Weil-Petersson Currents

ICTS 2018

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Then the tangent space of $\mathcal{T}_p$ at X can be identified with the space of harmonic Beltrami differentials on X

$$T_X(\mathcal{T}_p) = H^1(X, \mathcal{T}_X) \simeq H^0(X, \Omega_X^{\otimes 2})^\vee$$

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Previously introduced by **Hans Petersson**.

$$\langle \phi dz^2, \psi dz^2 \rangle_{PW} = \int_X \frac{\phi dz^2\, \overline{\psi}\, \overline{dz}^2}{g dz \overline{dx}} = \int_X \frac{\phi \overline{\psi}}{g} dz \overline{dz}$$

(where $g dz \overline{dz}$ Poincaré metric)

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Substantially different methods

Generalized Petersson-Weil metrics

Approach

Given a holomorphic family (of vector bundles, complex manifolds, etc.) parameterized by S.
Kodaira-Spencer map for $s \in S$

$$\rho_s : T_s S \to \{\text{infinitesimal deformations}\} = Def(D)$$

Equip the vector space of infinitesimal deformations with natural L^2-inner product (typically induced on a cohomology group by distinguished metric on given object).

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Generalizations to moduli spaces of stable holomorphic vector bundles, to Douady spaces, moduli space of canonically polarized manifolds, and for families of polarized Calabi-Yau manifolds.

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$\Rightarrow$intrinsically defined hermitian metric G^{WP} on parameter space S.

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$\Rightarrow$Kähler property of G^{WP}, Kähler form ω^{WP} descending to corresponding moduli space.

Synopsis: Fiber integrals for Weil-Petersson forms

Wolpert (1986)

$(f : \mathcal{X} \to S, g_{\mathcal{X}/S})$ holomorphic family of compact Riemann surfaces $\{\mathcal{X}_s = f^{-1}(s)\}_{s \in S}$, with Poincaré metrics g_s. Then

$$\omega^{WP} = \frac{1}{2\pi} \int_{\mathcal{X}/S} c_1^2(\mathcal{X}/S, g).$$

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Idea: Realize ω^{WP} as integral of a real, closed $(n+1, n+1)$ over family of compact n-dimensional manifolds.

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Fact:

$$T_s S \ni v \mapsto (v \lrcorner F)|X \times \{s\}$$

harmonic representative of $\rho_s(v)$.

Fix $\det(\mathcal{E}_s)$, $r = rk(\mathcal{E})$, then simplest form ($\lambda = 0$):

Fiber integral formula

$$\frac{1}{2\pi^2}\omega^{WP} = -\frac{1}{r}\int_{X\times S/S} ch_2(End(\mathcal{E}), h) \wedge \omega^{n-1}.$$

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where

$$ch_2(End(\mathcal{E}), h) = \frac{1}{2} tr\left(\frac{\sqrt{-1}}{2\pi}F \wedge \frac{\sqrt{-1}}{2\pi}F\right).$$

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Theorem (Biswas-Sch.)

Let $\mathcal{E} \in Coh(X \times S)$, $\mathcal{O}_S$-flat, S a polydisk. Let $\mathcal{E}_s$ be stable and locally free for $s \in S \backslash A$. Then ω^{WP} extends to S as a closed, positive $(1,1)$-current.

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Heat equation

Let $s \in S' = S \backslash A$. For all $t \geq 0$ there a differentiable endomorphisms $h(t)$ of $\mathcal{E}_s$ with $\det(h(t)) = 1$ s.t.

$$\frac{\partial h}{\partial t} h^{-1} = -(\Lambda_X F_s - \lambda id_{\mathcal{E}_s}).$$

$$h(0) = h_0 \text{ initial metric}$$

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$$R_2(t) = \int_0^t \mathrm{tr}\big(F(\tau)\cdot\Lambda_X F(\tau)\big)\,d\tau$$

over all of $X \times S'$

$$\frac{1}{2}\left(\mathrm{tr}(F(t)\wedge F(t)\right) = \mathrm{tr}(F_0\wedge F_0)) + \sqrt{-1}\,\overline{\partial}\partial R_2(t)$$

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Boundedness of **Donaldson functional**

$$M(t) = \int_X R_2(t) \wedge \omega_X^n \leq 0 \qquad 0 \leq t < \infty$$

$\Rightarrow$ claim

Douady spaces (Axelsson-Biswas-Sch.)

Given Kähler manifold (Z, ω_Z), Douady space **D** of embedded *n*-dimensional subspaces:

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\mathcal{X} & \stackrel{i}{\hookrightarrow} & Z \times \mathbf{D} \\
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Varouchas $\Rightarrow$ continuous potential for ω^{WP}.

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Hurwitz space (Axelsson-Biswas-Sch.)

Objects

Riemann surfaces $pr : X \to \mathbb{P}_1$ simple coverings.

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Fiber integral

$$\omega^{WP} \simeq \int_{\mathcal{X}/S} c_1(\mathcal{X}/S, g) \wedge pr^* c_1(\mathbb{P}_1, h)$$

Extension Theorem holds

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Moduli of canonically polarized manifolds (Sch.)

Objects

$$(X, \omega_X), \text{ s.t. } K_X \text{ ample}$$

$$\omega_X \text{ Kähler-Einstein: } Ric(\omega_X) = -\omega_X$$

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Degenerating families:

- Extend $\omega_{\mathcal{X}}$ as a positive current
- Push forward to parameter space

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Universal (degenerating) family

$$\begin{array}{ccc} \mathcal{X} & \xhookrightarrow{\ \Phi\ } & \mathbb{P}_N \times S \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle pr} \\ & & S \end{array}$$

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$$\omega^n_{\mathcal{X}/S} = e^{u+F}(\omega^0_{\mathcal{X}/S})^n$$

with

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Let $f : \mathcal{X} \to S$ be singular over $S' = S\backslash V(\sigma)$. Then (from embedding)

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Consequence:

$\omega_{\mathcal{X}'/S'}$ possesses bounded potentials (locally w.r.t. to $\mathcal{X}$)

hence extends as a positive current.

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Descend to moduli space

Finally

$$\omega^{WP} = \int_{\mathcal{X}/S} (\omega_{\mathcal{X}})^{n+1} = \int_{\mathcal{X}/S} (\omega^0_{\mathcal{X}} + \sqrt{-1}\partial\overline{\partial} u)^{n+1}$$

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First integral possesses continuous potential by Varouchas' theorem.

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Present work for SLC singularities by Takayama, Tosatti, Rong-Zhang, a.o.

Degenerating families of Calabi-Yau manifolds

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Calabi-Yau manifold X: $c_{1,\mathbb{R}}(X) = 0$

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Fujiki-Sch. ’90

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$$\omega^{WP} = -\int_{\mathcal{X}/S} 2\pi c_1(\mathcal{X}/S, g) \wedge \omega^n_{\mathcal{X}} \text{ with norm. } \int_{\mathcal{X}/S} \omega^{n+1}_{\mathcal{X}} = 0$$

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Then $\exists$ Kähler form $\widetilde{\omega}_{\mathcal{X}}$ s.t. $\widetilde{\omega}_{\mathcal{X}} | \mathcal{X}_s = \omega_{\mathcal{X}_s}$ Ricci flat.

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Validity of the assumptions

Cheeger '70, Cheeger-Yau '80

The Green's function is bounded, if the diameter (of the fibers $\mathcal{X}_s$) is bounded from above.

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The diameter is bounded for polarized families of Calabi-Yau manifolds, under mild assumptions for the type of degeneration.

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M. Păun: Demailly's approximation theorem of psh. functions by fctns. with algebraic singularities. Ohsawa-Takegoshi Thm.

On S' for any form representing the polarization, in particular for ω:

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Theorem 2 and wedge product of currents Bedford-Taylor.

Corollary

ω^{WP} extends to S as a positive, closed current.

Extension Theorem

Kähler form $\omega = \omega_{\mathcal{X}}$

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In terms of Euclidean volume forms

$$\omega^n \wedge f^* dV_S = \chi \cdot dV_{\mathcal{X}} = e^{\psi_U} \cdot dV_{\mathcal{X}}$$

with $\psi_U = -\infty$ at singular points of f.

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Corresponding curvature form on X':

$$\Theta_{h^{\omega}_{X/S}}(K_{X/S}) = \sqrt{-1}\partial\overline{\partial}\psi_U | X' = \sqrt{-1}\partial\overline{\partial} \log\left(\omega^n \wedge f^* dV_S\right).$$

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Ricci-flat volume form

exists a unique function $\eta_s \in C^\infty(X_s)$ s.t.

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Here the Kähler property of $\mathcal{X}$ is being used.

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Curvature of relative canonical bundle near singular points

$$\sqrt{-1}\partial\overline{\partial}(\eta + \psi_U) = \sqrt{-1}\partial\overline{\partial}\eta + \Theta_{h^{\omega}_{X'/S'}}(K_{X'/S'}) = \Theta_{h^{\rho}_{X'/S'}}(K_{X'/S'}).$$

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Aim.

Bound the potential $\theta := \eta + \psi_U$

Ohsawa-Takegoshi Extension Theorem

Let $\psi : \Omega \subset \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ a psh. function on a bounded pseudoconvex domain. Then there exists a constant $C > 0$ (only depending on $diam(\Omega)$) s.t. given a hyperplane $H \subset \mathbb{C}^n$

$$\forall f \in \mathcal{O}(\Omega \cap H) \text{ with } \int_{\Omega \cap H} e^{-\psi} |f|^2 d\lambda < \infty$$

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$\exists F \in \mathcal{O}(\Omega)$ s.t. $F|\Omega \cap H = f$ and

$$\int_{\Omega} e^{-\psi}|F|^2 d\lambda \leq C \int_{\Omega\cap H} e^{-\psi}|f|^2 d\lambda.$$

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- Apply the Ohsawa-Takegoshi Theorem to $|\sigma_k|^2$.
- Use extra variable $\mathbb{Z} \ni m \to \infty$ in weight function $e^{-m\theta}$.

Hilbert space of holomorphic functions

Definition

$m > 1$, $s \in S'$ **weighted norm** on $\mathcal{O}(U_s)$:

$$\|\phi\|^2_{m,s} := \int_{U_s} |\phi|^2 \, e^{-m\,\theta_s} \rho_s^n. \quad \text{for } \phi \in \mathcal{O}(U_s).$$

Hilbert space

$$\mathcal{H}_s^{(m)} = \{\phi \in \mathcal{O}(U_s); \|\phi\|^2_{m,s} < \infty\}.$$

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Hilbert space

$$\mathcal{H}_s^{(m)} = \{\phi \in \mathcal{O}(U_s); \|\phi\|^2_{m,s} < \infty\}.$$

$$e^{\theta_s} dV_X / f^*(dV_S) = \rho_s^n \text{ on } \mathcal{X}'$$

$$\text{so that } \|\phi\|^2_{m,s} = \int_{U_s} |\phi|^2 \, e^{-(m-1)\theta_s} dV_{X_s}$$

where $dV_{X_s} = dV_X / f^* dV_S$ on U_s.

Theorem (cf. Demailly)

$$\theta_s(x) = \lim_{m\to\infty}\left(\sup\left\{\frac{1}{m}\log|\phi(x)|^2 ; \phi \in \mathcal{H}_s^{(m)} \text{ with } \|\phi\|^2_{m,s} \le 1\right\}\right)$$

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By general theory

$$\varphi_m^s(x) = \sup \left\{ \frac{1}{m} \log |\phi(x)|^2 : \phi \in \mathcal{H}_s^{(m)} \text{ s.t. } \|\phi\|^2_{m,s} \leq 1 \right\}$$

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$$\leq \frac{n!}{\pi^n r^{2n}} \sup_{\zeta \in B_r} \left(e^{m\theta_s} \right) \int_{B_\varepsilon} |\phi(\zeta)|^2 \, e^{-m\theta_s} d\lambda(\zeta).$$

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Let $C_s := \sup_{B_r} d\lambda/(\rho_s)^n$. Then it follows that

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|\phi(x)|^2 &\leq \frac{n! C_s}{\pi^n \varepsilon^{2n}} \sup_{\zeta \in B_\varepsilon} \left(e^{m\theta_s} \int_{B_\varepsilon} |\phi(\zeta)|^2 \, e^{-m\theta_s} (\rho_s)^n \right) \\
&\leq \frac{n! C_s}{\pi^n \varepsilon^{2n}} \sup_{\zeta \in B_\varepsilon} \left(e^{m\theta_s} \right).
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Hence

$$\frac{1}{m}\log|\phi(x)|^2 \leq \sup_{\zeta\in B_\varepsilon}\theta_s(\zeta) + \frac{1}{m}\left(\log\frac{n!}{\pi^n\varepsilon^{2n}} + \log C_s\right).$$

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Conversely, the Ohsawa-Takegoshi Theorem will be applied to the extension of functions to the ambient space that are given at a point.

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Finally let $m \to \infty$.

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Note the order of limit processes: First let $m \to \infty$. The constants C_s, which are unbounded as the singular figers are approached, are eliminated in this way.

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Proof of the Extension Theorem

Let $\phi_s \in \mathcal{H}_s^{(m)}$ with $\|\phi_s\|_s^2 \leq 1$. Hölder inequality:

$$\begin{aligned}\int_{U_s} |\phi_s|^{2/m}\, dV_{X_s} &= \int_{U_s} |\phi_s|^{2/m}\, e^{-\theta_s} \rho_s^n \\ &\leq \left(\int_{U_s} |\phi_s|^2\, e^{-m\theta_s} \rho_s^n \right)^{\frac{1}{m}} \left(\int_{U_s} \rho_s^n \right)^{\frac{m-1}{m}} \\ &\leq (\mathrm{vol}(X_s))^{\frac{m-1}{m}} \leq C\end{aligned}$$

$C = \max(1, \mathrm{vol}(\mathrm{X_s}))$, independent of m and s.

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- There exists a numerical constant $C_0 > 0$ independent of m and s such that

$$\int_U |F|^{2/m} dV_z \leq C_0 \int_{U_s} |\phi|^{2/m} \frac{dV_z}{p^* dV_s} \leq C_0 \cdot C.$$

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Mean value inequality applicable

$$|F(x)|^{2/m} \leq C_r \int_{B_r(x)} |F(z)|^{2/m} dV_z \leq C_r \cdot C_0 \cdot C$$

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Use argument for all fixed $\theta_s,\ s \in S'$.

θ_s are uniformly bounded on $W \cap X'$ where $W \subset\subset U$.

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$\Rightarrow$ curvature $\Theta_{h^\rho_{X/Y}}(K_{X/Y})|_{X_0}$ extends as a d-closed positive real $(1,1)$-current on the total space X completing the proof.

Application to the extension of positive line bundles on moduli spaces

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singular hermitian metric h' s.t. $\omega_{Y'} = 2\pi c_1(L', h')$ is positive.

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Then $\exists$ modification $\tau : Z \to Y$, isomorphism over Y' s.t. (L', h') extends to Z as a holomorphic line bundle with singular hermitian metric.

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Note that the extended curvature current may be different from the given one.

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Projective embeddings

Sch.-Tsuji 2004

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Theorem (Păun-Sch. 2017)

Let X be a compact Moishezon manifold, and L a big line bundle on X. Then the restricted volume of L along a closed, irreducible complex subspace V is positive, if and only if V is not contained in $E_{nK}(L)$. For high multiples of L, the linear system $|mL|$ embeds the complement of the non-Kähler locus into a projective space.

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Definition

Let $m > 0$, then the image of the restriction map is denoted by

$$H^0(X|V, mL) = \mathrm{Im}\left(H^0(X, mL) \to H^0(V, mL|V)\right).$$

The restricted volume of L on V is

$$\mathrm{vol}_{X|V}(L) = \overline{\lim_{m\to\infty}}\left(\frac{\dim H^0(X|V, mL)}{m^d/d!}\right).$$

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Given any positive current T, the union of all Lelong sublevel sets $E_c(T) = \{x \in X; \nu(x, T) \geq c\}$ is denoted by

$$E_+(T) = \bigcup_{c>0} E_c(T).$$

The classes α containing Kähler currents α are called *big*.

Definition

Let X be a compact, complex manifold. The non-Kähler locus of a big class equals

$$E_{nK}(\alpha) := \bigcap E_+(T),$$

where T runs through all Kähler currents representing α.

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Proposition

Let X be a compact, complex manifold, and L a big line bundle. Let $V \subset X$ denote an irreducible, reduced, closed, complex subspace, whose restricted volume is positive

$$\mathrm{vol}_{X|V}(L) > 0.$$

Then for $m \gg 0$ the linear system $|W_m|$ with $W_m = H^0(X|V, mL)$ yields a map that is birational onto the image.