ENTANGLEMENT RELATIONS & BULK LOCALITY

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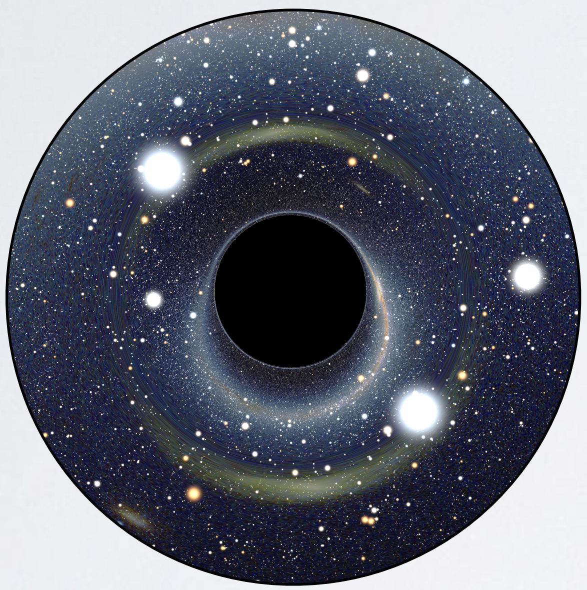
AdS/CFT at 20 and Beyond

AdS/CFT at 20

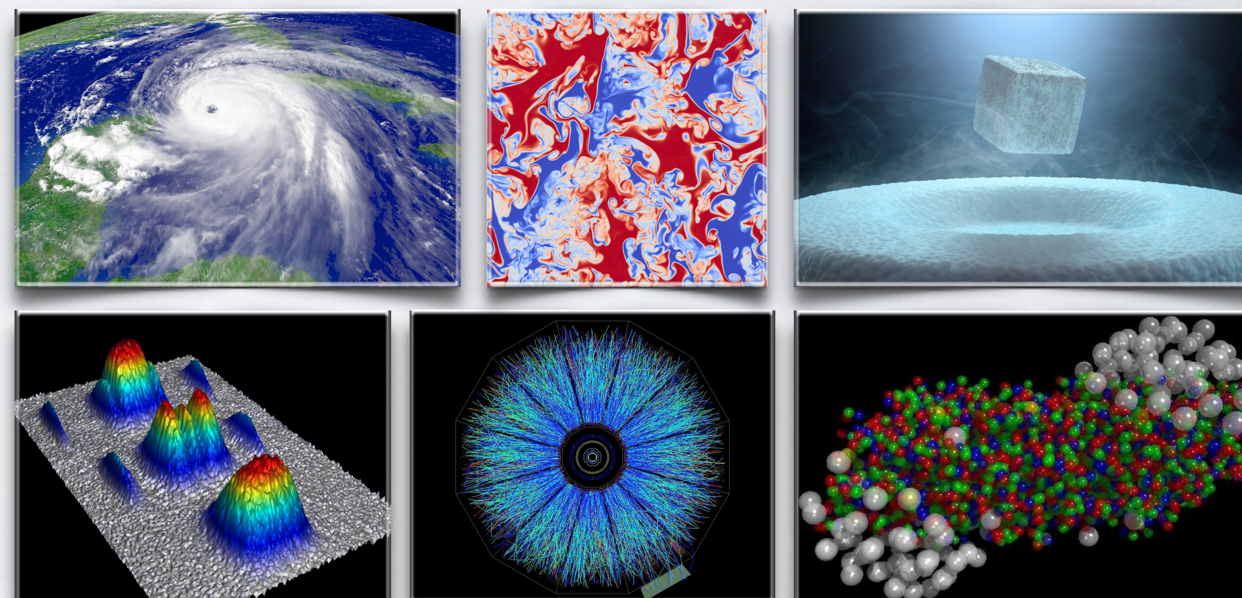
String theory (gravity) $\Longleftrightarrow$ field theory (no gravity)

"in bulk" = higher dimensions — *"on boundary" = lower dimensions*

describes gravitating systems, e.g. black holes



describes experimentally accessible systems



Invaluable tool to:

- Study strongly interacting field theory (hard, but describes many systems) by working with higher-dimensional gravity on AdS (easy).
- Study quantum gravity in AdS (hard, but needed to understand spacetime) by using the field theory (easy for certain things)

Pre-requisite:

We need to understand the AdS/CFT dictionary…

- How does bulk spacetime emerge from the CFT?
  - Which CFT quantities give the bulk metric?
  - What determines bulk dynamics (Einstein's eq.)?
  - How does one recover a local bulk operator from CFT quantities?
- What part of bulk can we recover from a restricted CFT info?
  - What bulk region does a CFT state (at a given instant in time) encode?
  - What bulk region does a spatial subregion of CFT state encode?
- (How) does the CFT "see" inside a black hole?
  - Does it unitarily describe black hole formation & evaporation process?
  - How does it resolve curvature singularities?

Recent hints / expectations: entanglement plays a crucial role…

Entanglement Entropy (EE)

Suppose we can divide a quantum system into a subsystem A and its complement B, such that the Hilbert space decomposes:

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Suppose we only have access to a subsystem A of the full system = A + B. The amount of entanglement is characterized by Entanglement Entropy S_A :

- reduced density matrix $\rho_A = \mathrm{Tr}_B\, |\psi\rangle\langle\psi|$

  (more generally, for a mixed total state, $\rho_A = \mathrm{Tr}_B \rho$)

- EE = von Neumann entropy $S_A = -\mathrm{Tr}\, \rho_A \log \rho_A \equiv S(A)$

Entanglement relations

- Sub-additivity (SA) $S(A) + S(B) \geq S(AB)$
  - Mutual information $I(A:B) \equiv S(A) + S(B) - S(AB) \geq 0$
  - ~ non-negative amount of correlation (qtm. & classical) betw. A & B

SA & SSA holds universally for any quantum system

- Strong Subadditivity (SSA) $S(AB) + S(BC) \geq S(B) + S(ABC)$
  - Conditional mutual information $I(A:C|B) \equiv I(A:BC) - I(A:B) \geq 0$
  - ~ amount of correlation is monotonic under inclusion

MMI does NOT hold universally…

- Monogamy of Mutual Information (MMI)

$$S(AB) + S(BC) + S(CA) \geq S(A) + S(B) + S(C) + S(ABC)$$

  - Tripartite information $I3(A:B:C) \equiv I(A:B) + I(A:C) - I(A:BC) \leq 0$
  - ~ super-additivity of mutual information

Entanglement monogamy

- But MMI does hold for genuine quantum entanglement
  - (as opposed to classical correlation)
  - since entanglement between A and B cannot be shared by C

- ⇝ represent entanglement by a thread connecting A & B
  - automatically implements monogamy
  - classical correlations could nevertheless emerge naturally…

    cf. `micro-equilibration' of entanglement [VH, Rota]:

- MMI holds for holographic states w/ geom. dual [Hayden, Headrick, Maloney]
  - ⇝ quantum entanglement dominates over classical correlation
  - ⇝ threads weave the fabric of bulk spacetime ?

Cf. "geometry from entanglement" [Van Raamsdonk], "ER=EPR" [Maldacena & Susskind], tensor networks [Swingle, Takayanagi, …]

Motivation

- Elucidate holography
  - Structure/characterization of CFTs (& states) w/ gravity dual
  - Fundamental nature of spacetime & its relation to entanglement:
  - (How) does entanglement "build" geometry?
- Bulk geometrizes entanglement relations…
  - e.g. SA, SSA, MMI very easy to prove geometrically
- …but simultaneously obscures their fundamental differences:
  - i.e. SSA is true for all quantum states, while MMI is not.
  - Why does MMI and SSA "look the same" from bulk standpoint?
- Understanding the distinction would elucidate the essence of HEE prescriptions & emergence of bulk spacetime.

Preview

- Q: Why does MMI and SSA "look the same" from the bulk?
  - Only $O(N^2)$
  - in geometric proof we can "cancel surfaces"
  - But is that the full story? Or is there a distinction already at the geometric level?
- Hint from "Bit Thread" description of HEE:
  - SSA trivial
  - MMI hitherto eluded proof
  - But follows easily from assumption of 'cooperative' flows [M. Headrick]
- *Q*: What guarantees existence of cooperative flows?
- *A*: Bulk locality…

OUTLINE

- Motivation & Preview
- Background
  - SSA & MMI proof using RT
  - Bit Threads
  - SSA proof using bit threads
- MMI proof using bit threads
  - Basic argument
  - Cooperative flow construction
- Generalizations
- Summary & Open questions

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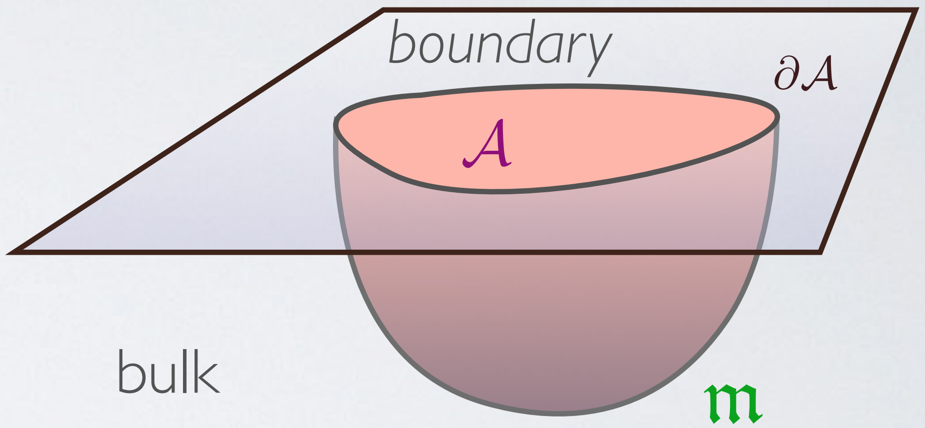
Holographic Entanglement Entropy

Proposal [RT=Ryu & Takayanagi, '06] for *static* configurations:

In the bulk, entanglement entropy $S_{\mathcal{A}}$ for a boundary region $\mathcal{A}$ is captured by the area of a minimal co-dimension-2 bulk surface $\mathfrak{m}$ *at constant t* anchored on entangling surface $\partial\mathcal{A}$ & homologous to $\mathcal{A}$

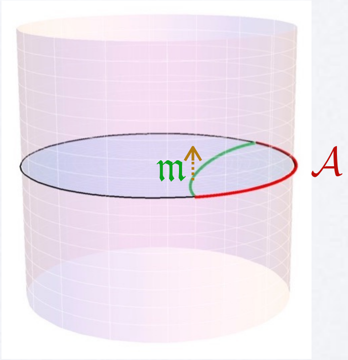


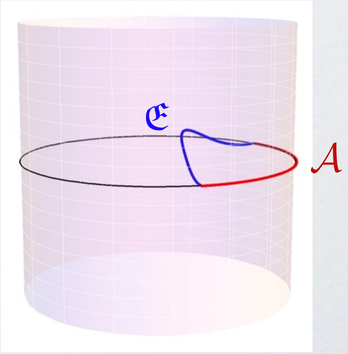
$$S_{\mathcal{A}} = \min_{\partial\mathfrak{m}=\partial\mathcal{A}} \frac{\text{Area}(\mathfrak{m})}{4\,G_N}$$

In *time-dependent* situations, RT prescription needs to be covariantized:

[HRT = VH, Rangamani, Takayanagi '07]

minimal surface $\mathfrak{m}$ at constant time → extremal surface $\mathfrak{E}$ in the full bulk

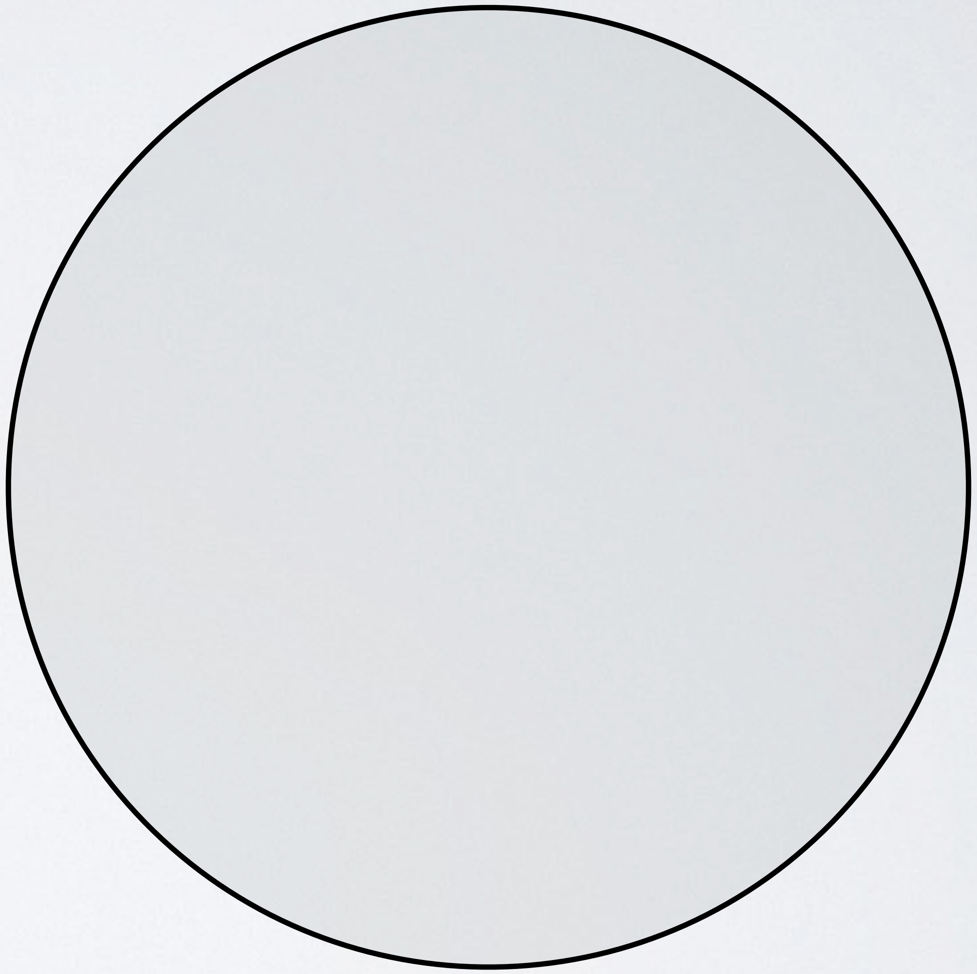




This gives a well-defined quantity in any (arbitrarily time-dependent asymptotically AdS) spacetime.

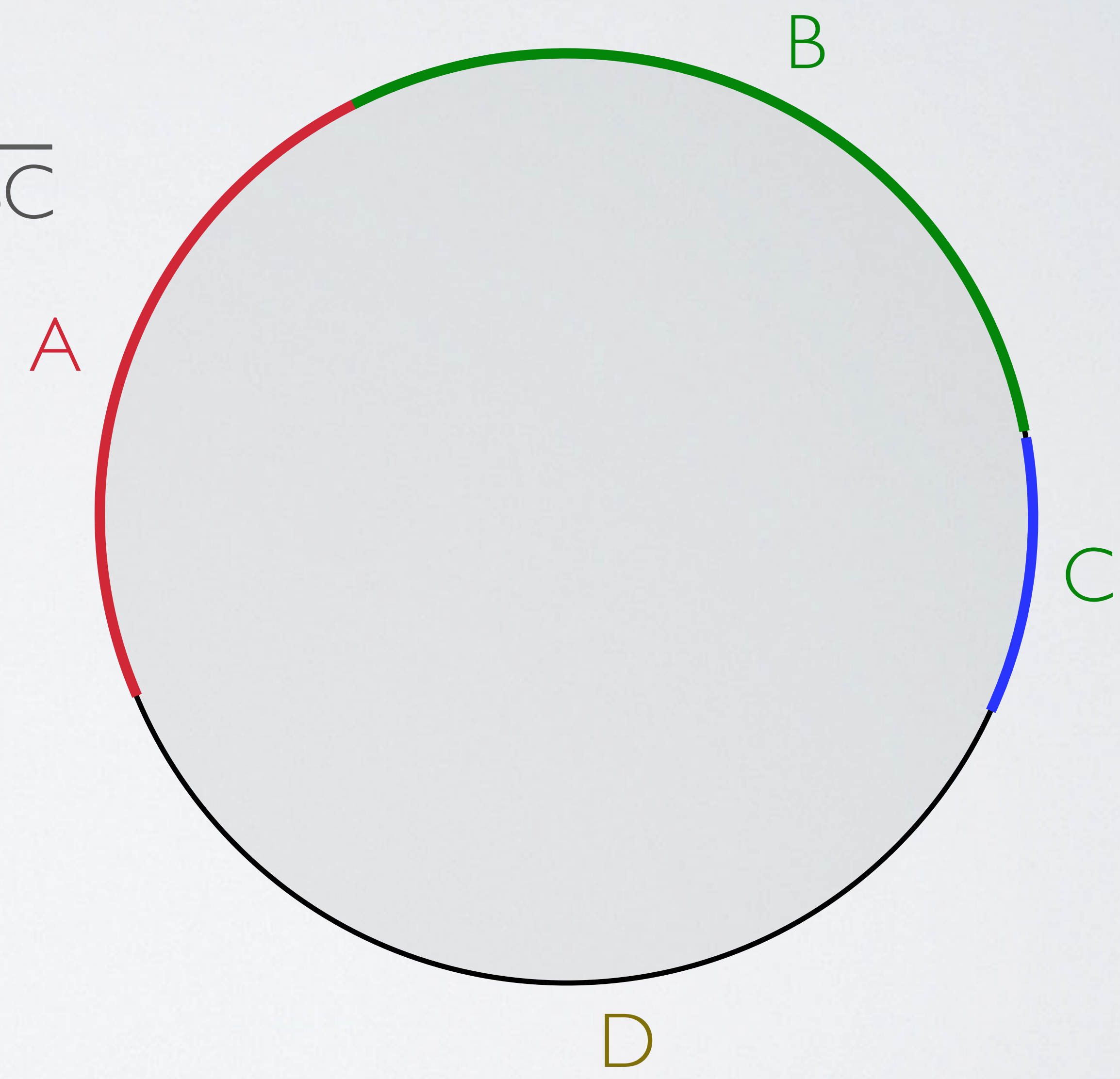
Geometrizing EE relations

- Set-up — example:
  - consider static slice of AdS_3



Geometrizing EE relations

- Set-up
  - consider static slice of AdS_3
  - partition into A,B,C, and $D=\overline{ABC}$



Geometrizing EE relations

- Set-up
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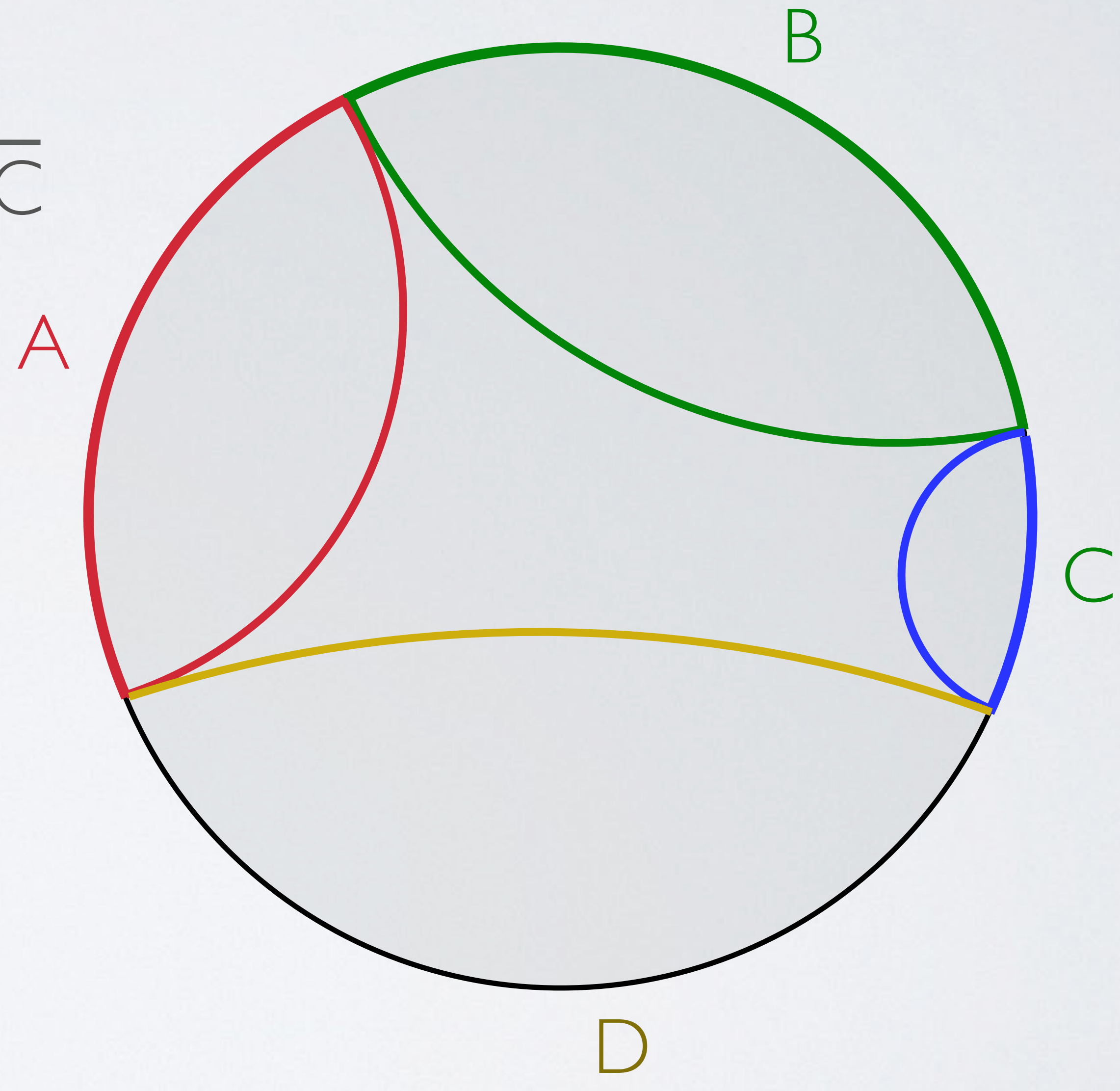
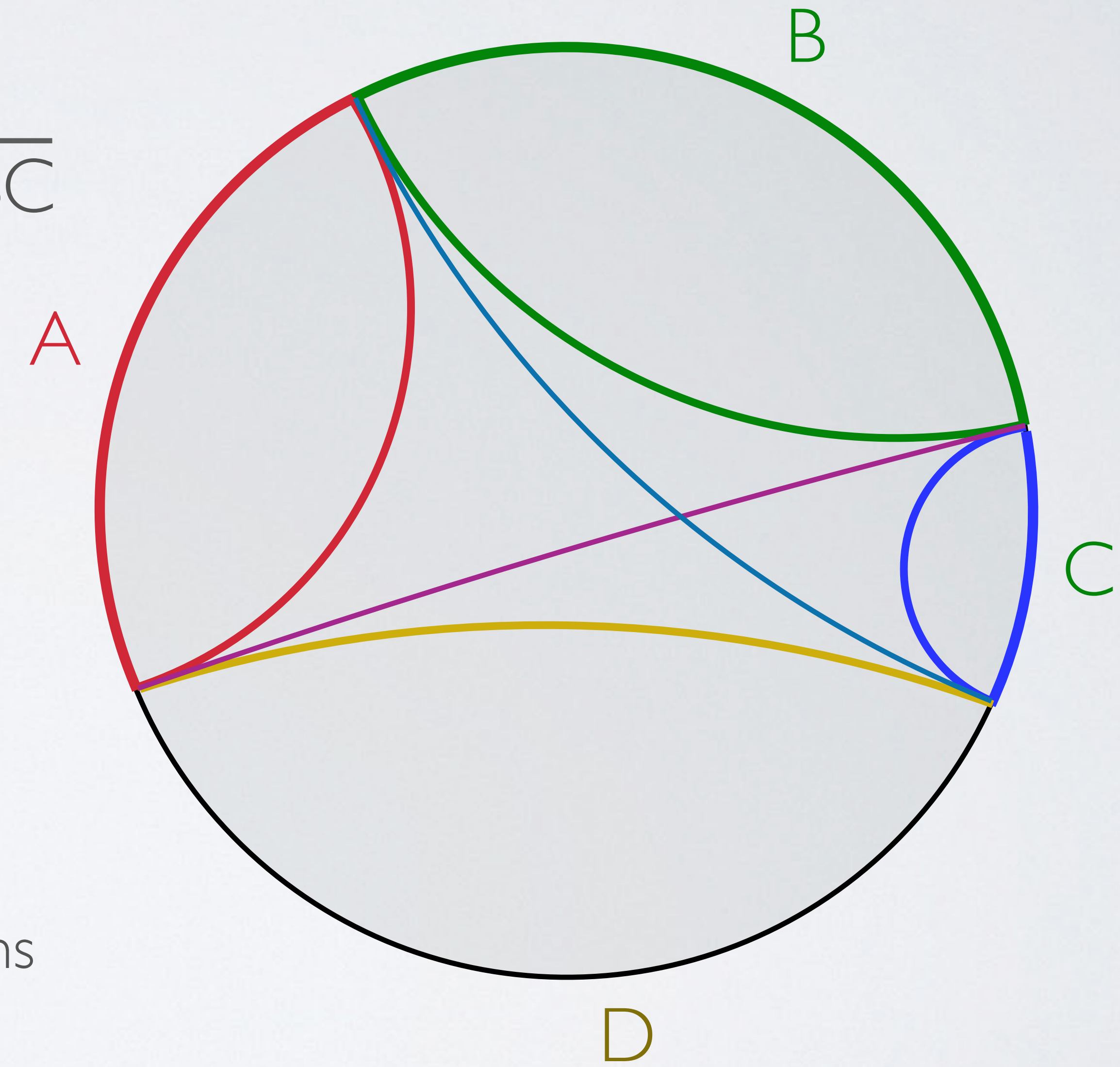
  - construct corresponding minimal surfaces



Geometrizing EE relations

- Set-up
  - consider static slice of AdS_3
  - partition into A,B,C, and D=$\overline{ABC}$
  - construct corresponding minimal surfaces
  - & ones for AB & BC



- UV divergences
  - EE has 'area-law' UV divergence
  - MI is UV finite for non-adjoining regions (but infinite for adjoining ones)
  - CMI can be UV finite even for adjoining regions
  - tripartite info. I3 is always UV finite for any regions

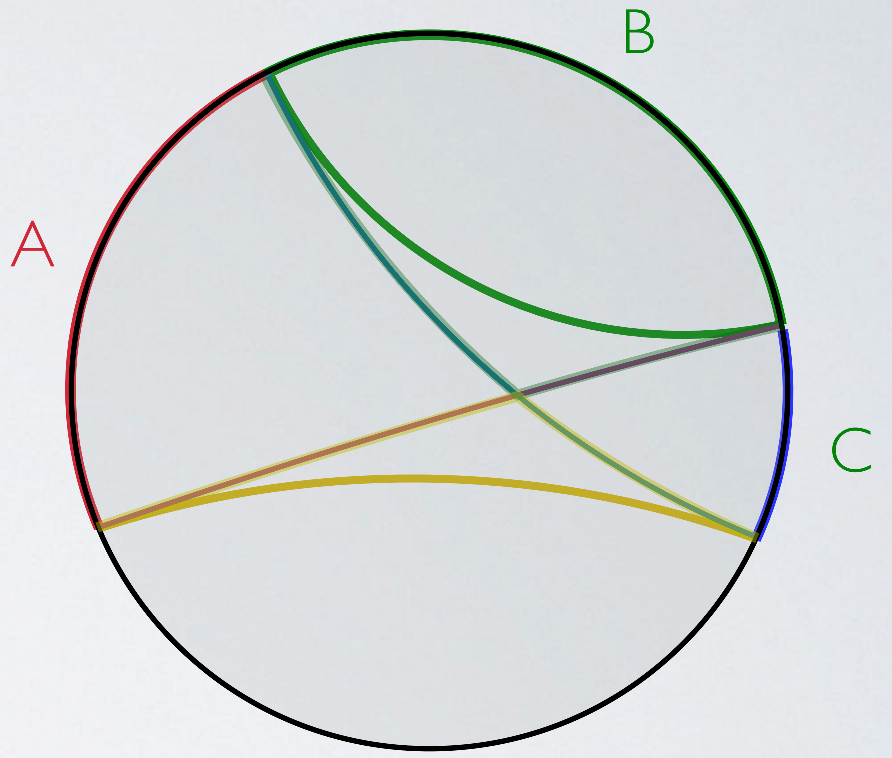
Proof of SSA & MMI in RT

- Strong subadditivity (SSA)

~ amount of correlation is monotonic under inclusion
(= positivity of conditional mutual information I(A:B|C))

$$S(AB)+S(BC) \geq S(B)+S(ABC)$$

follows from area comparison…



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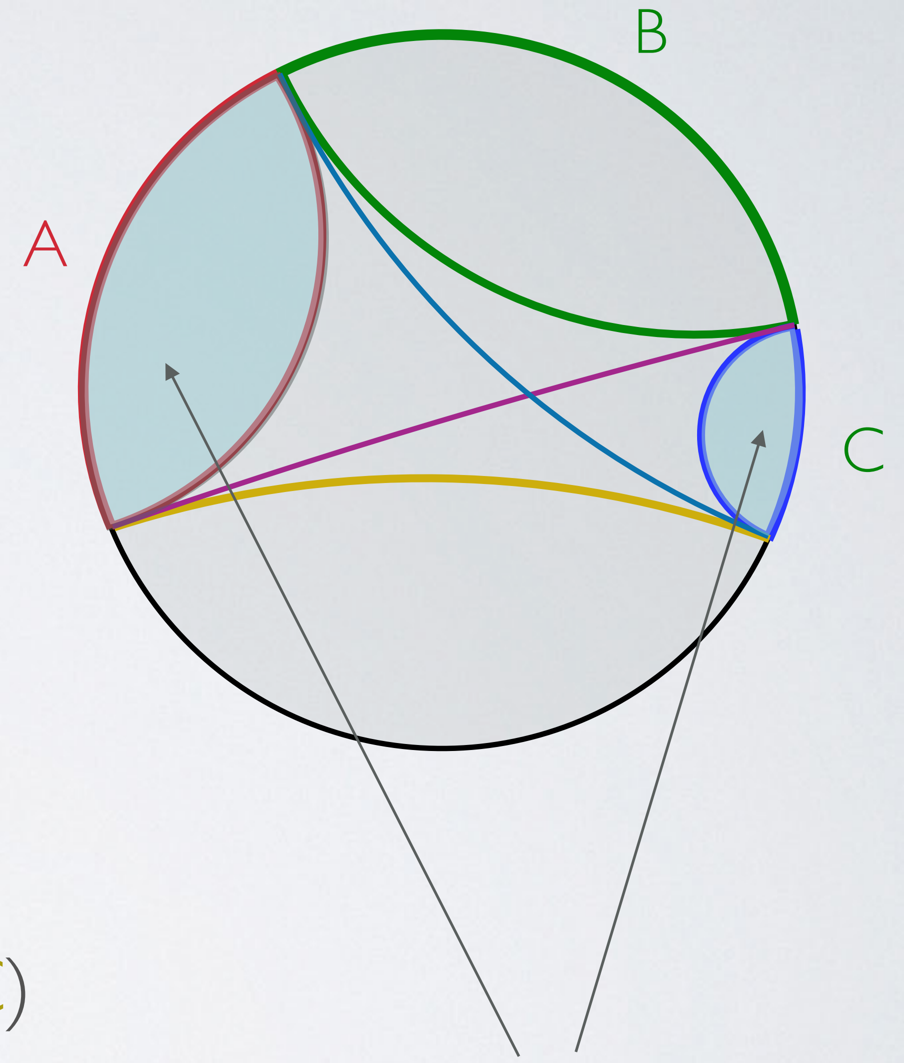
- Monogamy of mutual information (MMI)

~ superadditivity of mutual information
(= negativity of tripartite information I3(A:B:C))

$$S(AB)+S(BC)+S(AC) \geq S(A)+S(B)+S(C)+S(ABC)$$

Suppose I(A:C)=0 $\Rightarrow$ surfaces cancel

reduces to the same proof as for SSA



homology region for AC

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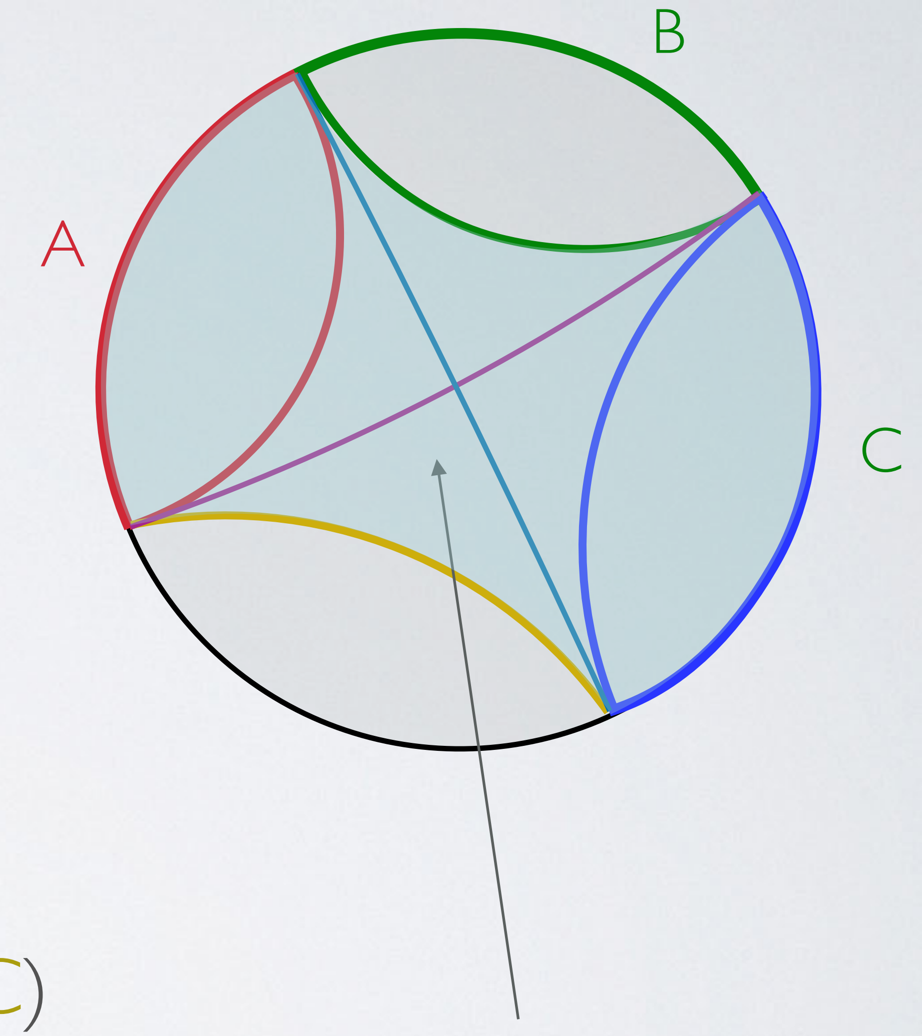
~ superadditivity of mutual information
(= negativity of tripartite information I3(A:B:C))

$$S(AB)+S(BC)+S(AC) \geq S(A)+S(B)+S(C)+S(ABC)$$

Alternately if I(B:D)=0 ⇒ other set of surfaces cancel

again reduces to the same proof as for SSA



homology region for AC

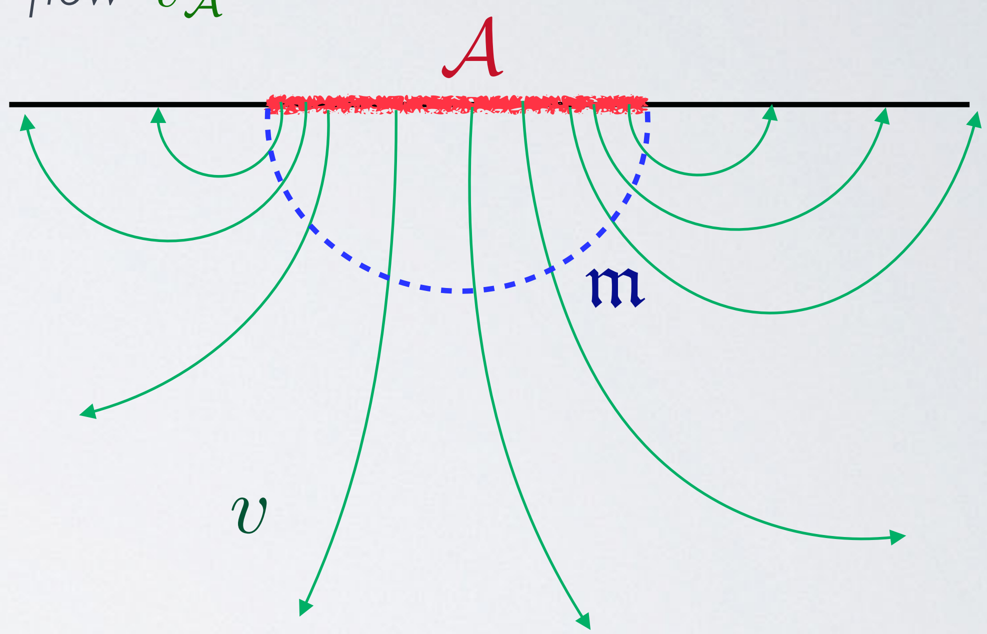
cf. [Hayden, Headrick, Maloney]

- Both SSA & MMI proved in HRT = maximin [Wall]

Bit thread picture of (static) EE

- Reformulate EE in terms of flux of flow lines [Freedman & Headrick, '16]
  - let v be a *flow* = vector field satisfying $\nabla \cdot v = 0$ and $|v| \leq 1$.
  - Then EE is given by max flux = flux of any *maximizer flow* $v_{\mathcal{A}}$

$$S(\mathcal{A}) = \max_v \int_{\mathcal{A}} v \quad = \int_{\mathcal{A}} v_{\mathcal{A}} \geq \int_{\mathcal{A}} v$$



  - By Max Flow - Min Cut theorem, equivalent to RT:
    (bottleneck for flow = minimal surface)
    detailed derivation in [Headrick & VH, '17]

- Useful reformulation of holographic EE
  - behaves more naturally
  - is more computationally efficient
  - ties better to QI quantities
  - provides more intuition — cf. suggested threads capturing quantum entanglement monogamy…

Bit threads - interpretation

cf [Freedman & Headrick, '16]

Nesting: ∃ common maximizer flow for nested regions

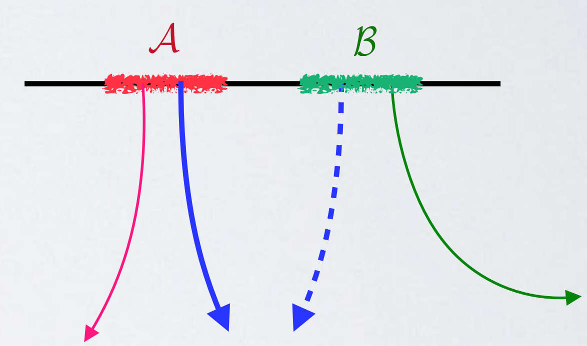
Suppose we maximize on AB.
— Then we can additionally maximize on either A or B, but not both.

- Conditional entropy H(A:B) = S(AB) - S(B)
  ~ bits in A which are uncorrelated with B
  = # of threads left on A when we measure B

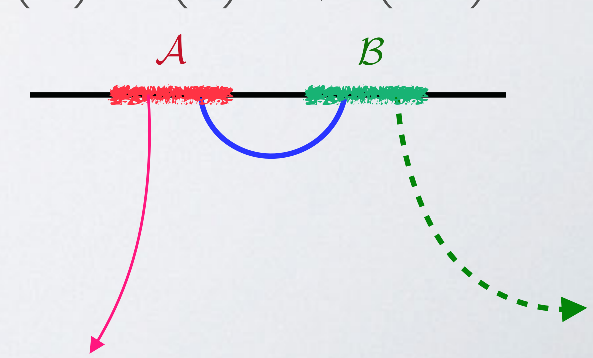
- Mutual information I(A:B) = S(A) + S(B) - S(AB)
  ~ correlations (redundancy) between A and B
  = # of threads which can flop between A and B

- Entangled qubits are threads between A and B
  which switch direction.

S(A)=S(B)=2, S(AB)=3:



S(A)=S(B)=2, S(AB)=1:



proof of SSA via bit threads

Consider conditional mutual information I(A:B|C)

= (max flux on A after maximinizing on C and ABC)
  - (min flux on A after maximinizing on C and ABC)
= # of threads moveable betw. A and B after maximizing on C and ABC

$\Rightarrow$ ≥ 0 $\Rightarrow$ SSA [Freedman & Headrick, '16]

Can't do the same for MMI:

e.g. using nesting & the basic properties of fluxes, can construct state which nevertheless violates MMI

$\Rightarrow$ need additional ingredients…

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Basic argument

- Recall MMI: $S(AB)+S(BC)+S(CA) \geq S(A)+S(B)+S(C)+S(ABC)$

- Consider the LHS:

$$S(AB)+S(BC)+S(CA) = \int_{AB} v_{AB} + \int_{BC} v_{BC} + \int_{CA} v_{CA}$$

maximizer flows for the corresponding regions

arbitrary flows

$$\geq \int_{AB} v_3 + \int_{BC} v_1 + \int_{CA} v_2$$

$$= \int_A (v_2+v_3) + \int_B (v_1+v_3) + \int_C (v_1+v_2)$$

- We want to find flows v_1, v_2, v_3 which would allow us to relate this to the RHS…

Basic argument

- If we can define

$$v_1 = \tilde{v}_1 \equiv \frac{1}{2}\left(v_{ABC} - v_A + v_B + v_C\right)$$
$$v_2 = \tilde{v}_2 \equiv \frac{1}{2}\left(v_{ABC} + v_A - v_B + v_C\right)$$
$$v_3 = \tilde{v}_3 \equiv \frac{1}{2}\left(v_{ABC} + v_A + v_B - v_C\right)$$

such that each $\tilde{v}_i$ is a flow,

- Then

$$\int_A (\tilde{v}_2 + \tilde{v}_3) + \int_B (\tilde{v}_1 + \tilde{v}_3) + \int_C (\tilde{v}_1 + \tilde{v}_2)$$
$$= \int_A (v_A + v_{ABC}) + \int_B (v_B + v_{ABC}) + \int_C (v_C + v_{ABC}))$$
$$= S(A) + S(B) + S(C) + S(ABC)$$

which is the desired RHS of MMI.

- Hence MMI holds, **provided** we can show that for any configuration of A,B,C, we can find corresponding maximizer flows $v_A, v_B, v_C,$ and $v_{ABC},$ such that $\tilde{v}_1, \tilde{v}_2,$ and $\tilde{v}_3$ are all flows simultaneously.
- We'll call such flows *cooperative flows*. [M.Headrick]

Main challenge

- The expressions

$$v_1 = \tilde{v}_1 \equiv \frac{1}{2}(v_{ABC} - v_A + v_B + v_C)$$
$$v_2 = \tilde{v}_2 \equiv \frac{1}{2}(v_{ABC} + v_A - v_B + v_C)$$
$$v_3 = \tilde{v}_3 \equiv \frac{1}{2}(v_{ABC} + v_A + v_B - v_C)$$

do not a-priori give flows, since the norm bound need not be satisfied:

$$\left.\begin{array}{r} |v_A| \leq 1 \\ |v_B| \leq 1 \\ |v_C| \leq 1 \\ |v_{ABC}| \leq 1 \end{array}\right\} \quad \Rightarrow \quad \left\{\begin{array}{l} |\tilde{v}_1| \leq 2 \\ |\tilde{v}_2| \leq 2 \\ |\tilde{v}_3| \leq 2 \end{array}\right.$$

- So we need large amount of simultaneous cancellations.
- Can't have e.g. $v_A = v_B$ for generic A and B since generally not compatible
- Could have v_{ABC} equal to at most one of $v_A, v_B,$ or v_C , but that does not suffice…

Detour

- If A and C are completely uncorrelated, i.e. if $I(A:C)=0$, then we can have simultaneous maximizer flows $v_A = v_C$.

- Then choosing additionally $v_B = v_{ABC}$ (using nesting), we verify flows:

$$\left.\begin{aligned} v_1 = \tilde{v}_1 &\equiv \frac{1}{2}(v_{ABC} - v_A + v_B + v_C) = v_B \\ v_2 = \tilde{v}_2 &\equiv \frac{1}{2}(v_{ABC} + v_A - v_B + v_C) = v_A \\ v_3 = \tilde{v}_3 &\equiv \frac{1}{2}(v_{ABC} + v_A + v_B - v_C) = v_B \end{aligned}\right\} \quad \Rightarrow \quad |\tilde{v}_i| \leq 1$$

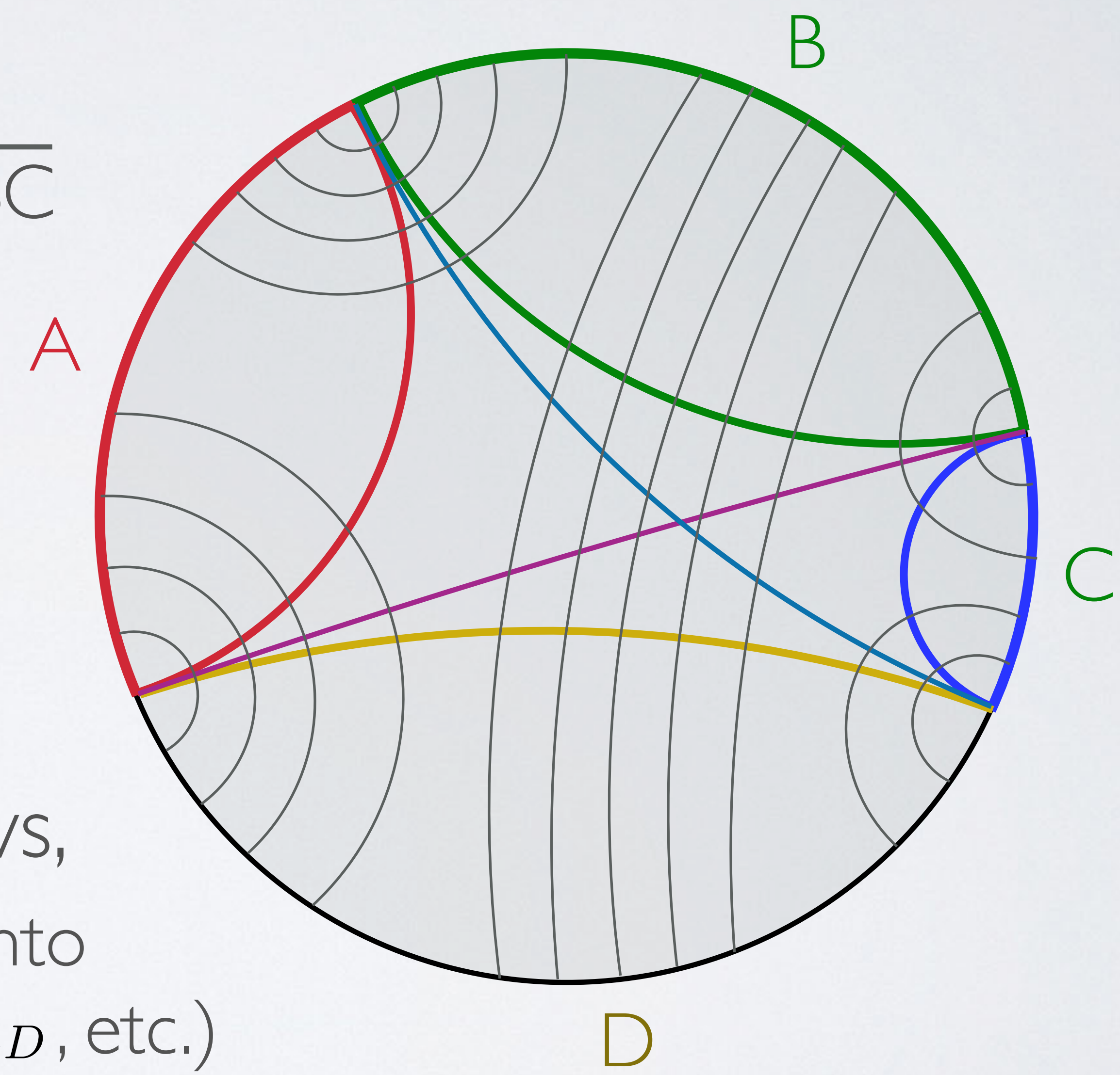
- However, if $I(A:C) \neq 0$, we could at best take $v_A = -v_C$ (by viewing C as nested in BCD and using $v_C = v_{BCD} = -v_A$)

- But now $\tilde{v}_1 = v_B - v_A$ and $\tilde{v}_3 = v_B + v_A$ are no longer flows…

- Hence we have to find a more robust method…

Cooperative flow construction

- Set-up
  - consider static slice of AdS_3
  - partition into A,B,C, and D=$\overline{ABC}$
  - construct corresponding minimal surfaces
  - & ones for AB & BC (WLOG assume AC = A+C)



- To construct cooperative flows,
  - separate each maximizer flow into "strands" (e.g. $v_A = v_{A\to B} + v_{A\to D}$, etc.)

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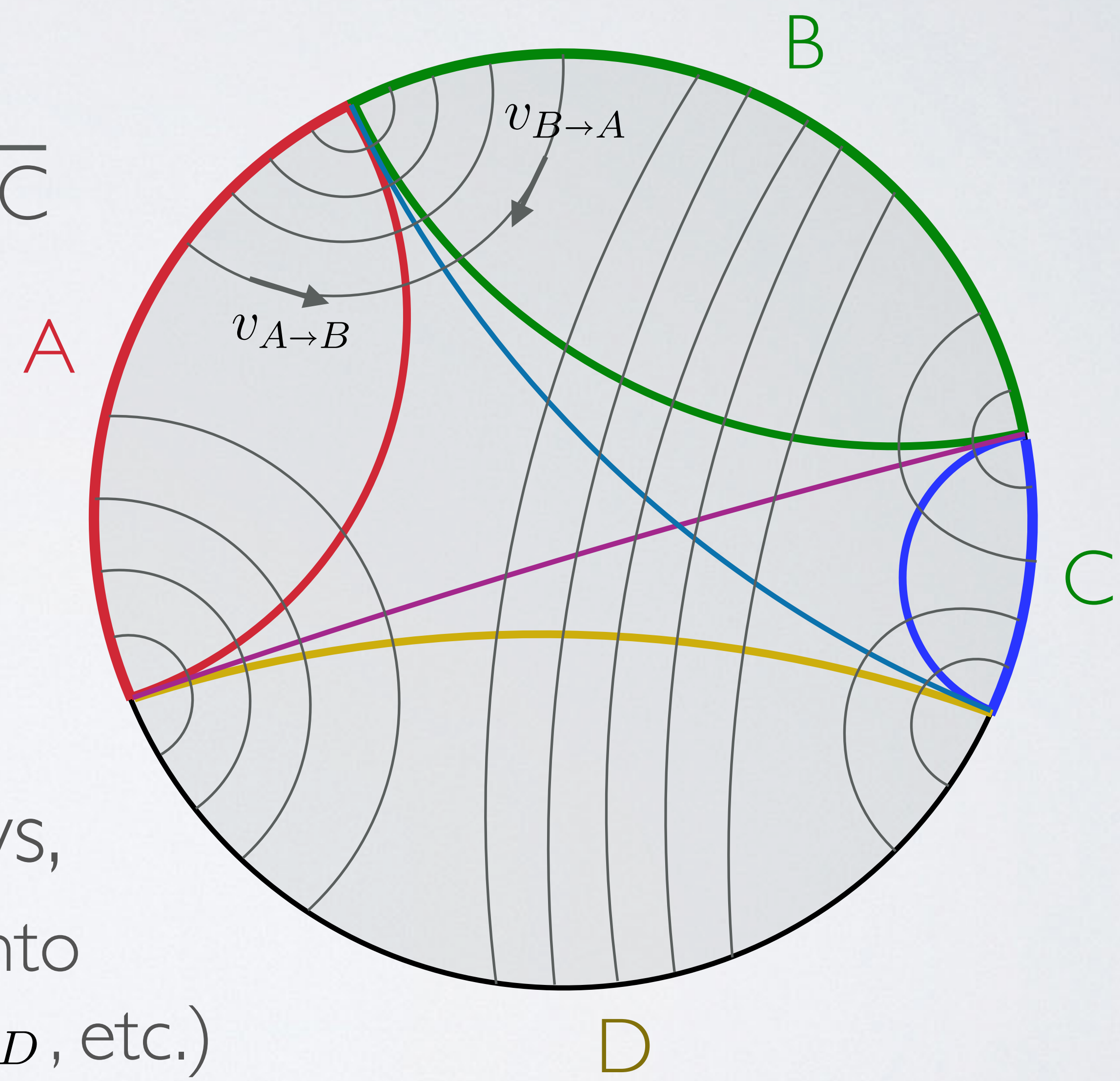
  - s.t. $v_{A\to B} = -v_{B\to A}$, etc.



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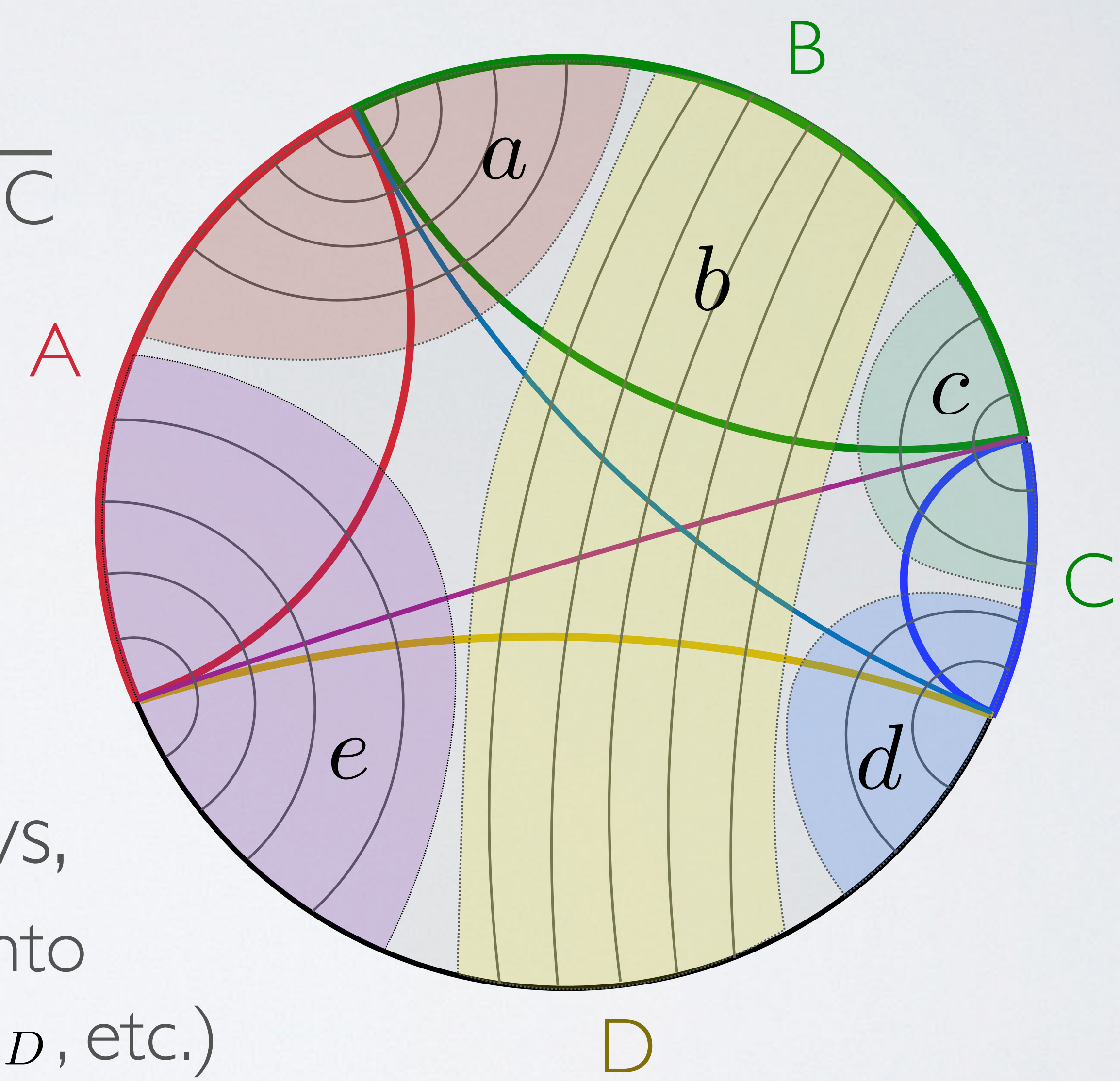
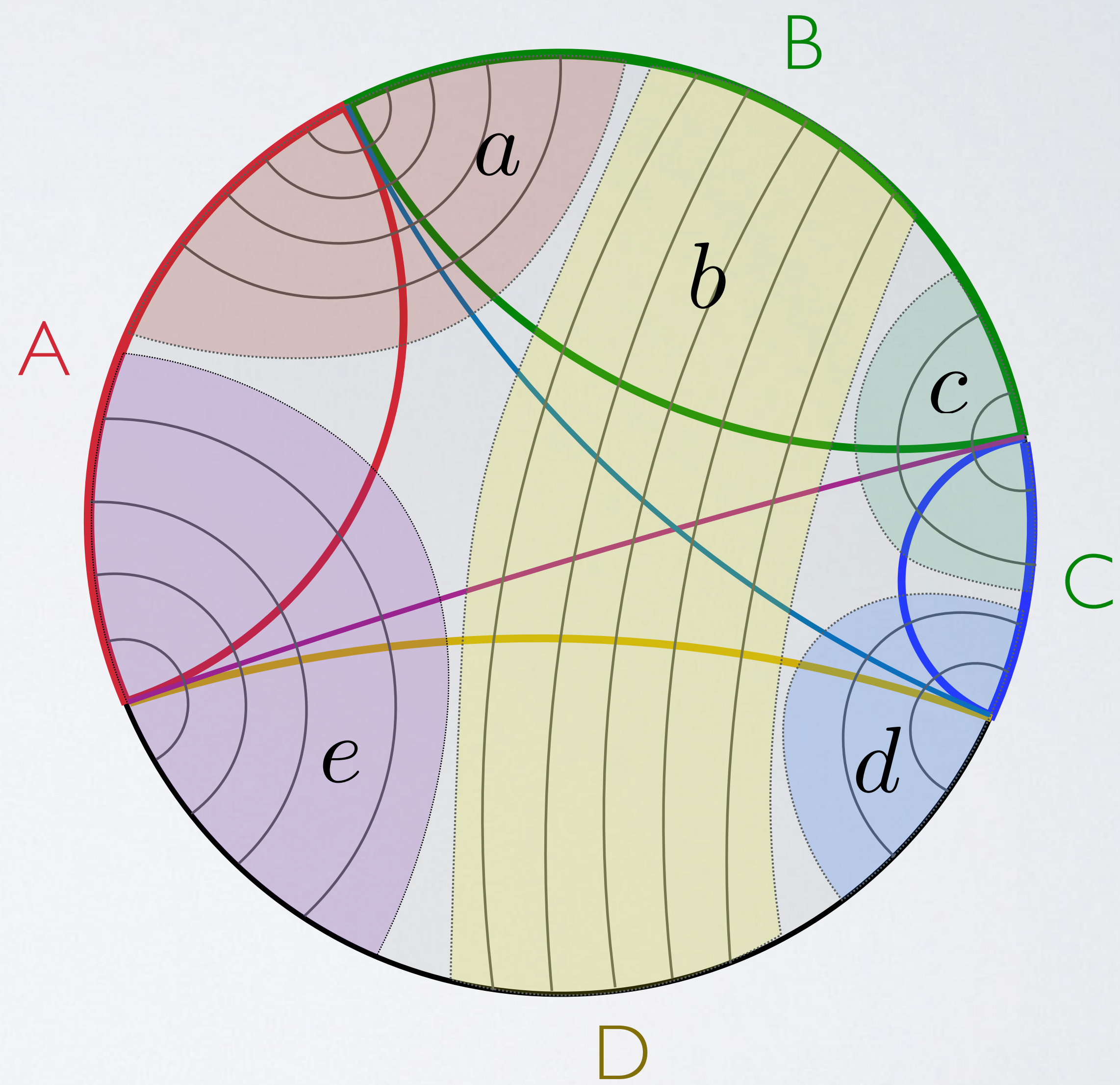


- To construct cooperative flows,
  - separate each maximizer flow into "strands" (e.g. $v_A = v_{A\to B} + v_{A\to D}$, etc.)
  - s.t. $v_{A\to B} = -v_{B\to A}$, etc.
  - w/ each strand confined to distinct bulk regions: a, b, c, d, e

Cooperative flow construction

- Then w/ maximizer flows split as follows:

	a	b	c	d	e
v_A	$v_{A\to B}$	0	0	0	$v_{A\to D}$
v_B	$v_{B\to A}$	$v_{B\to D}$	$v_{B\to C}$	0	0
v_C	0	0	$v_{C\to B}$	$v_{C\to D}$	0
v_{ABC}	0	$v_{B\to D}$	0	$v_{C\to D}$	$v_{A\to D}$



- The corresponding $\tilde{v}_i$'s are given by

	a	b	c	d	e
$\tilde{v}_1$	$v_{B\to A}$	$v_{B\to D}$	0	$v_{C\to D}$	0
$\tilde{v}_2$	$v_{A\to B}$	0	$v_{C\to B}$	$v_{C\to D}$	$v_{A\to D}$
$\tilde{v}_3$	0	$v_{B\to D}$	$v_{B\to C}$	0	$v_{A\to D}$

- So despite being sum of several flows, each $\tilde{v}_i$ is indeed a flow in the full bulk.

Cooperative flow construction

- Crux: How can we guarantee that such separation into distinct bulk regions is always possible?

- NB. threads maximally constrained on corresponding minimal surfaces; require

$$\left.\begin{aligned}
\text{Area}\,[\mathfrak{m}_A] &= \text{Area}\,[\mathfrak{m}_{A\to B}] + \text{Area}\,[\mathfrak{m}_{A\to D}] \\
\text{Area}\,[\mathfrak{m}_B] &= \text{Area}\,[\mathfrak{m}_{B\to A}] + \text{Area}\,[\mathfrak{m}_{B\to C}] + \text{Area}\,[\mathfrak{m}_{B\to D}] \\
\text{Area}\,[\mathfrak{m}_C] &= \text{Area}\,[\mathfrak{m}_{C\to B}] + \text{Area}\,[\mathfrak{m}_{C\to D}] \\
\text{Area}\,[\mathfrak{m}_D] &= \text{Area}\,[\mathfrak{m}_{D\to A}] + \text{Area}\,[\mathfrak{m}_{D\to B}] + \text{Area}\,[\mathfrak{m}_{D\to C}] \\
\text{Area}\,[\mathfrak{m}_{A\to B}] &= \text{Area}\,[\mathfrak{m}_{B\to A}] \\
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\text{Area}\,[\mathfrak{m}_{B\to D}] &= \text{Area}\,[\mathfrak{m}_{D\to B}] \\
\text{Area}\,[\mathfrak{m}_{C\to D}] &= \text{Area}\,[\mathfrak{m}_{D\to C}]
\end{aligned}\right\}$$

= 9 eqns for 10 unknowns

$\rightsquigarrow$ 1-param. family of solutions

- Elsewhere we can 'comb' flows away from each other using minimal surface foliations…

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Generalizations

- General regions
  - regions composed of multiple components require more strands but same principle applies.
- General static asymp. AdS_3 geometries
  - since minimal surfaces of nested regions don't intersect, same…
- Higher dimensions
  - nesting of minimal surfaces still applies, and strands can be braided through each other, so less constraining…
- Time dependence
  - use covariant bit threads: since still 1-d, seemingly less constraining.
- More partitions
  - more subtle…

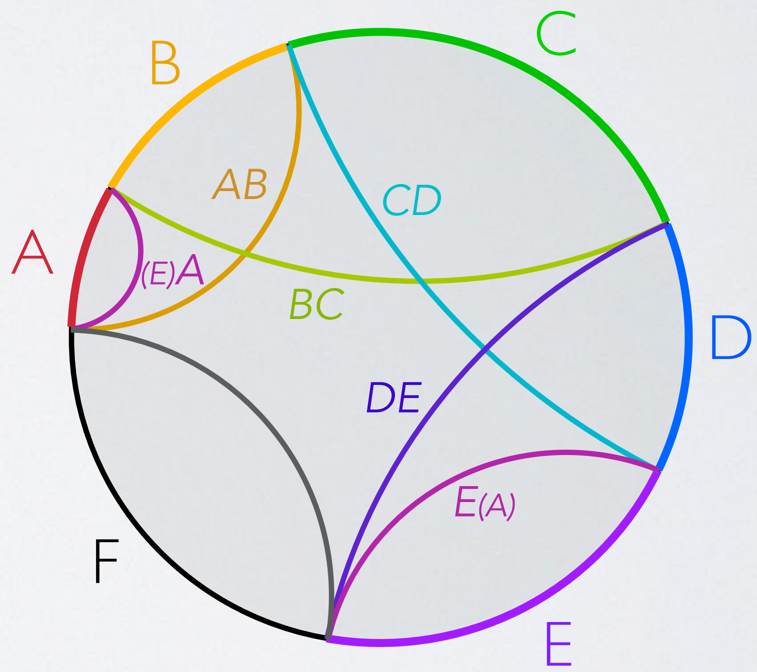
5-region cyclic inequality

- As example w/ more partitions, consider the 5-region cyclic inequality: cf. holographic entropy cone [Bao, Nezami, Ooguri, Stoica, Sully, Walter, '15]

$$S(ABC)+S(BCD)+S(CDE)+S(DEA)+S(EAB) \geq$$
$$S(AB)+S(BC)+S(CD)+S(DE)+S(EA)+S(ABCDE)$$

- Similar argument as above would give this provided

$$v_1 = \frac{1}{3}\left(v_{AB}+v_{BC}+v_{CD}-2v_{DE}+v_{EA}+v_{ABCDE}\right)$$
$$v_2 = \frac{1}{3}\left(v_{AB}+v_{BC}+v_{CD}+v_{DE}-2v_{EA}+v_{ABCDE}\right)$$
$$v_3 = \frac{1}{3}\left(-2v_{AB}+v_{BC}+v_{CD}+v_{DE}+v_{EA}+v_{ABCDE}\right)$$
$$v_4 = \frac{1}{3}\left(v_{AB}-2v_{BC}+v_{CD}+v_{DE}+v_{EA}+v_{ABCDE}\right)$$
$$v_5 = \frac{1}{3}\left(v_{AB}+v_{BC}-2v_{CD}+v_{DE}+v_{EA}+v_{ABCDE}\right)$$



are all flows everywhere.

- However now can't use a single set of strands since intersecting minimal surfaces — need overlaid strands…

OUTLINE

- Motivation & Preview
- Background
  - SSA & MMI proof using RT
  - Bit Threads
  - SSA proof using bit threads
- MMI proof using bit threads
  - Basic argument
  - Cooperative flow construction
- Generalizations
- Summary & Open questions

Summary

- Demonstrated existence of cooperative flows
  - explicit construction for 4 simple regions in static asymp. AdS_3
- Crucial ingredient: bulk locality
  - => local regions foliated by (piecewise) minimal surfaces
  - => can "comb" threads into strands = cooperative flows
- Flows (bit threads) more useful than RT
- SSA vs. MMI:
  - SSA proof didn't require cooperative flows => more general
  - MMI proof uses cooperative flows
  - => bulk locality used more crucially for MMI than for SSA

Open Questions

- Extent of generalizations
- Role of time (cf. covariant bit threads)
- Better re-formulation
- Mapping out entropy cone
- Other (multipartite) entanglement measures
- Relation to entanglement of purification


Thank you