The particles at null-infinity

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Introduction

- Minkowski space metric in spherical polar coordinates can be written as,

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2$$

where

$$d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$$

is the metric on a unit sphere.

- Now define retarded coordinate u as $u = t - r$.
- In these coordinates

$$ds^2 = -du^2 - 2dudr + r^2 \frac{4dzd\bar{z}}{(1+z\bar{z})^2} \quad (1)$$

where

$$z = \frac{x^1 + ix^2}{r + x^3} = \tan\frac{\theta}{2} e^{i\phi} \quad (2)$$

- Future null-infinity can be reached by taking $r \to \infty$ at fixed $(u, z, \bar{z})$.
- So $(u, z, \bar{z})$ are coordinates at future null infinity. They are known as Bondi coordinates.

- For our purpose it is convenient to think of $SL(2,\mathbb{C})$ as the Lorentz group. This is actually a double cover of $SO(3,1)$.
- Now define the hermitian matrix X as,

$$X = \begin{pmatrix} x^0 - x^3 & x^1 + ix^2 \\ x^1 - ix^2 & x^0 + x^3 \end{pmatrix}, \ \det X = -x_\mu x^\mu \tag{3}$$

- An $SL(2,\mathbb{C})$ matrix M acts on X as,

$$X \to X' = MXM^\dagger = \begin{pmatrix} x'^0 - x'^3 & x'^1 + ix'^2 \\ x'^1 - ix'^2 & x'^0 + x'^3 \end{pmatrix} \tag{4}$$

- This defines the action of $SL(2,\mathbb{C})$ on space-time via,

$$x'^\mu = \Lambda^\mu{}_\nu(M) x^\nu \tag{5}$$

- Now under a $SL(2,\mathbb{C})$ transformation M, the Bondi coordinates transform as $(r,u,z,\bar{z}) \to M(r,u,z,\bar{z})$.
- In the limit $r \to \infty$, at fixed $(u,z,\bar{z})$, the transformation simplifies drastically,

$$M(u,z,\bar{z}) = \left(\frac{u\,(1+z\bar{z})}{|az+b|^2+|cz+d|^2}\,,\frac{az+b}{cz+d}\,,\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}} \right) \tag{6}$$

where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \tag{7}$$

- Similarly under space-time translation by a four vector l^{μ},

$$T(l)(u,z,\bar{z}) = (u + f(z,\bar{z},l), z, \bar{z}) \tag{8}$$

where

$$f(z,\bar{z},l) = \frac{(l^0 - l^3) - (l^1 - il^2)z - (l^1 + il^2)\bar{z} + (l^0 + l^3)z\bar{z}}{1+z\bar{z}} \tag{9}$$

- So the coordinates $(u, z, \bar{z})$ at infinity are closed under the action of the Poincare group.
- The action of the Poincare group at null-infinity in coordinates $(u, z, \bar{z})$ will be called the **Asymptotic Poincare Group**.
- The sphere at infinity parametrized by the stereographic coordinates $(z, \bar{z})$ is also known as the Celestial sphere.
- Note that the Lorentz group $SL(2, \mathbb{C})$ acts on the Celestial Sphere as the group of **Global Conformal Transformations**.

Goal

- Find out a new basis for massless single particle quantum states which **naturally transform under the Asymptotic Poincare Group**. What "natural" means will be clear later.
- Construct a **Manifestly Poincare Invariant Unitary QFT** on the three dimensional space parametrized by $(u, z, \bar{z})$ using these states.
- Construct this space abstractly i.e without any reference to the $(3 + 1)$ dimensional Minkowski space, to the extent possible.

Results

- Consider massless particles in $(3+1)$ dimensional Minkowski space.
- Single particle quantum states of massless particles can be written as $|p^\mu, \sigma\rangle$, where $p_\mu p^\mu = 0$ and σ is the helicity.
- Poincare transformation (a^μ, Λ) which acts on the Minkowski space as, $X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$, acts on the state as,

$$U(\Lambda)\,|p^\mu, \sigma\rangle = e^{i\sigma\theta(\Lambda, p^\mu)}\,|\Lambda^\mu{}_\nu p^\nu, \sigma\rangle \tag{10}$$

$$e^{-ia.P}\,|p^\mu, \sigma\rangle = e^{-ia.p}\,|p^\mu, \sigma\rangle \tag{11}$$

- The states are normalized as,

$$\langle p_1, \sigma_1 | p_2, \sigma_2\rangle = (2\pi)^3 2E_1 \delta^3(\vec{p}_1 - \vec{p}_2)\delta_{\sigma_1\sigma_2} \tag{12}$$

- Now the Lorentz group $SL(2,\mathbb{C})$ acts on a momentum vector p^μ in the same way it acts on a Minkowski space point.
- Define,

$$P = \begin{pmatrix} p^0 - p^3 & p^1 + ip^2 \\ p^1 - ip^2 & p^0 + p^3 \end{pmatrix} \tag{13}$$

- An $SL(2,\mathbb{C})$ matrix M acts on P as,

$$P \to P' = MPM^\dagger = \begin{pmatrix} p'^0 - p'^3 & p'^1 + ip'^2 \\ p'^1 - ip'^2 & p'^0 + p'^3 \end{pmatrix} \tag{14}$$

- In component form,

$$p'^\mu = \Lambda^\mu{}_\nu(M)p^\nu \tag{15}$$

- In four dimensions the space of null momentum *directions* is a two-sphere. The stereographic coordinate of the two-sphere can be defined as,

$$z = \frac{p^1 + ip^2}{p^0 + p^3}, \; p_\mu p^\mu = 0 \tag{16}$$

- One can check that under $SL(2, \mathbb{C})$ transformation M, z transforms as,

$$z \to z' = \frac{p'^1 + i p'^2}{p'^0 + p'^3} = \frac{az + b}{cz + d} \tag{17}$$

- Now if we introduce spherical polar coordinates in momentum space then we can write a null vector P^μ as,

$$p^\mu = (E, E\sin\theta\cos\phi, E\sin\theta\sin\phi, E\cos\theta) \tag{18}$$

In this prametrization (θ, ϕ) become coordinates on the two-sphere. Its relation to the stereographic coordinates z is given by,

$$z = \tan\frac{\theta}{2}e^{i\phi} \tag{19}$$

- Now define the following states for a massless particle of helicity σ as,

$$\left|h, \bar{h}, z, \bar{z}\right\rangle = \frac{1}{\sqrt{8\pi^4}}\left(\frac{1}{1+z\bar{z}}\right)^{1+i\lambda} U\big(R(z,\bar{z})\big)\int_0^\infty dE\ E^{i\lambda}\left|E,0,0,E,\sigma\right\rangle, \quad (20)$$

where

$$U\big(R(z,\bar{z})\big) = e^{-i\phi J_3}e^{-i\theta J_2}e^{i\phi J_3},\ z = \tan\frac{\theta}{2}e^{i\phi},\ \lambda \in \mathbb{R} \quad (21)$$

and

$$h = \frac{1+i\lambda-\sigma}{2},\ \bar{h} = \frac{1+i\lambda+\sigma}{2},\ \lambda \in \mathbb{R} \quad (22)$$

- These states are orthonormal, i.e,

$$\left\langle h, \bar{h}, z, \bar{z} \middle| h', \bar{h}', z', \bar{z}' \right\rangle = \delta_{\sigma\sigma'}\delta(\lambda-\lambda')\delta^2(z-z') \quad (23)$$

- These states form a complete set, i.e,

$$1 = \int_{-\infty}^{\infty} d\lambda \int d^2z \left|h, \bar{h}, z, \bar{z}\right\rangle \left\langle h, \bar{h}, z, \bar{z}\right| \quad (24)$$

- Under a (Lorentz) $SL(2,\mathbb{C})$ transformation Λ the states transform as,

$$\boxed{U(\Lambda)\left|h,\bar{h},z,\bar{z}\right\rangle = \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\left|h\ ,\bar{h}\ ,\frac{az+b}{cz+d}\ ,\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right\rangle} \tag{25}$$

- So these states transform like (quasi-)primary of the $SL(2,\mathbb{C})$ which acts on the celestial sphere as the group of global conformal transformations. The scaling dimensions are $(h,\bar{h})$.

- This is a Unitary representation, i.e,

$$\left(U(\Lambda)\left|h,\bar{h},z,\bar{z}\right\rangle, U(\Lambda)\left|h',\bar{h}',z',\bar{z}'\right\rangle\right) = \left(\left|h,\bar{h},z,\bar{z}\right\rangle, \left|h',\bar{h}',z',\bar{z}'\right\rangle\right) \tag{26}$$

- These representations are known as the Unitary Principal Continuous series representation of the $SL(2,\mathbb{C})$.

- The difference with the standard Wigner representation is that in the present case one considers the **little group of a null momentum direction** instead of the little group of a null momentum vector.
- The states $\{|p, \sigma\rangle\}$ form an irreducible representation of the Poincare group $ISO(3,1)$. Under the action of the Lorentz group $SO(3,1)$ or $SL(2,\mathbb{C})$ this irreducible representation $\{|p, \sigma\rangle\}$ decomposes into an infinite number of **irreducible representations of $SL(2,\mathbb{C})$ indexed by** $(h, \bar{h})$.
- A massless particle **described by the state** $\left|h, \bar{h}, z, \bar{z}\right\rangle$ **can be thought of as living on the two-sphere in momentum space** with coordinates $(z, \bar{z})$ and $(h, \bar{h})$ are internal quantum numbers of the particle.

Action of Space-time Translation

- Think of the state $\left|h,\bar{h},z,\bar{z}\right\rangle$ as the **position eigenket of a particle living on the sphere**.

- So we define the following Heisenberg picture states,

$$\left|h,\bar{h},u,z,\bar{z}\right\rangle = e^{iHu}\left|h,\bar{h},z,\bar{z}\right\rangle \tag{27}$$

where H is the Hamiltonian.

- Under Poincare transformation this family of states transform as,

$$\boxed{U(\Lambda)\left|h,\bar{h},u,z,\bar{z}\right\rangle = \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\left|h\;,\bar{h}\;,\frac{u\;(1+z\bar{z})}{|az+b|^2+|cz+d|^2},\frac{az+b}{cz+d}\;,\frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right\rangle} \tag{28}$$

and

$$\boxed{e^{-il.P}\left|h,\bar{h},z,\bar{z},u\right\rangle = \left|h,\bar{h},u+f(z,\bar{z},l),z,\bar{z}\right\rangle} \tag{29}$$

where

$$f(z,\bar{z},l) = \frac{(l^0-l^3)-(l^1-il^2)z-(l^1+il^2)\bar{z}+(l^0+l^3)z\bar{z}}{1+z\bar{z}} \tag{30}$$

- So the states $\left\{ \left| h, \bar{h}, u, z, \bar{z} \right\rangle \right\}$ can be thought of as living in the three dimensional space-time with coordinates $(u, z, \bar{z})$ on which the Poincare group acts as,

$$\Lambda(u, z, \bar{z}) = \left(\frac{u\ (1 + z\bar{z})}{|az + b|^2 + |cz + d|^2}\ , \frac{az + b}{cz + d}\ , \frac{\bar{a}\bar{z} + \bar{b}}{\bar{c}\bar{z} + \bar{d}} \right) \tag{31}$$

$$T(l)(u, z, \bar{z}) = (u + f(z, \bar{z}, l), z, \bar{z}) \tag{32}$$

- The coordinate u plays the role of time.

- Now this is precisely the way Poincare group acts at null-infinity in Minkowski space in Bondi coordinates. Therefore, from this point of view *the massless particles can be thought of as living at null-infinity and transform under the* **asymptotic Poincare group**.

Transition Amplitude

- The basic transition amplitude for a single massless particle, $\langle h, \bar{h}, u, z, \bar{z} | h', \bar{h}', u', z', \bar{z}' \rangle$, is given by,

$$\langle h, \bar{h}, z, \bar{z} | \, e^{-iH(u-u')} \, | h', \bar{h}', z', \bar{z}' \rangle = \frac{\delta_{\sigma\sigma'}}{2\pi} \frac{\Gamma\big(i(\lambda' - \lambda)\big)}{(1+z\bar{z})^{i(\lambda'-\lambda)}} \frac{\delta^2(z'-z)}{\big(-i(u'-u+i\epsilon)\big)^{i(\lambda'-\lambda)}} \quad (33)$$

- This has various interesting properties.
- Due to momentum conservation the amplitude is **ultralocal in the z direction**. It can only move in time.
- The amplitude is manifestly Poincare invariant.
- Under space-time translation $(u, z, \bar{z}) \to (u + f(z, \bar{z}), z, \bar{z})$. Due to $\delta^2(z' - z)$ the difference $(u' - u)$ remains invariant. So the amplitude is space-time translation invariant.
- One can similarly show that the amplitude is covariant under (Lorentz) $SL(2, \mathbb{C})$ transformation.
- Moreover the representations are all **Unitary** as can be easily checked.
- **Due to ultralocality** the amplitude has a **hidden symmetry** under which $u \to u + f(z, \bar{z})$ where $f(z, \bar{z})$ **is now an arbitrary smooth function on the sphere**. This is "supertranslation" but note that this is **not an asymptotic symmetry**. This is just a geometrical symmetry of the $(u, z, \bar{z})$ space just like ordinary translations.

Creation and Annihilation Fields

- Let us now introduce Heisenberg-Picture creation operator $A^{\dagger}_{\lambda,\sigma}(u,z,\bar{z})$ corresponding to the states $|\lambda,\sigma,u,z,\bar{z}\rangle$ such that

$$U(\Lambda)\, A^{\dagger}_{\lambda,\sigma}(u,z,\bar{z})\; U(\Lambda)^{-1} = \frac{1}{(cz+d)^{2h}}\frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\, A^{\dagger}_{\lambda,\sigma}\left(\frac{u\;(1+z\bar{z})}{|az+b|^2+|cz+d|^2}, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right)$$

and

$$e^{-il.P}A^{\dagger}_{\lambda,\sigma}(u,z,\bar{z})e^{il.P} = A^{\dagger}_{\lambda,\sigma}(u+f(z,\bar{z},l),z,\bar{z})$$

- If we use the definition of the states $|\lambda,\sigma,u,z,\bar{z}\rangle$ given by

$$|\lambda,\sigma,u,z,\bar{z}\rangle = \frac{1}{\sqrt{8\pi^4}}\left(\frac{1}{1+z\bar{z}}\right)^{1+i\lambda} U\big(R(z,\bar{z})\big)\int_0^{\infty} dE\;E^{i\lambda}e^{iEu}\,|E,0,0,E,\sigma\rangle \tag{34}$$

then we can simply write down,

$$\boxed{A^{\dagger}_{\lambda,\sigma}(u,z,\bar{z}) = \frac{1}{\sqrt{8\pi^4}}\left(\frac{1}{1+z\bar{z}}\right)^{1+i\lambda}\int_0^{\infty} dE\;E^{i\lambda}e^{iEu}\;a^{\dagger}(p,\sigma)} \tag{35}$$

- This is analogous to the relation

$$\phi^{-}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2|\vec{p}|} e^{-ip.x} a^{\dagger}(p)$$

  which defines the creation field for a scalar field in Minkowski space.

- So we would interpret $A^{\dagger}_{\lambda,\sigma}(u, z, \bar{z})$ **creation field for a particle with quantum numbers** (λ, σ) **on the** $(u, z, \bar{z})$ **space-time**.

- They are **tensors or primaries** of the inhomogeneous $SL(2, \mathbb{C})$, **for any helicity** σ.

- Similarly the annihilation fields can be constructed by taking hermitian conjugate.

- The (anti-) commutator between creation and annihilation fields is given by,

$$[A_{\lambda,\sigma}(u, z, \bar{z}), A^{\dagger}_{\lambda',\sigma'}(u', z', \bar{z}')]_{\pm} = \frac{\delta_{\sigma\sigma'}}{2\pi} \frac{\Gamma\big(i(\lambda' - \lambda)\big)}{(1 + z\bar{z})^{i(\lambda'-\lambda)}} \frac{\delta^2(z' - z)}{\big(- i(u' - u + i0+)\big)^{i(\lambda'-\lambda)}} \tag{36}$$

- It seems that one can imagine a "QFT" living on a three dimensional space with coordinates $(u, z, \bar{z})$.

- The basic objects of such a field theory are "local operators" which are tensors of the inhomogeneous $SL(2, \mathbb{C})$, i.e,

$$U(\Lambda)\Phi_{h,\bar{h}}(u, z, \bar{z})\, U(\Lambda)^{-1} = \frac{1}{(cz+d)^{2h}} \frac{1}{(\bar{c}\bar{z}+\bar{d})^{2\bar{h}}}\, \Phi_{h,\bar{h}}\left(\frac{u\,(1+z\bar{z})}{|az+b|^2+|cz+d|^2}, \frac{az+b}{cz+d}, \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}\right) \tag{37}$$

and

$$e^{-il.P}\Phi_{h,\bar{h}}(u, z, \bar{z})e^{il.P} = \Phi_{h,\bar{h}}(u + f(z, \bar{z}), z, \bar{z}) \tag{38}$$

- The field theory does not have $ISO(1, 2)$ invariance but the Hilbert space of this field theory carries a **unitary** representation of $ISO(1, 3)$.

- One can consider correlation functions of the form,

$$\langle\Omega| \prod_{i=1}^{n} \phi_i(P_i)\, |\Omega\rangle \tag{39}$$

where $i \sim (\lambda_i, \sigma_i)$ and $P_i = (u_i, z_i, \bar{z}_i)$. $|\Omega\rangle$ is the Poincare invariant vacuum.

- Poincare invariance puts constraints on these correlation functions. For example it is easy to check that **4-point function vanishes unless the cross-ratio of the 4 points** (z_1, z_2, z_3, z_4) **is real**.

Some Applications

- Let us consider the following charges defined in the Hilbert space of a free massless particle of helicity σ,

$$\boxed{T_f = \int d\mu(p)\ E\ f(z,\bar{z})a^\dagger(E,z,\bar{z},\sigma)a(E,z,\bar{z},\sigma) = T_f^\dagger} \tag{40}$$

where $d\mu(p)$ is the Lorentz invariant measure given by

$$d\mu(p) = \frac{d^3\vec{p}}{(2\pi)^3 2|\vec{p}|} = \frac{E^2 dE}{(2\pi)^3 2E}\ \frac{4d^2z}{(1+z\bar{z})^2} \tag{41}$$

and we have used the following parametrisation of a null momentum p ,

$$p = E\left(1, \frac{z+\bar{z}}{1+z\bar{z}}, \frac{-i(z-\bar{z})}{1+z\bar{z}}, \frac{1-z\bar{z}}{1+z\bar{z}}\right) \to p^2 = 0 \tag{42}$$

- $f(z,\bar{z})$ is an arbitrary real smooth function on the 2-sphere.

- The operator T_f has the following properties :
  1)
  $$[H, T_f] = 0 \tag{43}$$
  So the charges are conserved for any function $f(z, \bar{z})$.
  2)
  $$[T_f, T_{f'}] = 0 \tag{44}$$
  for arbitrary f and f'.
  3) The last important property is,
  $$[T_f, a^\dagger(E, z, \bar{z}, \sigma)] = Ef(z, \bar{z})a^\dagger(E, z, \bar{z}, \sigma) \tag{45}$$
  $$[T_f, a(E, z, \bar{z}, \sigma)] = -Ef(z, \bar{z})a(E, z, \bar{z}, \sigma) \tag{46}$$

- Let us now consider the unitary operator $U_f = e^{-iH_f}$. This unitary operator is a "Supertranslation - operator" in the $(u, z, \bar{z})$ space.

- It is easy to see using the commutator of T_f with the momentum space creation/annihilation operators that ,

$$e^{iT_f} A^{\dagger}_{\lambda,\sigma}(u, z, \bar{z}) e^{-iT_f} = \frac{1}{\sqrt{8\pi^4}} \left(\frac{1}{1 + z\bar{z}} \right)^{1+i\lambda} \int_0^{\infty} dE \ E^{i\lambda} e^{iE(u+f(z,\bar{z}))} \ a^{\dagger}(p, \sigma)$$
$$= A^{\dagger}_{\lambda,\sigma}(u + f(z, \bar{z}), z, \bar{z}) \quad (47)$$

- For infinitesimal transformation,

$$f(z, \bar{z}) \frac{\partial A^{\dagger}_{\lambda,\sigma}}{\partial u} = i[T_f, A^{\dagger}_{\lambda,\sigma}] \quad (48)$$

- So T_f is supertranslation generator on free massless particles.

- Let us now see what is the algebra generated by the Lorentz generators and supertranslation generators which also include global space-time translations.

- One can easily show that,

$$\boxed{U(\Lambda)^{-1} T_f U(\Lambda) = T_{f'}, \quad f'(z,\bar{z}) = \frac{|az+b|^2 + |cz+d|^2}{1+z\bar{z}}\ f(\Lambda z, \Lambda \bar{z})}$$

- Let us define the following basis of generators as,

$$T_f = T_{pq}, \quad f = \frac{2z^p \bar{z}^q}{1+z\bar{z}}, \quad T^\dagger_{pq} = T_{qp}$$

where (p,q) are integers.

- Now after simple manipulations we get the following commutators,

$$\boxed{[L_n, T_{pq}] = \left(\frac{n+1}{2} - p\right) T_{p+n,q}, \quad [\bar{L}_n, T_{pq}] = \left(\frac{n+1}{2} - q\right) T_{p,q+n}, \quad p,q \in \mathbb{Z}}$$

$$\boxed{[L_m, L_n] = (m-n)L_{m+n}, \quad [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}, \quad [T_{pq}, T_{p'q'}] = 0, \quad [L_m, \bar{L}_n] = 0, \quad L^\dagger_n = -\bar{L}_n}$$

Here $m,n = 0, \pm 1$ and $(L_n, \bar{L}_n)$ are the (Lorentz) $SL(2,\mathbb{C})$ generators.

- This is essentially the algebra of the BMS group without superrotations.

- So the Fock space of free massless particles in four dimensions carries a **Unitary representation** of the BMS algebra.

Space-time Realization

- Now since we know how the charges act on creation and annihilation operators in momentum space we can also also write down the action of the unitary operator e^{-iT_f} on quantum fields defined on Minkowski space.
- Let us consider a massless scalar field in Minkowski space. The field corresponding to this is,

$$\phi(x) = \int d\mu(p)\left(e^{ip.x}a(p) + e^{-ip.x}a^{\dagger}(p)\right) \tag{49}$$

- So the transformed field can be written as,

$$\phi_f(x) = e^{iT_f}\phi(x)e^{-iT_f} = \int d\mu(p)\left(e^{ip.x}e^{iE_p f(z,\bar{z})}a(p) + e^{-ip.x}e^{-iE_p f(z,\bar{z})}a^{\dagger}(p)\right), \quad E_p = |\vec{p}| \tag{50}$$

- Now if we choose $f(z,\bar{z})$ to be of the following form,

$$f(z,\bar{z},l) = \frac{(l^0 - l^3) - (l^1 - il^2)z - (l^1 + il^2)\bar{z} + (l^0 + l^3)z\bar{z}}{1 + z\bar{z}} \tag{51}$$

then

$$\phi_f(x^{\mu}) = \phi(x^{\mu} + l^{\mu}) \tag{52}$$

So it is a space-time translation by four-vector l^{μ}.
- But, there are infinitely many functions $f(z,\bar{z})$ for which there is no such simple geometric interpretation in terms of space-time.

U(1) Kac-Moody

- Now consider a charged massless scalar particle. The usual expression for the conserved charge of the particle can be written as,

$$Q = e\int d\mu(p)\big(a^\dagger(p)a(p) - b^\dagger(p)b(p)\big) \tag{53}$$

- Now just like supertranslation we can consider the following charges given by,

$$Q_f = e\int d\mu(p)f(z,\bar{z})\big(a^\dagger(p)a(p) - b^\dagger(p)b(p)\big) = Q_f^\dagger \tag{54}$$

- Q_f is obviously conserved and $[Q_f, Q_{f'}] = 0$ for all f.

- The commutator of the charge with the cretaion-annihilation operators is given by,

$$[Q_f, a^\dagger(p)] = ef(z,\bar{z})a^\dagger(p), \quad [Q_f, a(p)] = -ef(z,\bar{z})a(p) \tag{55}$$

$$[Q_f, b^\dagger(p)] = -ef(z,\bar{z})a^\dagger(p), \quad [Q_f, b(p)] = ef(z,\bar{z})a(p) \tag{56}$$

- On the creation and annihilation fields on the $(u, z, \bar{z})$ space the unitary operator e^{-iQ_f} acts like,

$$e^{-iQ_f} A^{\dagger}_{\lambda}(u, z, \bar{z}) e^{iQ_f} = e^{-ief(z,\bar{z})} A^{\dagger}_{\lambda}(u, z, \bar{z}), \quad e^{-iQ_f} A_{\lambda}(u, z, \bar{z}) e^{iQ_f} = e^{ief(z,\bar{z})} A_{\lambda}(u, z, \bar{z})$$

$$e^{-iQ_f} B^{\dagger}_{\lambda}(u, z, \bar{z}) e^{iQ_f} = e^{ief(z,\bar{z})} B^{\dagger}_{\lambda}(u, z, \bar{z}), \quad e^{-iQ_f} B_{\lambda}(u, z, \bar{z}) e^{iQ_f} = e^{-ief(z,\bar{z})} B_{\lambda}(u, z, \bar{z})$$

- So in the $(u, z, \bar{z})$ space e^{-iQ_f} acts by local phase multiplication. So we get an infinite dimensional global $U(1)$ symmetry. **There is one copy of Global $U(1)$ at every point of the sphere**.

- The Lorentz transformation of the charge is given by,

$$U(\Lambda)^{-1} Q_f U(\Lambda) = Q_{f'}, \quad f'(z, \bar{z}) = f(\Lambda z, \Lambda \bar{z})$$

- Now we define the "moments",

$$Q_{pq} = e \int d\mu(p) z^p \bar{z}^q \left(a^{\dagger}(p) a(p) - b^{\dagger}(p) b(p) \right) = Q^{\dagger}_{qp}, \quad (p, q) \in \mathbb{Z} \tag{57}$$

- The commutator of Q_{pq} with the Lorentz generators is given by,

$$\boxed{[L_n, Q_{pq}] = -p \; Q_{p+n,q} \; , \quad [\bar{L}_n, Q_{pq}] = -q \; Q_{p,q+n} \; , \quad [Q_{pq}, Q_{p'q'}] = 0} \tag{58}$$

- To connect it to the more familiar "holomorphic" and "antiholomorphic" charges, we can define $Q_p = Q_{p,q=0}$ and $\bar{Q}_q = Q_{p=0,q}$ which satisfy,

$$[L_n, Q_p] = -pQ_{p+n}, \quad [\bar{L}_n, Q_p] = 0, \quad [Q_p, Q_q] = 0 \tag{59}$$

$$[L_n, \bar{Q}_q] = 0\ , \quad [\bar{L}_n, \bar{Q}_q] = -q\bar{Q}_{q+n}, \quad [\bar{Q}_p, \bar{Q}_q] = 0 \tag{60}$$

$$[Q, \bar{Q}] = 0 \tag{61}$$

  This is level zero U(1) Kac-Moody algebra.

- So the true symmetry of a massless charged scalar is a $U(1)$ Kac-Moody algebra.

- **It is not usually discussed because it is difficult to see this using a classical action. We do not know how this symmetry acts on off-shell objects**.

- In the interacting theory this infinite dimensional global symmetry is realized as the **large gauge transformations**.

- This is encoded in the relation between soft theorems and asymptotic symmetries.

- **In a sense these infinite dimensional Global symmetries are precursor to gauge interactions in the bulk because their presence in the interacting theory requires (soft) photons or (soft) gravitons. This seems to be one of the lessons of the connection between soft theorems and asymptotic symmetries.**