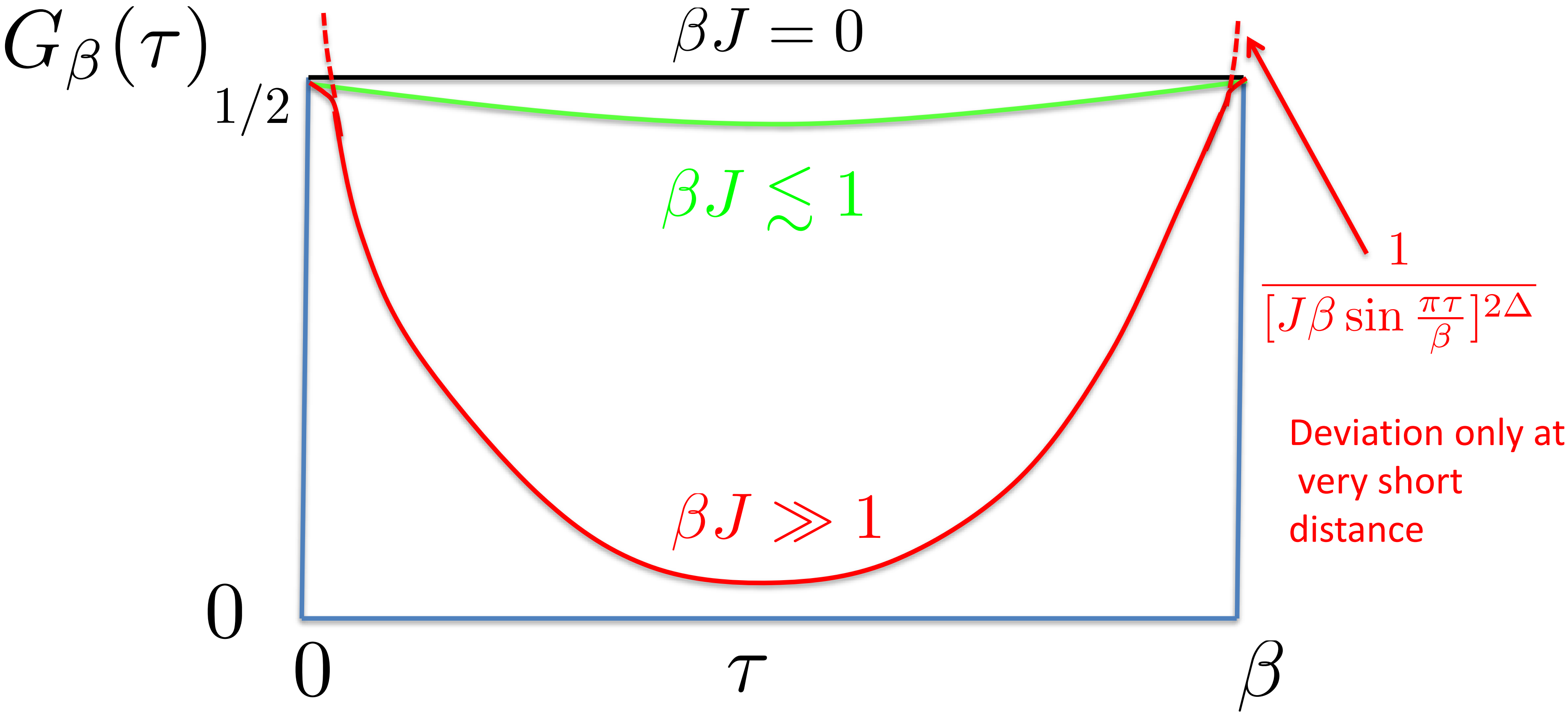
Example: Go from zero the temperature to a finite temperature solution

$$G(\tau, \tau') \propto \frac{1}{(\tau - \tau')^{2\Delta}}$$

$$f(\tau) = \tan \frac{\pi \tau}{\beta}$$

$$G_f = \left[\frac{\pi}{\beta \sin \frac{\pi \tau}{\beta}} \right]^{2\Delta}$$

General form of the propagator



Large N effective action

Integrate out the fermions and the couplings to obtain an effective action for the singlets, the fermion bilinears.

$$S = \frac{N}{2}\left[\log\det(\partial_t - \Sigma) - \int d\tau d\tau' \Sigma(\tau,\tau')G(\tau,\tau') + \frac{J^2}{4}G(\tau,\tau')^4\right]$$

Outline of the derivation

$$Z = \int dj \int D\psi \exp\{\int dt \left[i \int \psi^i \dot{\psi}^i + j_{lkmr} \psi^l \psi^k \psi^m \psi^l \right] - j_{lkmr}^2 N^3 / J^2\}$$

Integrate over j_{lkmr}

$$Z = \int d\psi \exp\{i \int dt \psi^l \dot{\psi}^l + N \int dt dt' [\frac{1}{N} \psi^l(t) \psi^l(t')]^4\}$$

Insert a 1

$$1 = \int DG \delta(G - \frac{1}{N} \psi^i(t) \psi^i(t')) = \int DG D\Sigma e^{i \int dt dt' \Sigma(t,t)(NG(t,t') - \psi^i(t) \psi^i(t'))}$$

Integrate out fermions

$$Z = \int DG D\Sigma \exp\{N \left[Pf(\partial_t - \Sigma) + \int dt dt' \left(G(t,t') \Sigma(t,t') + J^2 G(t,t')^4 \right) \right]\}$$

Large N effective action

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It is non-local in time. The bilocal terms come from the integral over the couplings.

This effective action is correct to leading orders, where we can ignore the replicas, $o(1/N^{q-1}$

Similar actions were obtained for usual O(N) vector models.

Equations of motion from this action are relatively simple integral equations that can be solved numerically.

At low energies the solution is simple $\qquad G_c(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}} \qquad \Delta = \frac{1}{4}$

It is scale invariant!

Scale vs conformal invariance

- Usually scale invariance → conformal invariance.
- In one dimensions: conformal invariance = full reparametrization symmetry.
- Is a symmetry of the low energy action

$$S = \frac{N}{2}\left[\log\det(\cancel{\partial_t} - \Sigma) - \int d\tau d\tau' \Sigma(\tau,\tau')G(\tau,\tau') + \frac{J^2}{4}G(\tau,\tau')^4\right]$$

If G is a solution, and we are given an arbitrary function f(τ), we can generate another solution:

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Emergent reparametrization symmetry

Example: Go from zero the temperature to a finite temperature solution

$$G(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

$$f(\tau) = \tan\frac{\pi\tau}{\beta}$$

$$G_f = \left[\frac{\pi}{\beta\sin\frac{\pi\tau}{\beta}}\right]^{2\Delta}$$

Zero modes of the action

Recall the conformal symmetry in the IR

$$G(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

$$G \longrightarrow G_f(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta} G(f(\tau), f(\tau'))$$

All these solutions have the same action in the strict IR limit.

Goldstone bosons → no action for f → would give a divergence if we do the path integra
over f.

Solution: remember that the symmetry is also slightly broken.

Nearly zero modes of the action

$$G(\tau,\tau') \propto \frac{1}{(\tau-\tau')^{2\Delta}}$$

$$G \longrightarrow G_f(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta} G(f(\tau), f(\tau'))$$

Deviations from the conformal solution were happening at short distances → expect that the effective action for f is local in time. We go from bilocal → local.

SL(2) invariance. $$f \to \frac{af+b}{cf+d}$$

Simplest action is ….

Schwarzian action

$$S = -\frac{N\alpha_s}{J} \int dt \, \mathrm{Sch}(f,t) \ , \qquad \mathrm{Sch}(f,t) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\frac{f''^2}{f'^2}$$

Numerical coefficient whose determination requires knowing the first deviation of the propagator from the IR conformal solution. Can be computed numerically.

ghosts ?

Thinking of SL(2) as a gauge symmetry → removes ghosts of the higher derivative action.
goldstones = coset = Reparametrizations/SL(2)

Low energy

$$\int \mathcal{D}G\mathcal{D}\Sigma e^{-S[G,\Sigma]} \to \int \mathcal{D}f e^{-S_f} \int \mathcal{D}'G\mathcal{D}'\Sigma e^{-S_{\mathrm{conf}}[G,\Sigma]}$$

Measure fixed by SL(2) symmetry

Bagrets, Altland, Kamenev
Stanford, Witten

Example: Four point function

disconnected

$$\langle \psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4)\rangle \propto$$

$$\int \mathcal{D}G\mathcal{D}\Sigma G(\tau_1,\tau_2)G(\tau_3,\tau_4)e^{-S[G,\Sigma]} = G_c(\tau_1,\tau_2)G_c(\tau_3,\tau_4) + \frac{1}{N}\frac{1}{S_2}$$

Inverse of the quadratic action. Since the leading conformal answer has zero modes, this is enhanced. The enhanced terms are given by the Schwarzian action

Enhancement factor

$$\langle 4pt\rangle \propto \int \mathcal{D}f G_{c,f}(\tau_1,\tau_2)G_{c,f}(\tau_3,\tau_4)e^{-S_f} = \frac{\beta J}{N} F\left(\frac{\tau_1}{\beta},\frac{\tau_2}{\beta},\frac{\tau_3}{\beta},\frac{\tau_4}{\beta}\right)$$

$$G_{c,f}(\tau,\tau') = [f'(\tau)f'(\tau')]^{\Delta}[f(\tau)-f(\tau')]^{-2\Delta}$$

Four point function

$$\frac{\langle \psi_i(\tau_1)\psi_i(\tau_2)\psi_j(\tau_3)\psi_j(\tau_4)\rangle}{\langle \psi_i(\tau_1)\psi_i(\tau_2)\rangle\langle \psi_j(\tau_3)\psi_j(\tau_4)\rangle} = 1 + \frac{\beta J}{N} F\left(\frac{\tau_i}{\beta}\right)$$

We can use this to compute lorentzian four point functions by analytic continuation.

Different analytic continuations → different orders in Lorentzian signature.

Of particular interest is to compute the out of time order correlator that is responsible for the growth of commutators.

Shenker, Stanford, Kitaev

$$\frac{\langle \psi_i(0)\psi_j(\tau)\psi_i(0)\psi_j(\tau)\rangle}{\langle \psi_i(0)\psi_i(0)\rangle\langle \psi_j(\tau)\psi_j(\tau)\rangle} \propto 1 + i\frac{\beta J}{N} e^{\frac{2\pi\tau}{\beta}}$$

Exponential growth
Saturating chaos bound

JM, Shenker, Stanford

Full four point function

Can be computed by summing some ladder diagrams and using the conformal symmetry, after removing the Schwarzian contribution.

$$\langle 4pt \rangle \propto \frac{1}{N}\left[\beta J F\left(\frac{\tau_i}{\beta}\right) + \tilde{F}\left(\frac{\tau_i}{\beta}\right) + \frac{1}{(\sin\frac{\pi\tau_{12}}{\beta}\sin\frac{\pi\tau_{34}}{\beta})^{2\Delta}} H(\chi)\right]$$

Conformal invariant part → contains information about the operator spectrum.

Anomalous dimensions of higher spin fields are of order one.

$$\psi_i \partial^{1+2m}\psi_i \quad \to \quad h_m = 2\Delta + 1 + 2m + \gamma_m$$

$$1 = -(q-1)\frac{\Gamma(\frac{3}{2}-\frac{1}{q})\Gamma(1-\frac{1}{q})}{\Gamma(\frac{1}{2}+\frac{1}{q})\Gamma(\frac{1}{q})}\frac{\Gamma(\frac{1}{q}+\frac{h}{2})}{\Gamma(\frac{3}{2}-\frac{1}{q}-\frac{h}{2})}\frac{\Gamma(\frac{1}{2}+\frac{1}{q}-\frac{h}{2})}{\Gamma(1-\frac{1}{q}+\frac{h}{2})}$$

Comparison with previous conformal quantum mechanics

$$\int dt(\dot{X}^2 + g/X^2)$$

De Alfaro, Fubini, Furlan

Michelson, Strominger, ….

Exact SL(2) symmetry acting on the dynamical variables.

No SL(2) invariant ground state.

Under a reparametrization the action changes as

$$\Delta S = \int d\tau X^2 \mathrm{Sch}(\mathrm{f}, \tau)$$

Different pattern of symmetry realization.

Is OK to describe brane probes in AdS_2, but does not seem to capture gravitational features properly.

Reparametrization symmetry

- SYK and AdS_2 both have an emergent, spontaneously broken and explicitly broken reparametrization symmetry.
- The spontaneous breaking, and the explicit breaking, both preserves an SL(2) gauge-like symmetry.

Questions

- The discussion was mostly through the Euclidean path integral.
- How should we think about this approximate symmetry in a Hilbert space context ?
- Is there a Virasoro <u>algebra</u> ?
- Is there any "central charge" to be computed ?

Reparametrizations in CFT_2

- In a CFT_2 we also have holomorphic reparametrizations
- $$z \to f(z), \;\; \bar{z} \to \bar{f}(\bar{z})$$
- Spontaneously broken by the vacuum to just $SL(2)^2$

- Goldstones: modes created by the stress tensor operator. They have a non-zero action consistent with conformal symmetry.
- Symmetry is not explicitly broken
- Symmetry algebra is deformed by the central charge (viewed as an operator with an expectation value = c) → Virasoro algebra.
- Only in the case that all other $\Delta \gg 1$ it dominates.

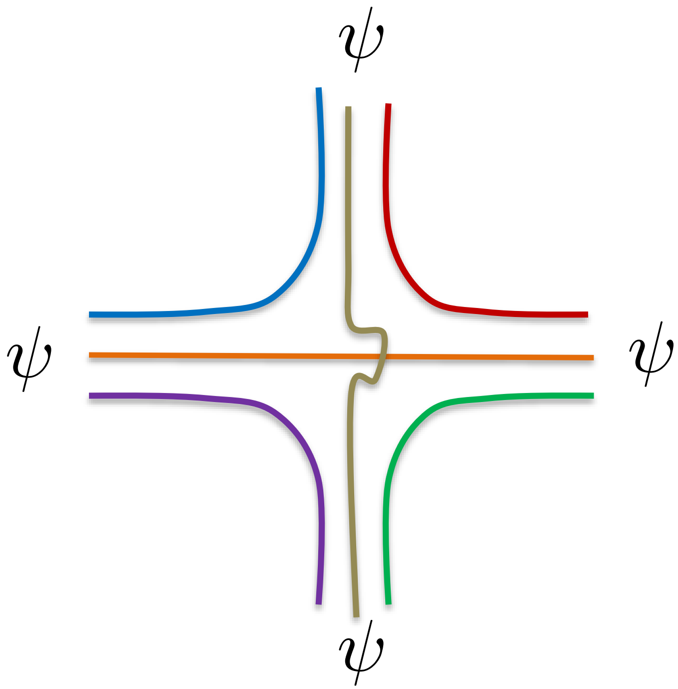
Version without disorder

Witten

Fields have three indices. 6 types of indices and each field has three of them

$$\psi_{ijk}$$

Gurau, Rivasseau, etc.



The interaction vertex has the indices contracted as above.

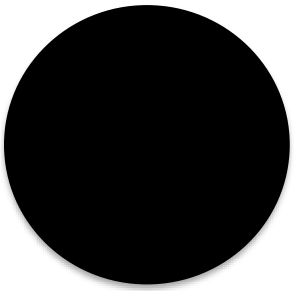
Same leading order diagrams as SYK. Different subleading order diagrams.

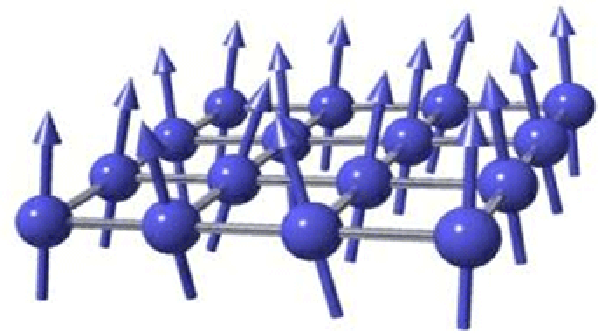
$$n^3 + n + \cdots$$

Conclusions

- SYK model is a simple solvable model (in the large N limit)
- It is strongly interacting and maximally chaotic
- Interesting nearly-conformal symmetry at relatively low energies.

Near extermal black holes





(Not a field theory.)

Extremal black hole $M \geq Q$ $M^2 \geq J$

Low energies, near horizon

AdS_2 region

Conformal quantum mechanics?

Conformal symmetry in quantum mechanics in a finite Hilbert space

- No go:

A. Strominger…

Density of states, scale invariance:

$$\rho(E) \propto \frac{1}{E}\ , \quad \text{or} \quad \delta(E) \qquad ?$$

Either divergent in IR or no dynamics.

Gravity in two dimensions

- No go:

Naïve two dimensional gravity :

$$\int \sqrt{g}(R - 2\Lambda) + S_M$$

Einstein term topological → no contribution to equations of motion.
Equations of motion → set stress tensor to zero.

No dynamics !

OK for extremal entropy.

Work by Ashok Sen and collaborators

Nearly AdS_2

Keep the leading effects that perturb away from AdS_2

Teitelboim Jackiw
Almheiri Polchinski

$$\int d^2x\sqrt{g}\phi(R+2) + \phi_0 \int d^2x\sqrt{g}R$$

Ground state entropy

Comes from the area of the additional dimensions, if we are getting this from 4 d gravity for a near extremal black hole.

$$\int \sqrt{g}\phi(R+2)$$

Equation of motion for ϕ → metric is AdS_2

Equation of motion for the metric → phi is almost completely fixed

$$ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2$$

$$\phi = \phi_h \cosh \rho$$

Value at the horizon

Position of the horizon.

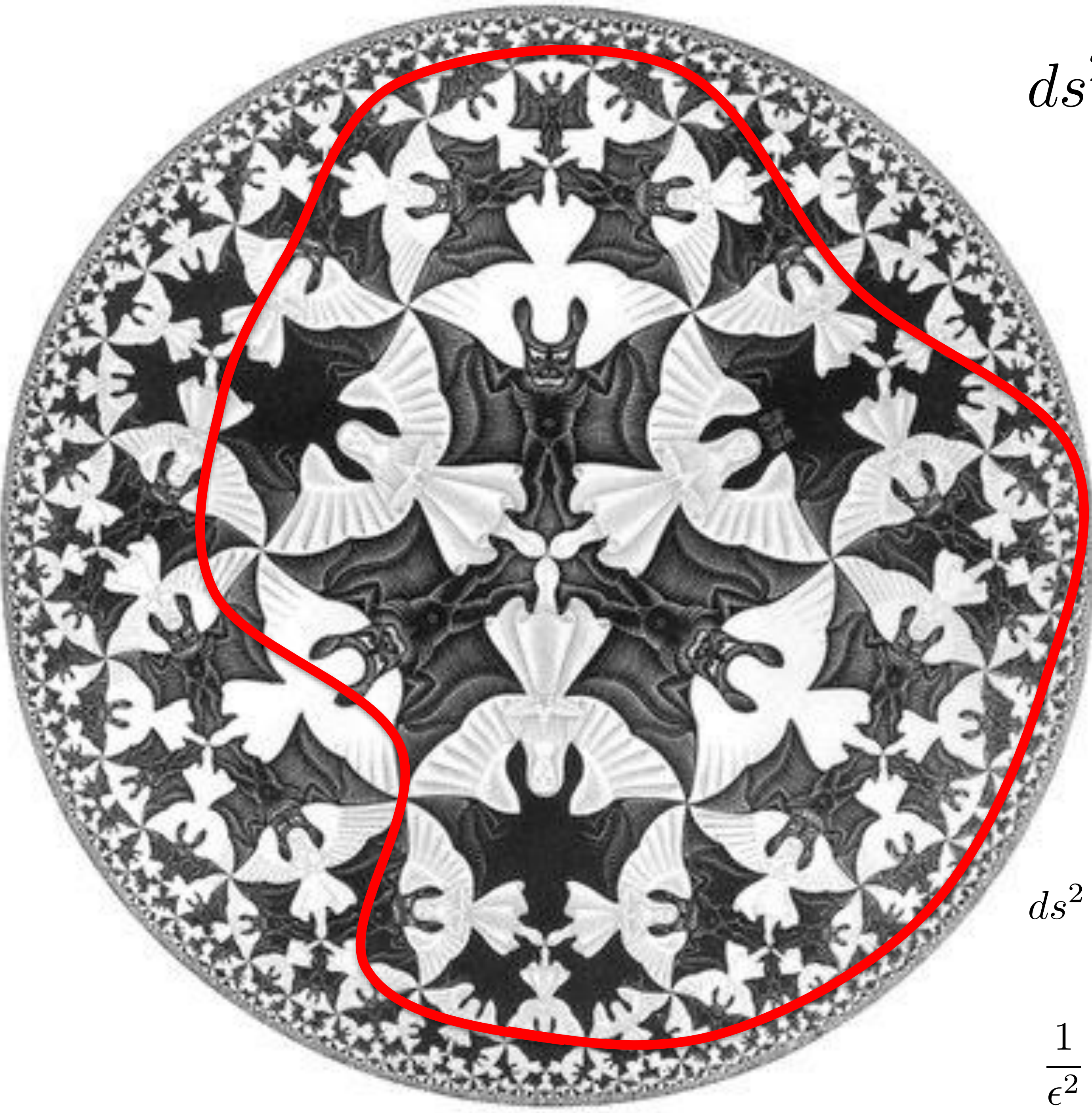
In the full theory: when ϕ is sufficiently large → change to a new UV theory

Asymptotic boundary conditions:

$$ds^2|_{Bdy} = \frac{1}{\epsilon^2} du^2$$

Fixed proper length

$$\phi|_{Bdy} = \frac{1}{\epsilon} \phi_r(u)$$



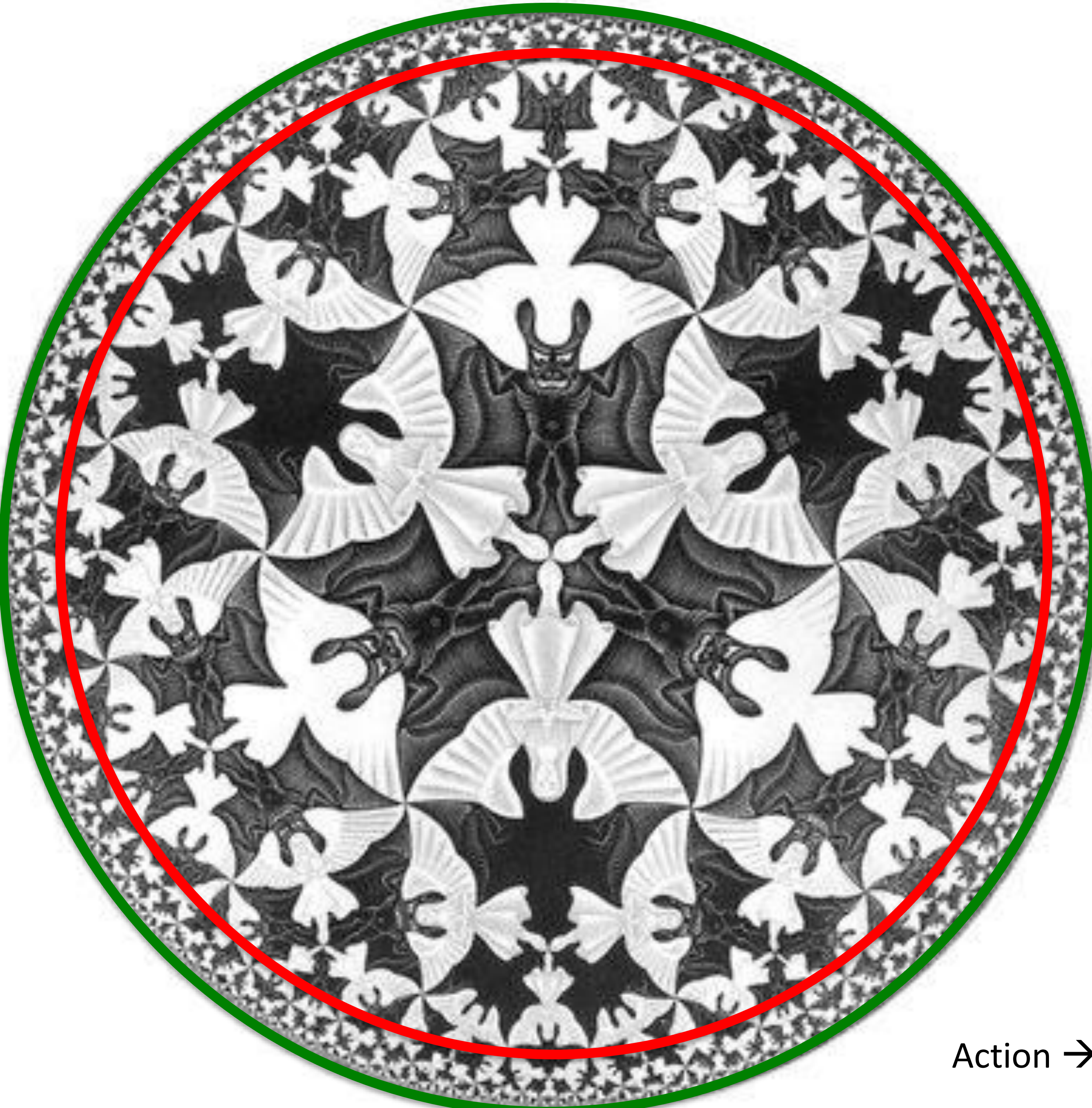
$$ds^2|_{Bdy} = \frac{1}{\epsilon^2} du^2$$

Infinite number of solutions.

$$ds^2 = d\rho^2 + \sinh^2 \rho d\tau^2$$

$$\rho(\tau)$$

$$\frac{1}{\epsilon^2} = (\rho'^2 + \sinh^2 \rho) \left(\frac{d\tau}{du}\right)^2$$



$$ds^2|_{Bdy} = \frac{1}{\epsilon^2} du^2$$

$$\phi|_{Bdy} = \frac{1}{\epsilon} \phi_r(u)$$

One one solution

Action → related to Schwarzian

$$S = \int d^2x \sqrt{g}\phi(R+2) - 2\int \frac{\phi_r(u)}{\epsilon^2} du K \to$$

$$S = \frac{1}{\epsilon^2} - \int du \phi_r(u) Sch(t, u)$$

$$t(u)$$

t = Usual AdS_2 time coordinate
u = Boundary system (quantum mechanical) time coordinate

Conclusions

- Nearly AdS_2 gravity is very simple.
- Fields propagate on a rigid AdS_2 space
- Gravitational effects are related to the position of the boundary → boundary graviton.
- Simple quantum mechanical degree of freedom.
- Same pattern of symmetry breaking as the SYK model.

Some qualitative relations

$$G_c(t_1, t_2) \propto \frac{1}{|t_1 - t_2|^{2\Delta}}$$

Background AdS_2 metric.

Both SL(2) invariant

Non-zero mode perturbations of G

Fields propagating on AdS_2

Nearly zero modes → t = f(u) , u is physical time , t = time set by the correlators, internal time.

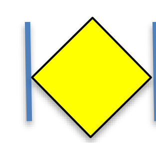
Gravitational interactions , via dilaton gravity → reduce to the same Schwarzian action. t = AdS_2 coordinate time, u = boundary proper time.

$$S[G, \Sigma]$$

$$S = S_{\rm dil.grav.} + S_{\rm matter}$$

$$G_c(t_L, t_R)$$

Wormhole or WdW patch of AdS_2



SYK model

Near extremal black holes

Low energies

Nearly AdS_2 gravity

$\underbrace{\text{Conformal invariant part}} + \underbrace{\text{reparametrizations}}$

$\underbrace{\text{QFT on } AdS_2} + \underbrace{\text{boundary dynamics}}$

Not the same

same

$$S = -C \int du \{ f(u), u \}$$

Schwarzian action

Boundary gravitons

Kitaev
JM, Stanford
Zhang, Suh

- Low temperature entropy
- Chaos exponent
- Wormhole traversability (location of horizon)

Emergent reparametrization symmetry which is spontaneously and explicitly broken

Properties fixed by the Schwarzian

- Common to $NAdS_2$ and in $NCFT_1$ (SYK, for example).
- Temperature dependence of the free energy $$S \propto \frac{N}{\beta J}$$
- Part of the four point function that comes from the explicit conformal symmetry breaking. This part leads to a chaos-like behavior with maximal growth in the commutator.

$$\text{growth of commutators} \sim \frac{1}{N}(\beta J)e^{2\pi t/\beta}$$

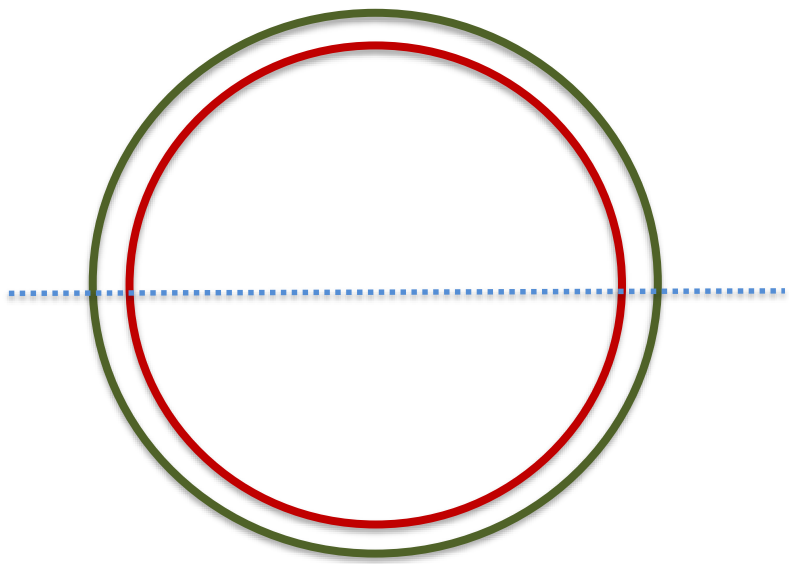
Kitaev

- Wormhole becomes traversable as we add a double trace interaction linking the two sides,

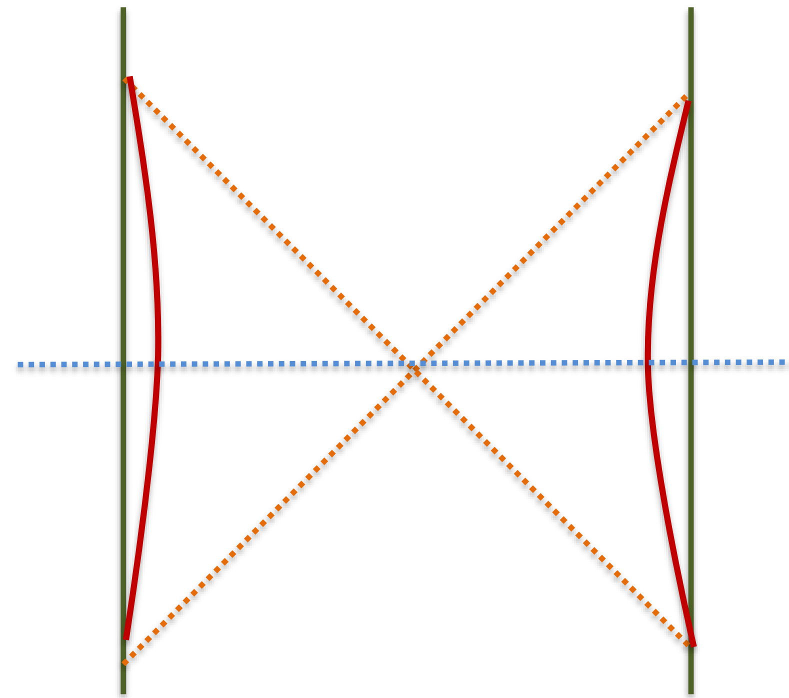
$$\int dt g(t) O_L(t) O_R(t)$$

Gao, Jafferis, Wall

Entangled states



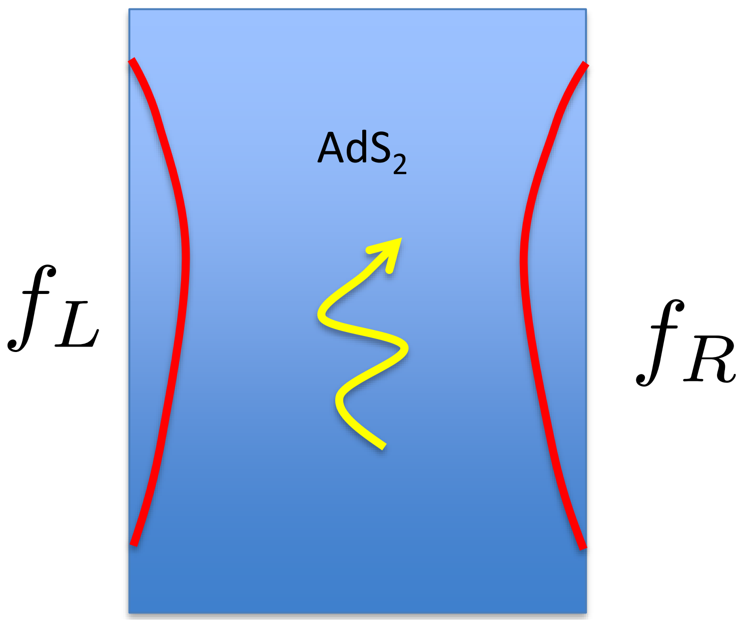
Euclidean black hole



Kruskal Schwarzschild AdS_2 wormhole

Thermofield double: $$|\Psi\rangle = \sum_n e^{-\beta E_n/2} |\bar{E}_n\rangle_L \times |E_n\rangle_R$$

Gravitational dynamics



$$\int \phi(R+2)$$

Rigid AdS$_2$

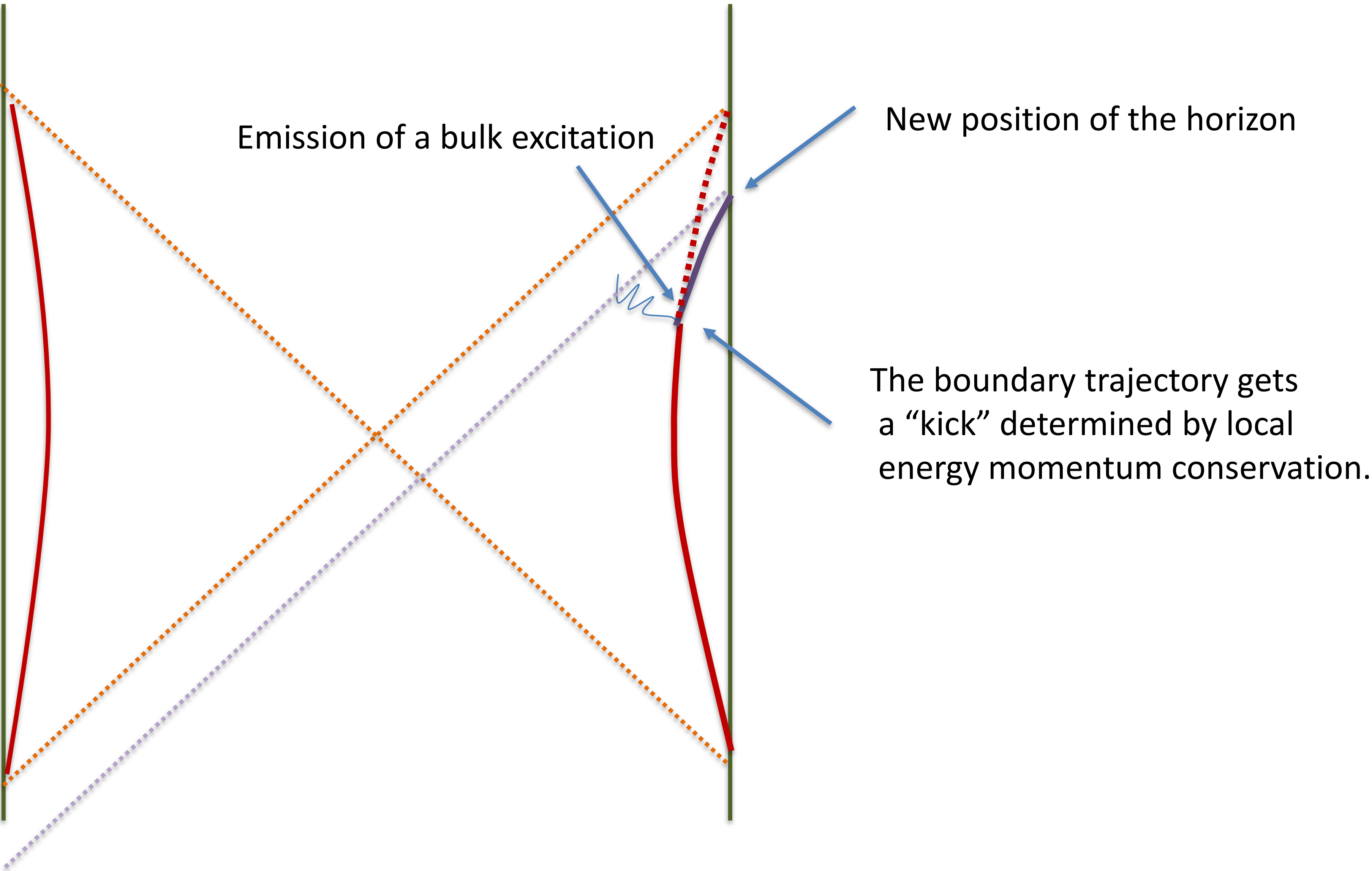
Physical boundary given by dilaton

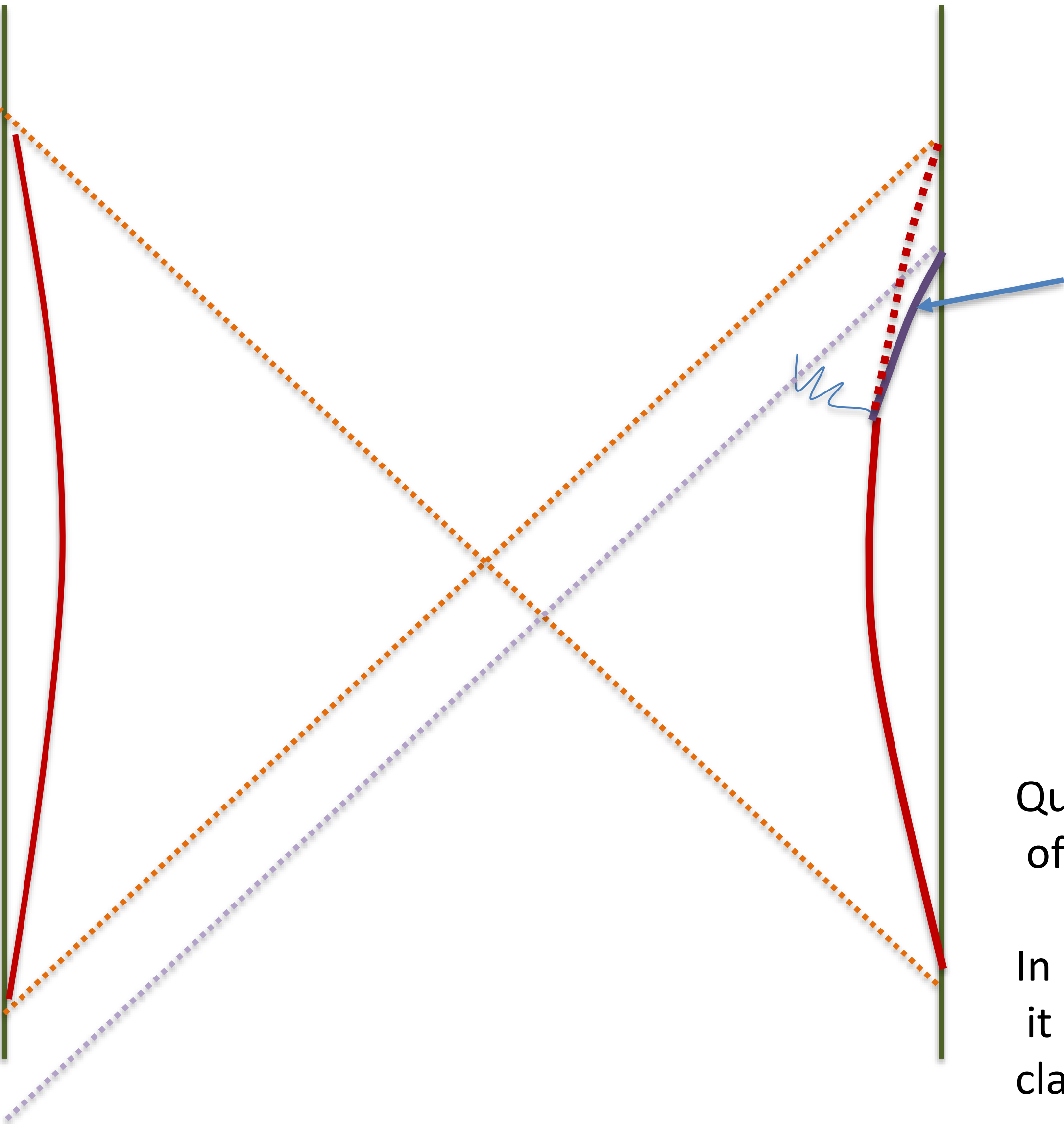
Dynamics is in the position of the boundary.

Boundary graviton: encodes the motion of the boundary.

$$(H_{f_L} \times H_{\rm bulk} \times H_{f_R})/SL(2,R)$$

Dynamics


Emission of a bulk excitation
New position of the horizon
The boundary trajectory gets
a “kick” determined by local
energy momentum conservation.



New trajectory diverges exponentially from the previous one

$$e^{\lambda t} = e^{2\pi T t}$$

This motion can be detected by OTOC and is directly related to the chaos exponent.

Quantum chaos = simple motion of a particle in AdS_2 , it is geometric.

In both the SYK model and gravity, it results from the motion of an essentially classical variable ! ~ motion in hyperbolic space.

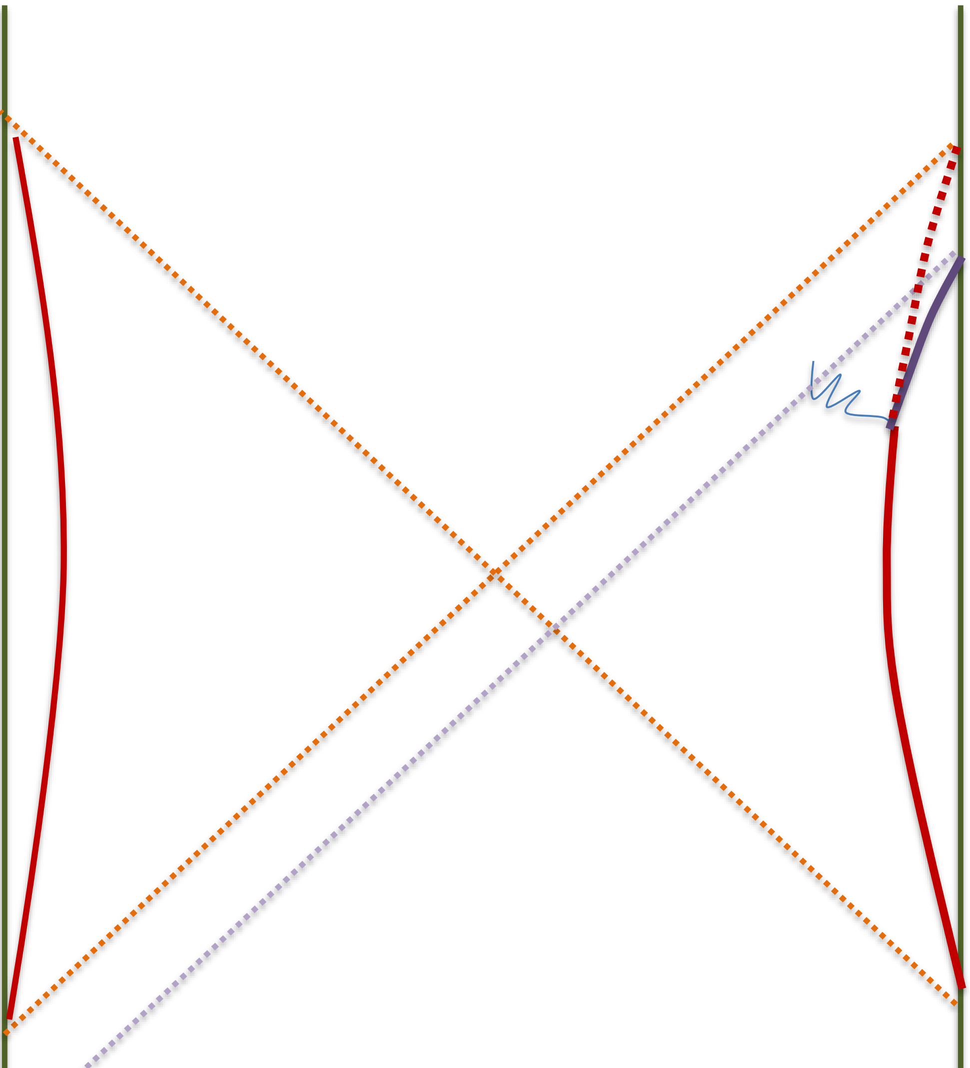
Quantum chaos from classical chaos

- The growth of out of time order correlators is related to the motion of a classical system.
- The one described by the Schwarzian action.
- Or the motion of the boundary particle.
- Roughly like motion in hyperbolic space : chaos from a geometric origin $\rightarrow$ structure of SL(2). Automatically maximal.

- The structure of the bulk is fixed and rigid. The boundary particle motion governs how this IR Hilbert space is embedded in the full exact Hilbert space. The same happens in SYK. The structure of the conformal solution is fixed and rigid, but the Schwarzian degree of freedom governs its precise embedding.
- Similar to hydrodynamics, where the fluid is locally the same but could be moving differently relative to the ambient space. Conservation of energy.

Entanglement and teleportation

No signals from one side to the other



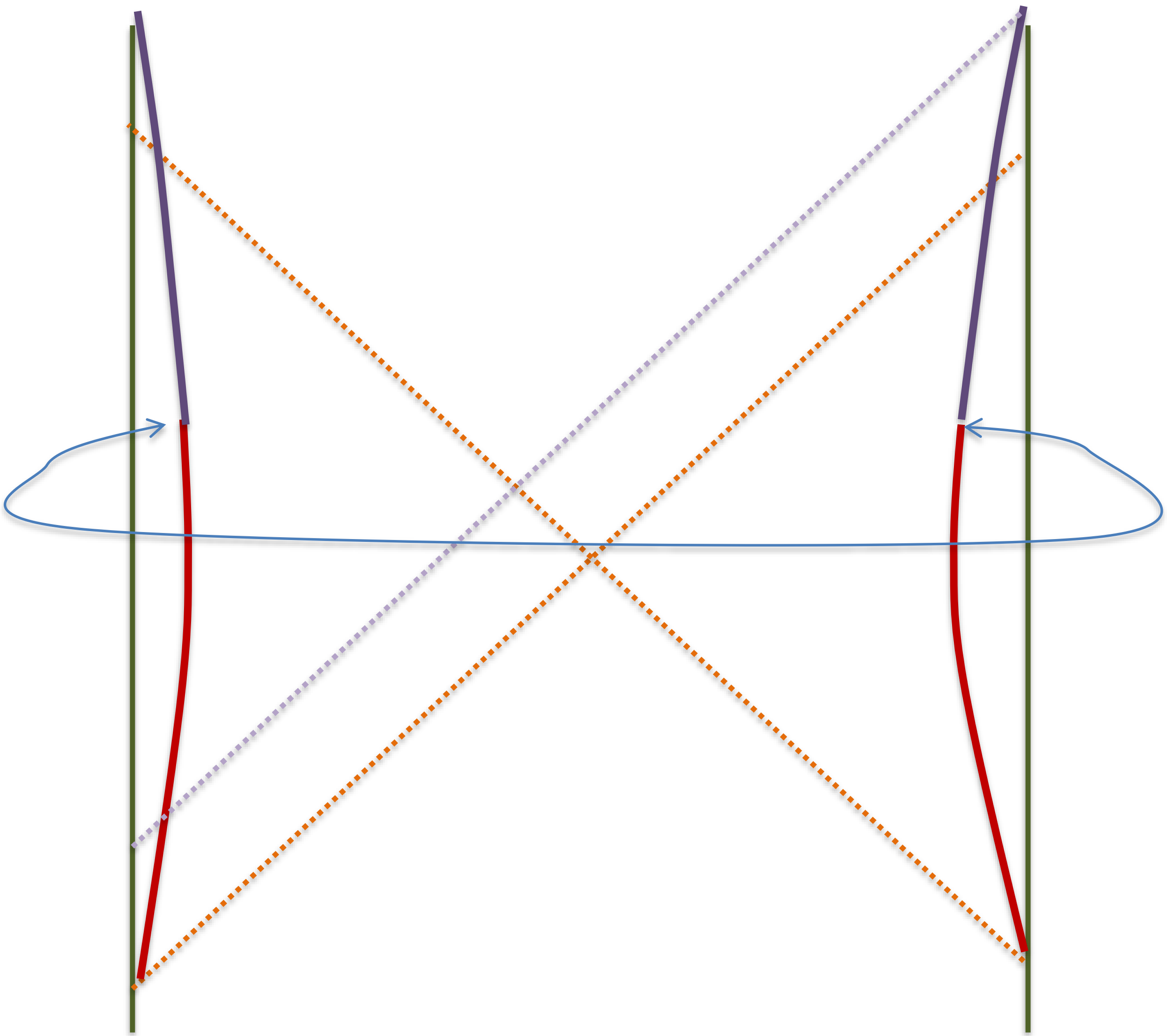
$$|\Psi\rangle = \sum_n e^{-\beta E_n/2} |\bar{E}_n\rangle_L \times |E_n\rangle_R$$

Kicks are always "outwards" →
no signal from one boundary to the other.

Consistent with entanglement.

Interaction between the two boundaries

Gao Jafferis Wall
(Susskind)



Insert this in the path integral

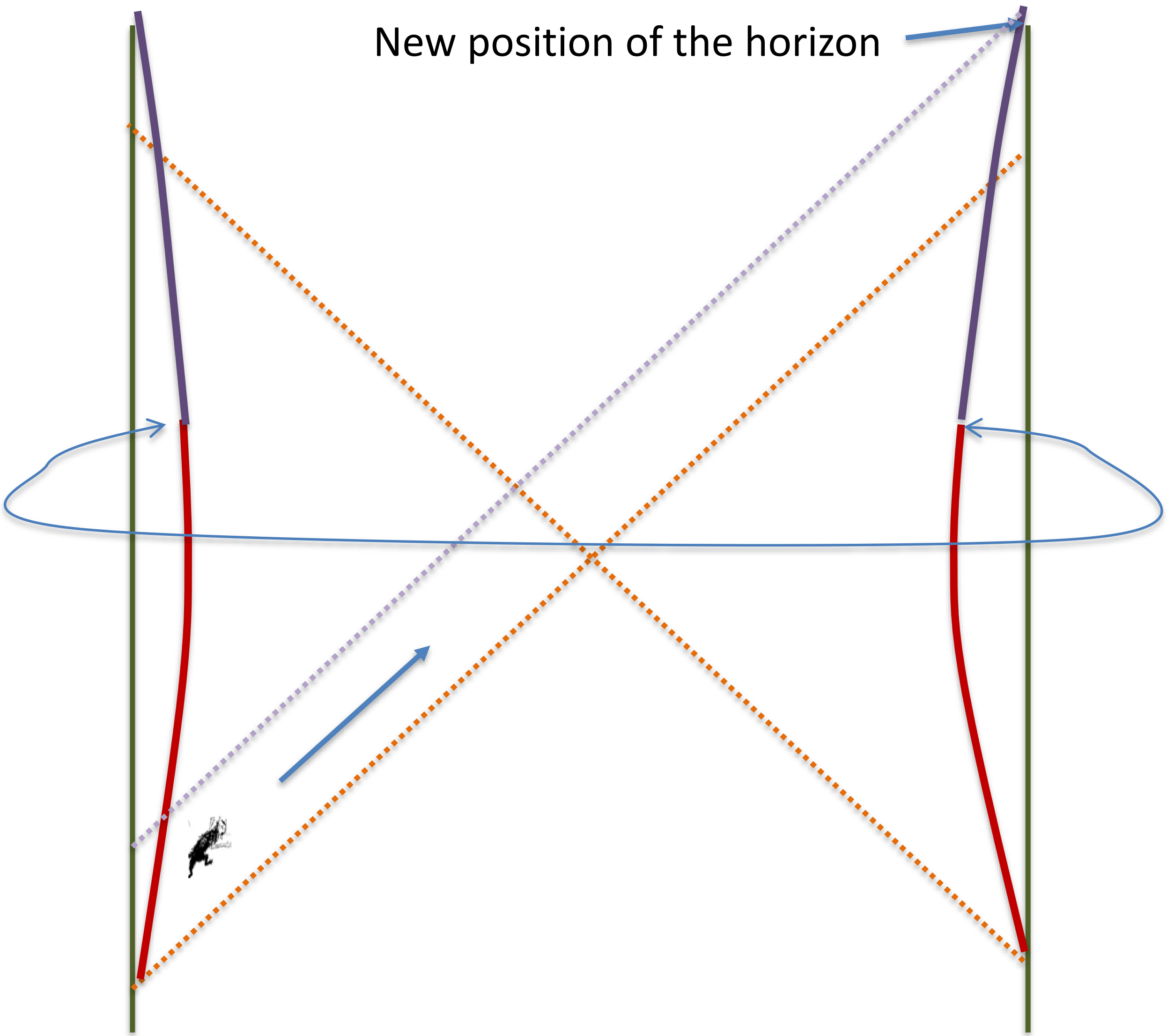
$$e^{ig\phi_L(t_L)\phi_R(t_R)}$$

approximate

$$e^{ig\langle\phi_L(t_L)\phi_R(t_R)\rangle}$$

Force between the two boundaries.
(Can be attractive for the right sign of g).
kicks the trajectories inwards

Interaction makes the wormhole traversable



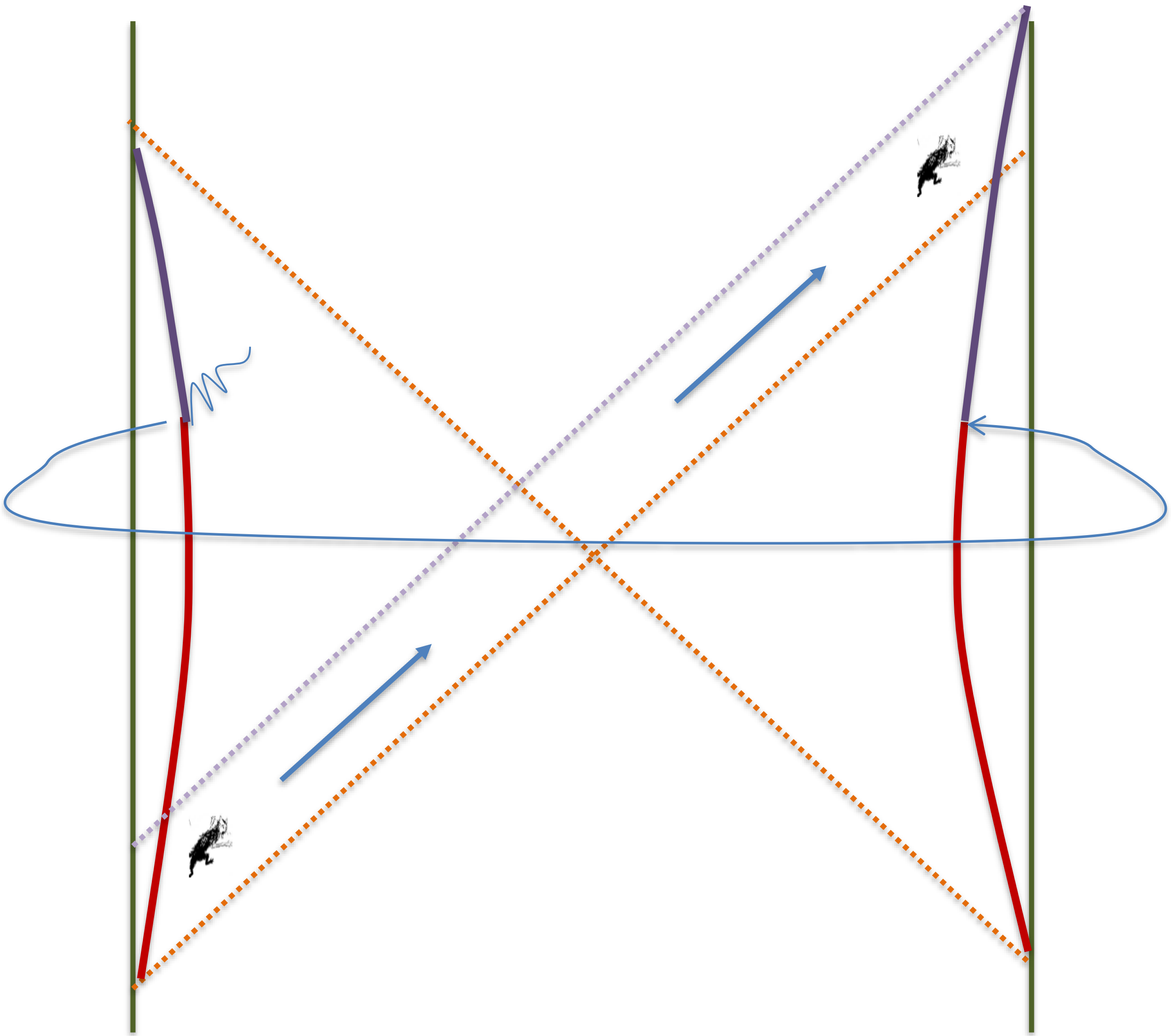
We can now send a signal
from the left to the right.

The wormhole has been
rendered traversable.

No contradiction because
we had a non-local interaction
between the two boundaries.

The point is not that it
we can send signals.
It is how signals get sent and
what they feel !

Quantum teleportation though the wormhole



$$e^{ig\phi_L(t_L)\phi_R(t_R)}$$

Measure $\phi_L \longrightarrow \sigma_L$

Act on the right with

$$e^{ig\sigma_L\phi_R(t_R)}$$

From the point of view of the right we get the same, whether we measure or not.

Conclusions

- Simple picture for teleportation in the gravity picture.
- Teleportation through the wormhole.

- What is the deep interior ?