# Geometric Quantization: Various Moduli spaces and the Toda system

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# A Favourite Quote

TRUTH COMES AS A CONQUEROR IF ONE DOESNOT MAKE FRIENDS WITH IT
-Rabindranath Tagore

# Outline of the talk

- Introduction to geometric quantization
- Geometric Quantization of finite Toda systems
- Quillen's Determinant line bundle
- Vortex moduli space and its quantum line bundle
- Two General Theorems
- Hitchin moduli space and its quantum bundles
- Further work completed/in progress

# 1 Geometric Quantization – a quick introduction

# Introduction to geometric quantization

- Axioms of Quantization: Given $(\mathcal{M}, \Omega)$, a symplectic manifold, is it possible to assign operators to functions on $\mathcal{M}$ which act on a Hilbert space such that:
  - $f \to \hat{f}$ is linear
  - $f =$ constant $\implies$ $\hat{f}$ is a multiplication operator
  - If $\{f_1, f_2\}_\Omega = f_3$, then $[\hat{f}_1, \hat{f}_2] = -i\hbar \hat{f}_3$.
  - Let $(\mathcal{M}, \Omega)$ be a symplectic manifold such that the symplectic form is integral. Geometric prequantization is a construction of a prequantum line bundle $\mathcal{L}$ on the symplectic manifold, whose curvature is proportional to the symplectic form $\Omega$.
  - This line bundle should come with a metric and the Hilbert space of the pre-quantization is the space of the square integrable sections of $\mathcal{L}$.

✦ To every $f \in C^\infty(\mathcal{M})$ we associate an operator acting on the Hilbert space, namely,

$$\begin{aligned} \hat{f} &= -i\hbar[X_f - \frac{i}{\hbar}\theta(X_f)] + f \\ &= -i\hbar\nabla^{\theta}_{X_f} + f \end{aligned}$$

where $X_f$ is the Hamiltonian vector field defined by $\Omega(X_f, \cdot) = -df$ and $\theta$ is a symplectic potential correponding to $\Omega$, i.e $\Omega = d\theta$ locally. Then if $f_1, f_2 \in C^\infty(\mathcal{M})$ and $f_3 = \{f_1, f_2\}_\Omega$, Poisson bracket of the two induced by the symplectic form, then $[\hat{f}_1, \hat{f}_2] = -i\hbar\hat{f}_3$ and the other two axioms are also satisfied.

✧ Quantization from Prequantization: The Hilbert space of square integrable sections of the line bundle is too huge. So geometric quantization involves searching for a polarization and taking polarized sections as the Hilbert space.

✧ Example: $\mathbb{R}^2$ , with $(p, q)$ coordinates.

$\Omega = dp \wedge dq$ is the symplectic form and $\theta = pdq$ the symplectic potential.

$\hat{p} = -i\hbar\frac{\partial}{\partial q}$, and

$\hat{q} = i\hbar\frac{\partial}{\partial p} + q$.

To get back the usual quantization, namely

$\hat{p} = -i\hbar\frac{\partial}{\partial q}$, and $\hat{q} = q$, we need to take polarized sections, i.e. functions which depend on $q$ alone.

# Real, Complex and holomorphic polarizations

✧ Real polarization: A real polarization of a symlectic manifold $(M, \omega)$ is a foliation of $M$ by Lagrangian submanifolds: i.e. a smooth distribution $P$ which is

(1) integrable: if $X, Y \in V_P(M)$ then $[X, Y] \in \mathrm{V}_P(M)$.

(2) Lagrangian: for each $m \in M$, $P_m$ is a Lagrangian subspace of $T_m M$.

✧ Complex polarization: A complex polarization of a symplectic manifold $(M, \omega)$ is a complex distribution $P$ on $M$ such that

(1) for each $m \in M$, $P_m \subset (T_m M)_{\mathbb{C}}$ is Lagrangian

(2) the dimension of $D = P \cap \bar{P} \cap TM$ is constant.

(3) P is integrable.

✧ Holomorphic polarization: A Kahler manifold is a complex manifold of real dimensin $2n$ with a symplectic structure $\omega$ and a complex structure $J$ which is compatible with $\omega$. The metric $g(\cdot, \cdot) = 2\omega(\cdot, J)$ makes $M$ into a Riemmanian manifold.

In local holomorphic coordinates $z^a$, $a = 1, ...n$, $\omega = i\omega_{ab} dz^a \wedge d\bar{z}^b$, where $\omega_{ab} = \bar{\omega}_{ba}$.

$M$ has two polarizations, the holomorphic polarization $P$ spanned by $\{\frac{\partial}{\partial z}\}$ and the anti-holomorphic polarization spanned by $\{\frac{\partial}{\partial \bar{z}}\}$.

✧ Definition: A smooth section $s : M \to L$ is said to be polarized if $\nabla_{\bar{X}} s = 0$ for every $X \in V_P(M)$.

✧ Example: $M = \mathbb{R}^{2n} \times \mathbb{C}^{n'}$. Then there are real coordinates $p_a$ and $q^b$ on $R^{2n}$ and complex coordinates $z^\alpha$ on $\mathbb{C}^{n'}$. Let $K(q, z, \bar{z})$ be a real function of $q$'s and $z$'s such that $det(\frac{\partial^2 K}{\partial z^\alpha \partial \bar{z}^\beta}) \neq 0$.

Then the following is a symplectic form on $M$.

$$\begin{aligned} \omega &= dp_a \wedge dq_a - \frac{i}{2} \frac{\partial^2 K}{\partial q_a \partial z_\alpha} dq_a \wedge dz_\alpha + \frac{i}{2} \frac{\partial^2 K}{\partial q_a \partial \bar{z}_\alpha} dq_a \wedge d\bar{z}_\alpha \\ & \quad + i \frac{\partial^2 K}{\partial z_\alpha \partial \bar{z}_\beta} dz_\alpha \wedge d\bar{z}_\beta \end{aligned} \tag{2}$$

(sum over repeated indices).

The symplectic potential is

$\theta = p_a dq^a - i\frac{\partial K}{\partial z^\alpha} dz^\alpha - \frac{i}{2}\frac{\partial K}{\partial q^a} dq^a.$

Polarised sections are functions of $q$ and $z$ alone – holomorphic in $z$.

This example is a typical case!

- By adding a constant to $K$ one can arrange that the Hermitian structure on $L$ is given by $(\phi, \phi) = \bar{\phi}\phi e^{-K/\hbar}$.

- We should like to quantize $M$ by replacing the Hilbert space $\mathcal{H}$ of prequantization by the subspace of sq. int. polarized sections of $L$.

  The first problem is the operator $\hat{f}$ corresponding to $f \in C^\infty(M)$ maps local polarized sections to polarized sections only if the flow of $X_f$ preserves $P$.

  This follows from

  $\nabla_{\bar{X}}(\hat{f}s) = \hat{f}(\nabla_{\bar{X}} s) - i\hbar\nabla_{[\bar{X}, X_f]} s.$

  So if $\hat{f}s$ is polarized whenever $s$ is polarized, then $[X, X_f] \in V_P(M)$ whenever $X \in V_P(M)$.

  Thus only a limited class of observables $f$ can be quantized directly.

- Holomorphic Quantization: If the prequantum bundle is holomorphic, one could take holomorphic sections as polarised sections. But for a general $f$, $\hat{f}$ doesnot take holomorphic sections to holomorphic sections. Only some functions can be quantized.

✧ Many successes in this: Harmonic osciallator, spin quantization, Witten's construction of Jone's polynomial from Chern-Simon's gauge theory and of course quantization of coadjoint orbits.

✧ Spin Quantization:

Example: The holomorphic quantization of $S^2$. $S^2$ is identified with $\mathbb{C}P^1$, a Kähler manifold. Using $z$ as the coodinates on $S^2$, the Kähler form is $\omega = i\frac{dz\wedge d\bar{z}}{(1+z\bar{z})^2}$. We take symplectic form to be $\Omega = n\omega$, where $n$ is an integer. This is integral and $\int_{S^2}\Omega = 2\pi n$ The symplectic potential can be taken to be $\mathcal{A} = \frac{in}{2}[\frac{zd\bar{z}-\bar{z}dz}{(1+z\bar{z})}]$. The covariant derivative acting on sections of the prequantum bundle is given by $\partial - i\mathcal{A}$.

✣ The holomorphic polarization condition is $(\partial_{\bar{z}} - i\mathcal{A}_{\bar{z}})\Psi = [\partial_{\bar{z}} + \frac{n}{2}\frac{z}{1+z\bar{z})}]\Psi = 0$. This can be solved as $\Psi = exp(-\frac{n}{2}log(1+z\bar{z}))f(z)$.

(Note that $nlog(1+z\bar{z})$ is the Kähler potential for $\Omega$. )

Th inner product is given by $< 1|2 >= i\alpha\int\frac{dz\wedge d\bar{z}}{2\pi(1+z\bar{z})^{n+2}}f_1^* f_2$.

Since $f(z)$ is holomorphic, we can see that a basis of non-singular wavefunctions is given by $f(z) = 1, z, z^2, ....z^n$, as higher powers will not give finite norm. The dimension of this Hilbert space is $(n+1)$.

✣ This is relevant to physics: a particle of spin $\frac{n}{2}$ has as phase space the coadjoint orbit of $SU(2)$ corresponding to a sphere and the Hilbert space of the quantization is exactly the one as above.

# Coadjoint Orbits

✧ Coadjoint actions of a Lie group $G$ whose Lie algebra is $\mathcal{G}$:

✧ The adjoint action $g \to gA$ induces a coadjoint action of $G$ on the dual space $\mathcal{G}^*$ by $g : \mathcal{G}^* \to \mathcal{G}^* : f \to fg$ where $fg(A) = f(gA)$ for $A \in \mathcal{G}$.

✧ Coadjoint Orbits:

Suppose that $G$ is connected. Under the coadjoint action $\mathcal{G}^*$ breaks up into orbits of the form

$M = \{fg | g \in G\}$, on each of which $G$ acts transitively. The orbits are connected because $G$ is connected.

✧ The first well known result is that they are all symplectic manifolds.

This is described as follows:

The action on $\mathcal{G}^*$ of the one-parameter subgroup generated by $A \in \mathcal{G}$ is the flow of the vector field $X_A \in V(\mathcal{G}^*)$, where the value of $X_A$ at $f$ is $f' \in T_f(\mathcal{G}^*) = \mathcal{G}^*$, where $f'(B) = f([A, B])$.

Let $f \in \mathcal{G}^*$ and let $M$ be the orbit through $f$. The vector fields $X_A$ span the $T_f M$. In fact, if $X_A$ and $X_{A'}$ are the same, then $f([A, B]) = f([A', B])$.

The symplectic form on the coadjoint orbit through $f \in \mathcal{G}^*$ is given by the formula $\omega(X_A, X_B) = \frac{1}{2} f([A, B])$.

In fact it depends only on $X_A$ and $X_B$ and not on $A, B$. It is invariant under the coadjoint action.

✧ The second result of Kostant and Souriau is that any symplectic manifold on which $G$ has a transitive Hamiltonian action is a covering space of an orbit in $\mathcal{G}^*$. In principle all elementary classical systems with symmetry group $G$ which admit a moment map can be found by analysing the orbits in $\mathcal{G}^*$.

✧ Kostant used the method of geometric quantization to quantize coadjoint orbits with an eye to constructing representations of $G$.

# Our work:

- Geometric Quantization of the finite Toda systems (joint with Dr. Saibal Ganguli)
- Extension of the construction of Quillen bundle to some other situations (various moduli spaces):
  - The vortex moduli space for a compact Riemann surface.
  - The Hitchin moduli space for a compact Riemann surface genus $g > 1$.
  - Dimensionally reduced Seiberg-Witten equations on a Riemann surface.
  - Dimensionally reduced generalized Seiberg-Witten equations on a Riemann surface, (joint work with Dr. Varun Thakre).
- Deriving a general theorem that the Quillen bundle is in some sense central to geometric quantization of compact symplectic and Kähler manifolds which are integral (jointly with Mathai Varghese):
  - Holomorphic Quillen determinant bundle on integral compact symplectic and Kähler manifolds

# Geometric Quantization of the finite Toda system

- Adler had showed that the Toda system can be given a coadjoint orbit descriptio[n]. We quantize the Toda system by viewing it as a single coadjoint orbit orbit of [the] multiplicative group of lower triangular matrices of determinant one with positiv[e] diagonal entries.
- We get a unitary representation of the group with square integrable polarized sec[-]tions of the quantization as the module. The eigenvectors of the Hamiltonian ar[e] given by the Whittacker functions, as in the literature.

# Determinant line bundle of Quillen

- The moduli space of solutions to various equations arising from physics often carry symplectic structure. In order to quantize some of these systems, we wish to quantize the moduli spaces using geometric quantization. In order to do that we need to understand the determinant line bundle of Quillen.

- $\mathcal{A} =$ space of unitary connections on a vector bundle $E$ associated to a principal $G$ bundle on a Riemann surface $\Sigma$. Can identify $\mathcal{A} = \mathcal{A}^{0,1}$ (since $A^{1,0*} = -A^{0,1}$) . On this infinite-dimensional space, we define the determinant line bundle of Quillen.

- One defines the Cauchy-Riemann operator on $\Sigma$ locally written as $\bar{\partial} + A^{0.1}$ which acts on sections of the vector bundle $E$. We denote it by $\bar{\partial}_A$. Construct a line bundle $\mathcal{L}$ on $\mathcal{A}^{0,1}$ as follows.

  The fiber on $A^{0,1}$ is $\det \bar{\partial}_A = \wedge^{\text{top}}(\text{Ker}\, \bar{\partial}_A)^* \otimes \wedge^{\text{top}}(\text{Coker}\, \bar{\partial}_A)$

- The dimension of $\text{Ker}\, \bar{\partial}_A$ jumps even locally but the whole object makes sense –after certain identifications (Quillen's ingenious construction). Quillen further shows that $\mathcal{L}$ carries a metric and a connection s.t. the Kähler potential is $\frac{\partial \zeta_A}{\partial s}(0)$ where $\zeta_A$ is the *zeta* function corresponding to the Laplacian of $\bar{\partial}_A$. The curvature turns out to be proportional to $\Omega(\alpha, \beta) = -\int_\Sigma \text{Tr}(\alpha \wedge \beta)$, the Kähler form on $\mathcal{A}^{0,1}$.

# Motivating example: geometric quantization of the moduli space of flat connections

- The motivating example in our context would be the geometric quantization of the moduli space of flat connections on a principal $G$-bundle $P$ on a compact Riemann surface $\Sigma$, using the Quillen determinant line bundle construction.
- The Hilbert space of quantization is given by some conformal blocks in a certain Conformal Field Theory. The literature is very vast but some names associated to this are Witten, Axelrod, Della Pietra, Narasimhan, Beauville, Drezet, Andersen.
- This example led to Witten's construction of the Jone's polynomial from the Chern's Simon's Gauge theory.
- Another example: Donaldson's construction of the Quillen bundle on the moduli space of ASD connections on Kahler surfaces.
- Yet another example: Quite recently, Eriksson and Romao have constructed Quillen bundle on vortex moduli spaces – a very algebro-geometric construction.

# The Vortex Moduli Space

- The vortex equations are as follows. Let $\Sigma$ be a compact Riemann surface and let $\omega = \frac{1}{2}h^2 dz \wedge d\bar{z}$ be $i$ times the volume form on it. Let $A$ be a unitary connection on a principal $U(1)$ bundle $P$

  Let $L$ be a complex line bundle associated to $P$. Let $\Psi$ be a section of $L$, i.e. $\Psi \in \Gamma(\Sigma, L)$ . There is a Hermitian metric $H$ on $L$.

  The pair $(A, \Psi)$ will be said to satisfy the vortex equations if

$$F(A) = \frac{1}{2}(1 - |\Psi|^2_H)\omega, \tag{1}$$

$$\bar{\partial}_A \Psi = 0, \tag{2}$$

  where $F(A)$ is the curvature of the connection $A$ and $\partial_A + \bar{\partial}_A$ is the covariant derivative operator

- Let $\mathcal{S}$ be the space of solutions to $(1)$ and $(2)$. There is a gauge group acting on the space of $(A, \Psi)$ which leaves the equations invariant. Locally it looks like $\mathrm{Maps}(\Sigma, U(1))$. If $g$ is an $U(1)$ gauge transformation then $(A_1, \Psi_1)$ and $(A_2, \Psi_2)$ are gauge equivalent if $A_2 = g^{-1}dg + A_1$ and $\Psi_2 = g^{-1}\Psi_1$. Taking the quotient by the gauge group of $\mathcal{S}$ gives the moduli space of solutions to these equations and is denoted by $\mathcal{M}$.

- $\int_\Sigma F(A) = 4\pi N$ where $N$ is an integer called the vortex number.

- By theorems of Taubes and Bradlow, one has that the moduli space of vortices is parametrized by the zeroes of $\Psi$. Thus the moduli space is the symmetric product, $\mathrm{Sym}^N(\Sigma)$. When the Riemann surface is $\mathbb{C}P^1$, the moduli space is complex projective space $\mathbb{C}P^N$.

# The metric and symplectic form

- On $\mathcal{M}$ one can define a metric which is the descendent of a metric on the configuration space, given by,

$$\mathcal{G}(X, Y) = \int_{\Sigma} *\alpha_1 \wedge \alpha_2 + i \int_{\Sigma} Re < \beta, \eta >_H \omega$$

- a complex structure induced by $\mathcal{I} = \begin{bmatrix} * & 0 \\ 0 & i \end{bmatrix} : T_p\mathcal{C} \rightarrow T_p\mathcal{C}$ where $* : \Omega^1 \rightarrow \Omega^1$ is the Hodge star operator on $\Sigma$ such that $*\alpha^{1,0} = -i\alpha^{1,0}$ and $*\alpha^{0,1} = i\alpha^{0,1}$

✦ We define the symplectic form to be the Marsden-Weinstein descendent of the form induced by the following symplectic form of the configuration space

$$\begin{aligned}\Omega(X,Y) &= -\frac{1}{2}\int_\Sigma \alpha_1 \wedge \alpha_2 + \frac{i}{2}\int_\Sigma Re < i\beta, \eta >_H \omega \\ &= -\frac{1}{2}\int_\Sigma \alpha_1 \wedge \alpha_2 - \frac{1}{4}\int_\Sigma (\beta H \bar{\eta} - \bar{\beta} H \eta)\omega\end{aligned}$$

In fact, $\mathcal{G}(\mathcal{I}X, Y) = 2\Omega(X,Y)$. This form is in fact a Kähler form on the $\mathcal{M}$.

# Quillen bundle on the vortex moduli space

- We denote the Quillen bundle $\mathcal{P} = \det(\bar{\partial}_A)$ which is well defined on $\mathcal{C} = \mathcal{A} \times \Gamma(L)$. We can give $\mathcal{P}$ a modified Quillen metric, namely, we multiply the Quillen metric $e^{-\zeta'_A(0)}$ by the factor $e^{-\frac{i}{4\pi}\int_M |\Psi|^2_H \omega}$, where recall $\zeta_A(s)$ is the zeta-function corresponding to the Laplacian of the $\bar{\partial}_A$ operator.

- The factor $e^{-\zeta'_A(0)}$ contributes $\frac{i}{2\pi}\left(-\frac{1}{2}\int_\Sigma \alpha_1 \wedge \alpha_2\right)$ to the curvature, and the factor $e^{-\frac{i}{4\pi}\int_M |\Psi|^2_H \omega}$ contributes $\frac{i}{2\pi}\left(-\frac{1}{4}\int_\Sigma (\beta H \bar{\eta} - \bar{\beta} H \eta)\omega\right)$ to the curvature.

  Thus we have the following:

  The curvature of $\mathcal{P}$ with the modified Quillen metric is indeed $\frac{i}{2\pi}\Omega$ on the affine space $\mathcal{C}$.

- One can show that the modified Quillen line bundle descends to the moduli space of vortices and indeed its curvature is $\frac{i}{2\pi}\Omega$ where $\Omega$ here is the descendent of the symplectic form on the affine space. We showed that this form is exactly the famous Samols-Manton-Nasir form on the moduli space.

- Now we restrict our attention to the case when the Riemann surface is $S^2$ of radius $R$. The moduli space is $\mathbb{C}P^N$. If $\frac{1}{2\pi}\Omega$ is integral, then $\frac{1}{2\pi}[\Omega] = [\omega_{FS}]$, (the cohomology class of the Fubini-Study Kahler form) if we fine tune the volume of the sphere!

  Thus in this case $[\frac{1}{2\pi}\Omega] = [\frac{1}{2\pi}\omega_{SMN}] = [\omega_{FS}]$.

Dimension of the Hilbert space of quantization of vortex moduli space:

✦ Theorem (D., S. Ganguli). The dimension of the Hilbert space of geometric quantization for $N$-vortices on a compact Riemann surface of genus $g$ is given by the holomorphic Euler characteristic of the quantum line bundle $L$ on the $Symm^N(\Sigma)$ (i.e. the vortex moduli space) which has curvature proportional to the Manton-Nasir form, $\omega_{MN}$, a Kähler form ( which is integral if the volume of the surface is $A = 4\pi k$, $k \in \mathbb{Z}^+$ ). The dimension of the Hilbert space is $\binom{k}{N}$, where $k = \frac{A}{4\pi}$ if $\frac{A}{4\pi} > max(N, g-1)$ .

# 2 Two General Theorems

# Holomorphic determinant line bundles on integral compact Kähler and symplectic manifolds

- Given an integral Kähler form $\omega$ on a compact Kähler manifold, there is a holomorphic line bundle $\mathcal{L}$ on the manifold with connection and curvature proportional to $\omega$ (setting for geometric quantization). We show in the following theorem that a high enough tensor power of $\mathcal{L}$ is indeed a Quillen determinant bundle.

- Theorem ( D., V. Mathai): Any compact Kähler manifold $M$ with integral Kähler form $\omega$, parametrizes a natural holomorphic family of Cauchy-Riemann operators $\{\bar{\partial}_a : a \in M\}$ on $CP^1$ such that the Quillen determinant line bundle $\det(\bar{\partial}) \cong \mathcal{L}^{\otimes k}$ as holomorphic line bundles, where $\mathcal{L}$ is the holomorphic line bundle determined by $\omega$, for some sufficiently large $k$.

- The strategy of the proof of the above theorem goes briefly as follows. We first establish the theorem for complex projective spaces $\mathbb{C}P^N$ for all integers $N$ and for $k = 1$. This is achieved by viewing $\mathbb{C}P^N$ as the moduli space of $N$-vortices on $\mathbb{C}P^1$.

- Every pt $[(A, \Psi)]$ on the moduli space of $N$-vortices on a sphere corresponds to a point $a \in \mathbb{C}P^N$ and determines a Cauchy-Riemann operator $\bar{\partial}_a$ on $\mathbb{C}P^1$. The Quillen determinant line bundle on $\mathbb{C}P^N$ denoted by $\det(\bar{\partial})$, (equipped with the modified metric which involves $\Psi$ also ) is isomorphic as a holomorphic line bundle to the hyperplane bundle which is the dual of the tautological bundle on $\mathbb{C}P^N$ and whose curvature is proportional to $\omega_{FS}$. This is because $[\frac{1}{2\pi}\Omega] = [\omega_{FS}]$.

- The next step is to apply the Kodaira embedding theorem into $\mathbb{C}P^N$ to establish the theorem in general for a Kahler manifold $M$ with integral Kahler form $\omega$. For $k$ large enough, there is a holomorphic emdedding $\phi_k$ of $M$ into $P(H^0(M, \mathcal{O}(\mathcal{L}^{\otimes k}))^*) = \mathbb{C}P^N$, (Kodaira embedding), such that $[\phi_k^*(\omega_{FS})] = k[\omega]$ where $\omega_{FS}$ is the Fubini-Study form on $\mathbb{C}P^N$. In fact, $\phi_k^*(H) \cong \mathcal{L}^{\otimes k}$, where $H$ is the hyperplane bundle on $\mathbb{C}P^N$ . But the hyperplane bundle is a Quillen determinant line bundle, as we saw before. Hence we have our theorem.

- Our second result is a symplectic analogue of our theorem, established using Gromov's embedding theorem.

  Theorem (D., V. Mathai): any compact symplectic manifold $M$ with integral symplectic form $\omega$, parametrizes a natural smooth family of Cauchy-Riemann operators $\{\bar{\partial}_a : a \in M\}$ on $\mathbb{C}P^1$ such that the determinant line bundle $\det(\bar{\partial}) \cong \mathcal{L}$ as complex line bundles, where $\mathcal{L}$ is the prequantum line bundle determined by $\omega$.

# An equivariant version

D.-Mathai conjecture on an equivariant version was proved by I. Biswas

Let $M$ be a compact Kähler $G$-manifold $M$ with integral Kähler form $\omega$, where $G$ is a compact Lie group. That is, $\omega$ is a $G$-invariant Kähler form on the $G$-manifold $M$ and all structures are $G$-invariant. Let $\mathcal{L}$ denote the holomorphic $G$-line bundle determined by $\omega$. Then $M$ parametrizes a natural holomorphic $G$-equivariant family of Cauchy-Riemann operators $\{\bar{\partial}_z : z \in M\}$ on a compact Kähler $G$-manifold $Z$ such that the Quillen determinant line bundle $\det(\bar{\partial}) \cong \mathcal{L}^{\otimes k}$ as holomorphic $G$-line bundles, for some sufficiently large $k$.

Similarly a symplectic analogue.

# Geometric quantization of the Hitchin moduli space

- The Hitchin moduli space $\mathcal{M}_H$ is the moduli space of solutions of a set of equations obtained by dimensionally reducing the Self-Dual Yang-Mills from $4$ to $2$ dimensions and defining them on a compact Riemann surface of genus $g > 1$. The moduli space has very rich structure. In particular, it has a hyperKähler structure, i.e. three symplectic forms (say, $\Omega$, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ ), derived from a metric $g$ and three complex structures $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ (Hitchin.)

- We showed that the three Kahler forms are integral and there are three prequantum line bundles corresponding to these Kahler forms, i.e. line bundles on the Hitchin moduli space $\mathcal{M}_H$ whose curvatures are proportional to these Kahler forms. These are all Quillen bundles with some modifications.

  For example the first Kahler form can be quantized by a determinant of a $\bar{\partial}_A$ operator, where the Quillen metric has been modified by terms involving the Higgs field $\Phi$.

  The first Kahler form is integral – private communication with I. Biswas

- ✦ Quantization:

  These line bundles are holomorphic w.r.t. their respective complex structures. One takes holomorphic sections of these line bundles to be the appropriate Hilbert spaces.

# Further Work Completed in this direction:

- Geometric Quantization of the moduli space of dimensional reduction of modified Seiberg-Witten equations on a Riemann Surface
- Studying the moduli space of dimensional reduction of generalized Seiberg-Witten equations (where the spinors take values in a hyperKähler manifold) and its geometric quantization. Joint work with Dr. Varun Thakre

# Talk is partly Based on:


- N.M.J. Woodhouse: Geometric Quantization, Second Edition, Clarendon Press, Oxford.
- D. Quillen: Determinants of Cauchy-Riemann operators over a Riemann surface; Functional Analysis and Its Application, 19, 31-34 (1985).
- R. Dey: HyperKähler prequantization of the Hitchin systems and Chern-Simons gauge theory with complex gauge group; Adv. Theor. Math. Phys. 11 (2007) 819-837; math-phy/0605027
- R. Dey: Geometric Quantization of the Hitchin System, Int. J. Geom. Methods Mod. Phys., 14, no. 4, 1750064 (2017)
- ✧ R. Dey, Geometric prequantization of the moduli space of the vortex equations on a Riemann surface; Journal of Mathematical Physics, vol. 47, issue 10, (2006), page 103501-103508; math-phy/0605025

  ✧ R. Dey, Erratum: Geometric prequantization of the moduli space of the vortex equations on a Riemann surface; Journal of Mathematical Phys. 50, 119901 (2009)

- R. Dey, V. Mathai: Holomorphic Quillen determinant bundle on integral compact Kähler manifolds, Quart. J. Math. 64 (2013), 785-794, Quillen Memorial Issue; arXiv:1202.5213v3

- R. Dey, S. Ganguli: The dimension of the Hilbert space of geometric quantization of vortices on a Riemann surface, to appear in Int. J. Geom. Methods Mod. Phys.

- R. Dey, S. Ganguli: Geometric Quantization of finite Toda systems and coherent states, Jour. Geom. Symm. Phys. vol 44,21-38, 2017.

- R. Dey and V. Thakre: Generalized Seiberg-Witten equations on a Riemann surface, Jour. Geom. Symm. Phys. vol 45, 47-66, 2017.

- R. Dey: Geometric prequantization of the modified Seiberg-Witten equations in 2-dimensions; Adv. Theor. Math. Phys. 13.5 (2009)

THANK YOU