Weil-Petersson Currents ICTS 2018

Georg Schumacher

Marburg



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Previously introduced by Hans Petersson.

$$\langle \phi dz^2, \psi dz^2
angle_{PW} = \int_X rac{\phi dz^2 \,\overline{\psi} \,\overline{dz}^2}{g dz \,\overline{dx}} = \int_X rac{\phi \overline{\psi}}{g} dz \,\overline{dz}$$

(where gdz dz Poincaré metric)



Wolpert



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Substantially different methods

Marburg

Generalized Petersson-Weil metrics

Approach

Given a holomorphic family (of vector bundles, complex manifolds, etc.) parameterized by *S*. Kodaira-Spencer map for $s \in S$

 $\rho_{s}: T_{s}S \rightarrow \{\text{infinitesimal deformations}\} = Def(D)$

Equip the vector space of infinitesimal deformations with natural L^2 -inner product (typically induced on a cohomology group by distinguished metric on given object).



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Generalizations to moduli spaces of stable holomorphic vector bundles, to Douady spaces, moduli space of canonically polarized manifolds, and for families of polarized Calabi-Yau manifolds.

Universität Marburg

\Rightarrow intrinsically defined hermitian metric G^{WP} on parameter space S.



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 \Rightarrow Kähler property of G^{WP} , Kähler form ω^{WP} descending to corresponding moduli space.



(B)

Synopsis: Fiber integrals for Weil-Petersson forms

Wolpert (1986)

 $(f : \mathcal{X} \to S, g_{\mathcal{X}/S})$ holomorphic family of compact Riemann surfaces $\{\mathcal{X}_s = f^{-1}(s)\}_{s \in S}$, with Poincaré metrics g_s . Then

$$\omega^{W\!P} = rac{1}{2\pi}\int_{\mathcal{X}/\mathcal{S}} c_1^2(\mathcal{X}/\mathcal{S},g).$$

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Idea: Realize ω^{WP} as integral of a real, closed (n + 1, n + 1) over family of compact *n*-dimensional manifolds.



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Fact:

$$T_s S \ni v \mapsto (v \lrcorner F) | X \times \{s\}$$

harmonic representative of $\rho_s(v)$.



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$$\frac{\partial h}{\partial t}h^{-1} = -(\Lambda_X F_s - \lambda i d_{\mathcal{E}_s}).$$

$$h(0) = h_0 \text{ initial metric}$$

Marburg

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over all of $X \times S'$

 $\frac{1}{2}\left(\operatorname{tr}(F(t)\wedge F(t)) = \operatorname{tr}(F_0\wedge F_0)\right) + \sqrt{-1}\ \overline{\partial}\partial R_2(t)$



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Boundedness of Donaldson functional

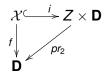
$$M(t) = \int_X R_2(t) \wedge \omega_X^n \leq 0 \qquad 0 \leq t < \infty$$

 \Rightarrow claim



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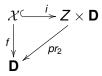
Given Kähler manifold (Z, ω_Z) , Douady space **D** of embedded *n*-dimensional subspaces:



universal, flat embedded family.



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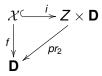


universal, flat embedded family. Over smooth locus D':

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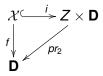


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Varouchas \Rightarrow continuous potential for ω^{WP} .

Objects

Riemann surfaces $pr : X \to \mathbb{P}_1$ simple coverings.



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Fiber integral

$$\omega^{WP} \simeq \int_{\mathcal{X}/S} c_1(\mathcal{X}/S,g) \wedge \textit{pr}^*c_1(\mathbb{P}_1,h)$$

Extension Theorem holds

Objects

 (X, ω_X) , s.t. K_X ample ω_X Kähler-Einstein: $Ric(\omega_X) = -\omega_X$



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Kodaira-Spencer map

Family $\mathcal{X} \rightarrow S, s \in S$

$$\rho_{s}: T_{s}S \rightarrow H^{1}(\mathcal{X}_{s}, \mathcal{T}_{\mathcal{X}_{s}})$$



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Curvature of the relative canonical bundle

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Degenerating families:

• Extend $\omega_{\mathcal{X}}$ as a positive current

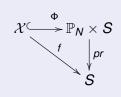


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Degenerating families:

- Extend ω_χ as a positive current
- Push forward to parameter space





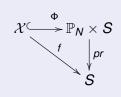
for
$$\Phi = \Phi_{|mK_{\mathcal{X}/S}|}$$



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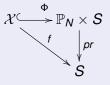
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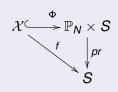
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Fubini-Study metric h^{FS} on $\mathcal{O}_{\mathbb{P}_N}(1)$, induces $(\Phi^* h^{FS})^{1/m}$ on $K_{\mathcal{X}/S}$, i.e. relative (initial) volume form $\Omega^0_{\mathcal{X}/S}$,





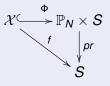
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s.t.





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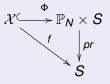
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Universität Marburg

Philipps



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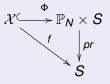
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The form $\omega_{\mathcal{X}}^{0}$ fails to be Kähler-Einstein

$$(\omega^0_{\mathcal{X}/\mathcal{S}})^n = e^{-F}\Omega^0_{\mathcal{X}/\mathcal{S}}$$



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C⁰-estimates

$$u|\mathcal{X}_{s} \leq sup(-F|\mathcal{X}_{s})$$



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1

Let $f : \mathcal{X} \to S$ be singular over $S' = S \setminus V(\sigma)$. Then (from embedding)

$$|\sigma(s)|^k \cdot sup(-F|\mathcal{X}_s) \leq C$$
 for some $k > 0$.



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Consequence:

 $\omega_{\mathcal{X}'/S'}$ possesses bounded potentials (locally w.r. to \mathcal{X}' Philipps Universität Marburg hence extends as a positive current. Georg Schumacher Marburg

Finally

$$\begin{split} \omega^{WP} &= \int_{\mathcal{X}/S} (\omega_{\mathcal{X}})^{n+1} = \int_{\mathcal{X}/S} (\omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\overline{\partial}u)^{n+1} \\ &= \int_{\mathcal{X}/S} (\omega_{\mathcal{X}}^0)^{n+1} + \sqrt{-1}\partial\overline{\partial} \int_{\mathcal{X}/S} \sum_{j=0}^n u(\omega_{\mathcal{X}}^0)^j (\omega_{\mathcal{X}}^0 + \sqrt{-1}\partial\overline{\partial}u)^{n-j}. \end{split}$$



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Present work for SLC singularities by Takayama, Tosatti, Rong-Zhang, a.o.

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Calabi-Yau manifold X:

 $c_{1,\mathbb{R}}(X)=0$



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A polarized family $(f : \mathcal{X} \to S, \lambda_{\mathcal{X}/S})$ defined by

 $\lambda_{\mathcal{X}/S} \in R^1 f_*(\Omega^1_{\mathcal{X}/S})(S)$ s.t. $\lambda_{\mathcal{X}/S} | \mathcal{X}_s$ are polarizations for the fibers \mathcal{X}_s .

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be the curvature form of $(\mathcal{K}_{\mathcal{X}/\mathcal{S}}, g^{-1})$.



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Fujiki-Sch. '90

$$\omega^{WP} = -\int_{\mathcal{X}/S} 2\pi c_1(\mathcal{X}/S, g) \wedge \omega_{\mathcal{X}}^n \text{ with norm. } \int_{\mathcal{X}/S} \omega_{\mathcal{X}}^{n+1} = 0$$

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 $f: \mathcal{X} \to S$ (polarized) family of Calabi-Yau manifolds Ricci-flat relative volume forms $\omega_{\mathcal{X}/S}^n = g(z, s)dV(z)$ (with normalization).



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Then

$$\Theta_{\mathcal{X}} = rac{1}{\operatorname{vol}(\mathcal{X}_{s})} f^{*} \omega_{WP}$$



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$G_s(z, w)$ the Green's functions for Laplacians \Box_s on fibers.



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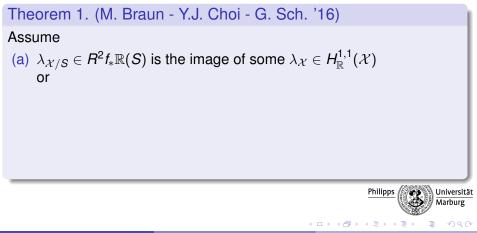
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Georg Schumacher

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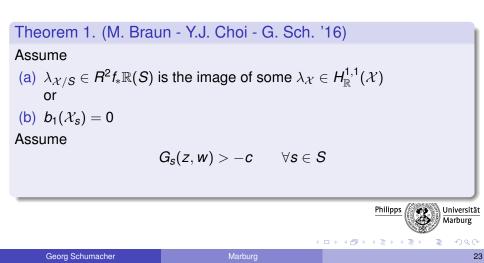
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Assume
 $G_s(z, w) > -c \quad \forall s \in S$
Then \exists Kähler form $\widetilde{\omega}_{\mathcal{X}}$ s.t. $\widetilde{\omega}_{\mathcal{X}} | \mathcal{X}_s = \omega_{\mathcal{X}_s}$ Ricci flat.

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Validity of the assumptions

Cheeger '70, Cheeger-Yau '80

The Green's function is bounded, if the diameter (of the fibers \mathcal{X}_s) is bounded from above.



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The diameter is bounded for polarized families of Calabi-Yau manifolds, under mild assumptions for the type of degeneration.



Let $f : \mathcal{X} \to S$ be a proper, open, surjective holomorphic mapping of Kähler manifolds $(\mathcal{X}, \omega_{\mathcal{X}})$ and $(S, \omega_{\mathcal{S}})$. Suppose that every regular fiber X_s is a Calabi-Yau manifold for $s \in S' = S \setminus W$



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Let $\widetilde{\omega}_{\mathcal{X}'}$ according to Theorem 1. Normalize

$$\rho = \widetilde{\omega}_{\mathcal{X}'} + f^* \widetilde{\omega}_{\mathcal{S}'} \quad \text{s.t. } \int_{\mathcal{X}'/\mathcal{S}'} \rho^{n+1} = 0.$$



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- (1) The metric h^{ρ} on the relative canonical bundle $K_{\chi'/S'}$ extends to $K_{\chi/S}$ as a singular hermitian metric, and
- (2) the curvature $\Theta_{h_{\mathcal{X}'/S'}^{\rho}}(K_{\mathcal{X}'/S'})$ of the relative canonical line bundle extends to \mathcal{X} as a *d*-closed positive (1, 1)-current.



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M. Păun: Demailly's approximation theorem of psh. functions by fctns. with algebraic singularities. Ohsawa-Takegoshi Thm.

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On S' for any form representing the polarization, in particular for ω :

$$\omega^{WP} = f_*(f^*(\omega^{WP}) \wedge \omega^n) = \int_{\mathcal{X}'/S'} f^*(\omega^{WP}) \wedge \omega^n = \operatorname{vol}(\mathcal{X}_s) \int_{\mathcal{X}/S} \Theta \wedge \omega^n$$



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Theorem 2 and wedge product of currents Bedford-Taylor.

Corollary

 ω^{WP} extends to *S* as a positive, closed current.



Extension Theorem

Kähler form $\omega = \omega_{\mathcal{X}}$



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Kähler form $\omega = \omega_{\mathcal{X}}$ Relative canonical bundle

$$K_{\mathcal{X}/S} = K_{\mathcal{X}} \otimes f^* K_S^{-1}$$



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Local coordinates

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In terms of Euclidean volume forms

$$\omega^{n} \wedge f^{*} dV_{S} = \chi \cdot dV_{\mathcal{X}} = e^{\psi_{U}} \cdot dV_{\mathcal{X}}$$

with $\psi_U = -\infty$ at singular points of *f*.

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$$\Theta_{h_{X/S}^{\omega}}(K_{X/S}) = \sqrt{-1}\partial\overline{\partial}\psi_U|X' = \sqrt{-1}\partial\overline{\partial}\log\left(\omega^n \wedge f^*dV_S\right).$$



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Ricci-flat volume form

exists a unique function $\eta_s \in C^{\infty}(X_s)$ s.t.

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Here the Kähler property of \mathcal{X} is being used.

Monge-Ampère equation $e^{\eta_s}\omega_s^n= ho_s^n \qquad ext{for each }s\in S'.$



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Curvature of relative canonical bundle near singular points

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Georg Schumacher

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Aim.

Bound the potential $\theta := \eta + \psi_U$

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Ohsawa-Takegoshi Extension Theorem

Let $\psi : \Omega \subset \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ a psh. function on a bounded pseudoconvex domain. Then there exists a constant C > 0 (only depending on $diam(\Omega)$) s.t. given a hyperplane $H \subset \mathbb{C}^n$

$$\forall f \in \mathcal{O}(\Omega \cap H) \text{ with } \int_{\Omega \cap H} e^{-\psi} |f|^2 d\lambda < \infty$$



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 $\exists F \in \mathcal{O}(\Omega) \text{ s.t. } F | \Omega \cap H = f \text{ and }$

$$\int_{\Omega} e^{-\psi} |F|^2 d\lambda \leq C \int_{\Omega \cap H} e^{-\psi} |f|^2 d\lambda.$$



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Idea





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• Approximate psh. function by a function of the form

 $\log \sum |\sigma_k|^2$ with σ_k holomorphic



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- Apply the Ohsawa-Takegoshi Theorem to $|\sigma_k|^2$.



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• Approximate psh. function by a function of the form

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- Apply the Ohsawa-Takegoshi Theorem to $|\sigma_k|^2$.
- Use extra variable $\mathbb{Z} \ni m \to \infty$ in weight function $e^{-m\theta}$.



Hilbert space of holomorphic functions

Definition

 $m > 1, s \in S'$ weighted norm on $\mathcal{O}(U_s)$:

$$\|\phi\|_{m,s}^2 := \int_{U_s} |\phi|^2 \, e^{-m\theta_s} \rho_s^n$$
 for $\phi \in \mathcal{O}(U_s)$.

Hilbert space

$$\mathcal{H}_{\boldsymbol{s}}^{(\boldsymbol{m})} = \{ \phi \in \mathcal{O}(\boldsymbol{U}_{\boldsymbol{s}}); \|\phi\|_{\boldsymbol{m},\boldsymbol{s}}^2 < \infty \}.$$



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$$e^{\theta_s} dV_X / f^*(dV_S) = \rho_s^n \text{ on } \mathcal{X}'$$

so that $\|\phi\|_{m,s}^2 = \int_{U_s} |\phi|^2 e^{-(m-1)\theta_s} dV_{X_s}$

where $dV_{X_s} = dV_X/f^*dV_S$ on U_s .

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By general theory

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Apply the mean-value inequality to a ball B_{ϵ} .



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Apply the mean-value inequality to a ball B_{ϵ} . $\phi \in \mathcal{H}_{s}^{(m)}$ with $\|\phi\|_{m,s}^{2} \leq 1$,

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Let $C_s := \sup_{B_r} d\lambda / (\rho_s)^n$. Then it follows that

$$\begin{split} |\phi(\boldsymbol{x})|^2 &\leq \frac{n! C_s}{\pi^n \varepsilon^{2n}} \sup_{\zeta \in B_{\varepsilon}} \left(\boldsymbol{e}^{m\theta_s} \int_{B_{\varepsilon}} |\phi(\zeta)|^2 \, \boldsymbol{e}^{-m\theta_s} (\rho_s)^n \right) \\ &\leq \frac{n! C_s}{\pi^n \varepsilon^{2n}} \sup_{\zeta \in B_{\varepsilon}} \left(\boldsymbol{e}^{m\theta_s} \right). \end{split}$$



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$$\frac{1}{m} \log |\phi(x)|^2 \leq \sup_{\zeta \in B_{\varepsilon}} \theta_s(\zeta) + \frac{1}{m} \left(\log \frac{n!}{\pi^n \varepsilon^{2n}} + \log C_s \right).$$



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ight|^2 e^{-2m heta_s(x)}.$$

Now

$$\varphi_m^s(x) \geq \theta_s(x) - \frac{\log C_s}{m}.$$

Finally let $m \to \infty$.

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Proof of the Extension Theorem Let $\phi_s \in \mathcal{H}_s^{(m)}$ with $\|\phi_s\|_s^2 \leq 1$. Hölder inequality:

$$\begin{split} \int_{U_s} |\phi_s|^{2/m} \, dV_{X_s} &= \int_{U_s} |\phi_s|^{2/m} \, e^{-\theta_s} \rho_s^n \\ &\leq \left(\int_{U_s} |\phi_s|^2 \, e^{-m\theta_s} \rho_s^n \right)^{\frac{1}{m}} \left(\int_{U_s} \rho_s^n \right)^{\frac{m-1}{m}} \\ &\leq (\operatorname{vol}(X_s))^{\frac{m-1}{m}} \leq C \end{split}$$

 $C = \max(1, \operatorname{vol}(X_s))$, independent of *m* and *s*.





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- There exists a numerical constant *C*₀ > 0 independent of *m* and *s* such that

$$\int_U |\mathcal{F}|^{2/m} \, dV_z \leq C_0 \int_{U_s} |\phi|^{2/m} \, rac{dV_z}{p^* dV_s} \leq C_0 \cdot C_s$$



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Mean value inequality applicable

$$|F(x)|^{2/m} \leq C_r \int_{B_r(x)} |F(z)|^{2/m} dV_z \leq C_r \cdot C_0 \cdot C$$

on $B_r(x)$.

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 \Rightarrow curvature $\Theta_{h_{X/Y}^{\rho}}(K_{X/Y})|_{X_0}$ extends as a *d*-closed positive real

(1, 1)-current on the total space X completing the proof.



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Theorem (Sch. '10/'16)

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Note that the extended curvature current may be different from the given one.

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Projective embeddings

Sch.-Tsuji 2004



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Projective embeddings

Sch.-Tsuji 2004

Theorem (Păun-Sch. 2017)

Let *X* be a compact Moishezon manifold, and *L* a big line bundle on *X*. Then the restricted volume of *L* along a closed, irreducible complex subspace *V* is positive, if and only if *V* is not contained in $E_{nK}(L)$. For high multiples of *L*, the linear system |mL| embeds the complement of the non-Kähler locus into a projective space.



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Let m > 0, then the image of the restriction map is denoted by

$$H^0(X|V, mL) = \operatorname{Im}\left(H^0(X, mL) \to H^0(V, mL|V)\right)$$

The restricted volume of *L* on *V* is

$$\operatorname{vol}_{X|V}(L) = \overline{\lim_{m \to \infty}} \left(\frac{\dim H^0(X|V, mL)}{m^d/d!} \right)$$



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The restricted volume of L on V is

$$\operatorname{vol}_{X|V}(L) = \overline{\lim_{m \to \infty}} \left(\frac{\dim H^0(X|V, mL)}{m^d/d!} \right)$$

Given any positive current *T*, the union of all Lelong sublevel sets $E_c(T) = \{x \in X; \nu(x, T) \ge c\}$ is denoted by

$$E_+(T)=\bigcup_{c>0}E_c(T).$$

The classes α containing Kähler currents α are called $big_{\mu\nu\rho\sigma}$



Let X be a compact, complex manifold. The non-Kähler locus of a big class equals

$$\Xi_{nK}(\alpha) := \bigcap E_+(T),$$

where T runs through all Kähler currents representing α .



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Proposition

Let X be a compact, complex manifold, and L a big line bundle. Let $V \subset X$ denote an irreducible, reduced, closed, complex subspace, whose restricted volume is positive

 $\operatorname{vol}_{X|V}(L) > 0.$

Then for $m \gg 0$ the linear system $|W_m|$ with $W_m = H^0(X|V, mL)$ yields a map that is birational onto the image.

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