Automorphism group of the moduli space of parabolic vector bundles

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Discussion Meeting on Analytic and Algebraic Geometry
International Centre for Theoretical Science
Bengaluru, 19-24 March 2018

Parabolic vector bundles

Let X be a smooth complex projective curve. Let $D = \{x_1, \dots, x_n\} \subset C$.

Definition

A **parabolic vector bundle** over (X, D) is a vector bundle E over X together with a weighted flag over the fiber $E|_x$ for every $x \in D$ called parabolic structure, i.e., a filtration by linear subspaces

$$E|_x = E_{x,1} \supsetneq E_{x,2} \supsetneq \cdots \supsetneq E_{x,l_x+1} = 0$$

together with a system of real weights $0 \leq \alpha_{x,1} < \alpha_{x,2} < \cdots < \alpha_{x,l_x} < 1$, denoted by α. We say that α is **full flag** if $l_x = r$ for all $x \in D$.

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Providing such a filtration on the fibers $E|_x$ is equivalent to giving a weighted filtration of E by subsheaves of the form

$$E = E_x^1 \supsetneq E_x^2 \supsetneq \cdots \supsetneq E_x^{l_x+1} = E(-x)$$

for every $x \in D$, where $0 \longrightarrow E_x^i \longrightarrow E \longrightarrow (E|_x / E_{x,i}) \longrightarrow 0$

Parabolic vector bundles

Definition as left continuous filtration

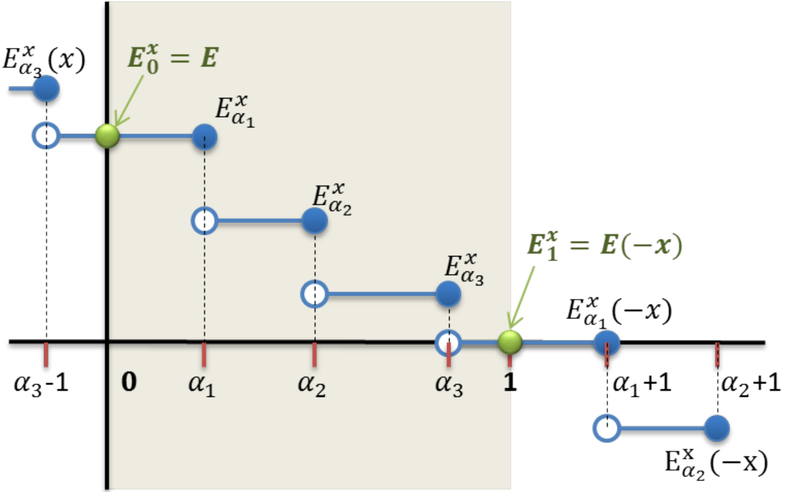
We can codify the parabolic structure at each $x \in D$ as a left continuous filtration E_α^x by sheaves indexed by $\alpha \in \mathbb{R}$, satisfying

1. If $\alpha \leq \beta$, $E_\alpha^x \supseteq E_\beta^x$.
2. For every α, there exists $\varepsilon > 0$ such that $E_{\alpha-\varepsilon}^x = E_\alpha^x$.
3. For every α, $E_{\alpha+1}^x = E_\alpha^x(-x)$.
4. $E_0^x = E$

$\alpha_i(x)$ is the i-th value of $\alpha \in [0,1)$ where the filtration E_α^x "jumps" and

$$0 \longrightarrow E_{\alpha_i(x)}^x \longrightarrow E \longrightarrow E|_x / E_{x,i} \longrightarrow 0$$

Left continuous filtration



Sheaves E_α^x $\longleftrightarrow$ Steps of the filtration

Jumps of E_α^x $\longleftrightarrow$ Parabolic weights

Stability condition

Parabolic degree

The parabolic degree of a parabolic vector bundle $(E, E_\bullet)$ is

$$\operatorname{pardeg}(E, E_\bullet) = \deg(E) + \sum_{x \in D} \sum_{i=1}^{l_x} \alpha_{x,i} \left(\operatorname{rk}(E_{x,i}) - \operatorname{rk}(E_{x,i+1}) \right)$$

We call $\operatorname{par}\mu(E, E_\bullet) = \operatorname{pardeg}(E, E_\bullet) / \operatorname{rk}(E)$ the parabolic slope.

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If $(E, E_\bullet)$ is a parabolic vector bundle and $F \subseteq E$ is a sub-bundle, the parabolic structure $E_\bullet$ induces a structure of parabolic structure on F taking the weighted filtration

$$F_\bullet = \{E_{x,i} \cap F|_x\} \qquad \beta_i(x) = \max_j\{\alpha_j(x) : F|_x \cap E_{x,j} = F_{x,i}\}$$

Stability

A parabolic vector bundle $(E, E_\bullet)$ is (semi-)stable if and only if for every sub-bundle F with the induced parabolic structure

$$\text{par}\mu(F, F_\bullet)(\leq) < \text{par}\mu(E, E_\bullet)$$

Generic weights

Definition (generic parabolic weights)

A parabolic system of weights α is called **generic** if there do not exist integers s and r' with $0 < r' < r$ and subsets $I(x) \subset \{1, \dots, r\}$ with $|I(x)| = r'$ such that

$$r' \sum_{x \in D} \sum_{i=1}^{r} \alpha_i(x) - r \sum_{x \in D} \sum_{i \in I(x)} \alpha_i(x) = s$$

Remark

If α is generic, then there are no strictly semistable parabolic vector bundles.

Otherwise, there exists $(F, F_\bullet) \subset (E, E_\bullet)$ with slope μ and $I_F(x)$ such that

$$\frac{\deg(F) + \sum_{x \in D} \sum_{i \in I_F(x)} \alpha_i(x)}{\mathrm{rk}(F)} = \mu = \frac{\deg(E) + \sum_{x \in D} \sum_{i=1}^{r} \alpha_i(x)}{\mathrm{rk}(E)}$$

Moduli space

Let ξ be a line bundle over X and let α be a rank r system of weights over (X, D). We denote by $\mathcal{M}(X, r, \alpha, \xi) = \mathcal{M}(r, \alpha, \xi)$ the moduli space of rank r stable parabolic vector bundles $(E, E_\bullet)$ over (X, D) with system of weights α and $\det(E) = \wedge^r E \cong \xi$.

Theorem (Boden, Yokogawa '99)

If α is a generic full flag system of weights then $\mathcal{M}(r, \alpha, \xi)$ is a smooth complex rational variety.

Theorem (Boden, Yokogawa '99)

If α is full flag then $\mathcal{M}(r, \alpha, \xi)$ admits a universal parabolic vector bundle $(\mathcal{E}, \mathcal{E}_\bullet)$ over $\mathcal{M}(r, \alpha, \xi) \times X$.

Questions

1. **Classify the automorphisms of $\mathcal{M}(X, r, \alpha, \xi)$.**
2. **When are $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ isomorphic?**
3. **Describe isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$.**

Answers for compact curves

Let $\mathcal{M}(X, r, \xi)$ be the moduli space of stable vector bundles E over X of rank r with $\det E \cong \xi$.

Torelli Theorem (Mumford, Newstead '68; Narasimhan, Ramanan '75)

If $\mathcal{M}(X, r, \xi) \cong \mathcal{M}(X', r', \xi')$ then $X \cong X'$.

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Given a vector bundle E over X we can perform some **"basic transformations"**

- If $\sigma : X \to X$ is an automorphism, take the **pullback** $E \longmapsto \sigma^* E$.
- If L is a line bundle over X, take the **tensor product** $E \longmapsto E \otimes L$.
- Take the **dual** of the vector bundle $E \longmapsto E^\vee$.

Theorem (Kouvidakis-Pantev '95, Biswas, Gómez, Muñoz '13)

The automorphisms of the moduli space $\Phi : \mathcal{M}(X, r, \xi) \to \mathcal{M}(X, r, \xi)$ are given by compositions of the previous "basic transformations".

Answers for compact curves

Clearly, basic transformations change the determinant of the vector bundle, so not every combination gives an automorphism.

If $\det(E) \cong \xi$

$$\det(\sigma^* E) \cong \sigma^* \xi$$
$$\det(E \otimes L) \cong \xi \otimes L^{\otimes r}$$
$$\det(E^\vee) \cong \xi^{-1}$$

The automorphisms of the moduli space are combinations of the form

$$\Phi(E) = \sigma^* E \otimes L \quad \text{or} \quad \Phi(E) = \sigma^* E^\vee \otimes L$$

for some $\sigma : X \to X$ and line bundle L satisfying

$$\sigma^* \xi \otimes L^{\otimes r} \cong \xi \quad \text{or} \quad \sigma^* \xi^{-1} \otimes L^{\otimes r} \cong \xi$$

respectively.

In particular, dualization is only possible if r divides $2d$.

Parabolic analogues of these basic transformations

Let $(E, E_\bullet)$ be a parabolic vector bundle

- If $\sigma : X \to X$ satisfies $\sigma(D) = D$, take the **pullback** $(E, E_\bullet) \longmapsto \sigma^*(E, E_\bullet)$
- If L is a line bundle over X, take the **tensor product** $E \longmapsto (E, E_\bullet) \otimes L$
- Take the **parabolic dual** $(E, E_\bullet) \longmapsto (E, E_\bullet)^\vee$ defined as follows. If E^x_α is the left-continuous filtration associated to $(E, E_\bullet)$ take the filtered bundle

$$(E^\vee)^x_\alpha = \lim_{t \to \alpha^+} (E^x_{-1-t})^\vee$$

Apart from changing the determinant, pullback and dualization may also **change the parabolic weights**.

Parabolic analogues of these basic transformations

Let $\alpha = \{\alpha_i(x)\}$ be the parabolic weights of $(E, E_\bullet)$ and let $\xi = \det(E)$. Assume that 0 is not a weight.

Transformation	Determinant	Weights
$\sigma^*(E, E_\bullet)$	$\sigma^*\xi$	$(\sigma^*\alpha)_i(x) = \alpha_i(\sigma(x))$
$(E, E_\bullet) \otimes L$	$\xi \otimes L^{\otimes r}$	$\alpha_i(x)$
$(E, E_\bullet)^\vee$	$\xi^{-1} \otimes \mathcal{O}_X(-rD)$	$(\alpha^\vee)_i(x) = 1 - \alpha_{r+1-i}(x)$

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Yes! We can use the parabolic filtrations to perform **Hecke transformations**!

Definition: Hecke transformation

Let E be a vector bundle over X. Let $x \in X$ and let $L \subset E|_x$ be a linear subspace. Define the **Hecke transformation** of E by L at x as the subsheaf $\mathcal{H}_x^L(E)$ fitting in the short exact sequence

$$0 \longrightarrow \mathcal{H}_x^L(E) \longrightarrow E \longrightarrow E|_x/L \longrightarrow 0$$

Hecke transformation

Let $x \in D$. Let $(E, E_\bullet)$ be a parabolic vector bundle and let $0 < l < r$. Consider the Hecke transformation of E by $E_{x,l+1}$ at x.

$$0 \longrightarrow H \longrightarrow E \longrightarrow E|_x / E_{x,l+1} \longrightarrow 0$$

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Restricting the sequence at x, if $N^\vee = \mathcal{O}_X(-x)|_x$, we get

$$\begin{tikzcd}
0 \arrow[r] & E|_x/E_{x,l+1} \otimes N^\vee \arrow[r] \arrow[d, equal] & H|_x \arrow[rr, "f"] \arrow[dr] & & E|_x \arrow[r] & E|_x/E_{x,l+1} \arrow[r] & 0 \\
& \mathrm{Tor}(E|_x/E_{x,l+1}, \mathcal{O}_x) & & E_{x,l+1} \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

thus, $H|_x$ gets an induced filtration

$$H|_x = f^{-1}(E_{x,l+1}) \supsetneq \cdots \supsetneq f^{-1}(E_{x,r}) \supsetneq \frac{E|_x}{E_{x,l+1}} \otimes N^\vee \supsetneq \cdots \supsetneq \frac{E_{x,l}}{E_{x,l+1}} \otimes N^\vee \supsetneq 0$$

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Definition: Parabolic Hecke transformation

Denote $(H, H_\bullet) = \mathcal{H}_x^l(E, E_\bullet)$.

Equivalent definition of Hecke transformation

Let $\{E_\bullet^x\}$ be the left continuous filtrations associated to $(E, E_\bullet)$. Take some ε with

$$\alpha_1(x) < \varepsilon < \alpha_2(x)$$

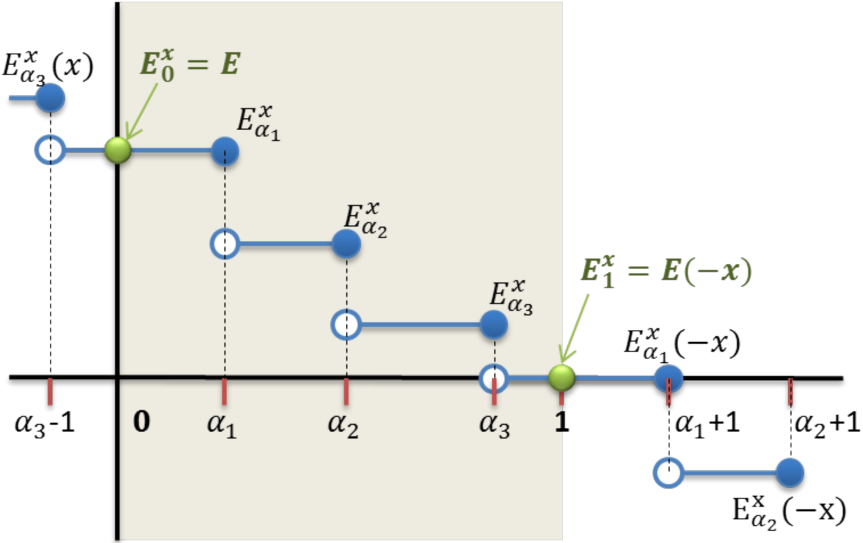
and consider the left continuous filtration $E_\alpha^x[\varepsilon]$ given by

$$(E_\bullet^x[\varepsilon])_\alpha = E_{\alpha-\varepsilon}^x$$

Notice that $((E_\bullet^x)[\varepsilon])_0 = E_\varepsilon^x = E_{\alpha_2(x)}^x = \mathcal{H}_x^{E_{x,2}}(E)$.
If the filtration $\{E_\bullet^x\}$ jumps at $\alpha_i(x)$ in the interval $[0,1)$, then the new filtration jumps at

$$\alpha[\varepsilon]_i(x) = \begin{cases} \alpha_{i+1}(x) - \varepsilon & i < r \\ \alpha_1(x) + 1 - \varepsilon & i = r \end{cases}$$

Equivalent definition of Hecke transformation



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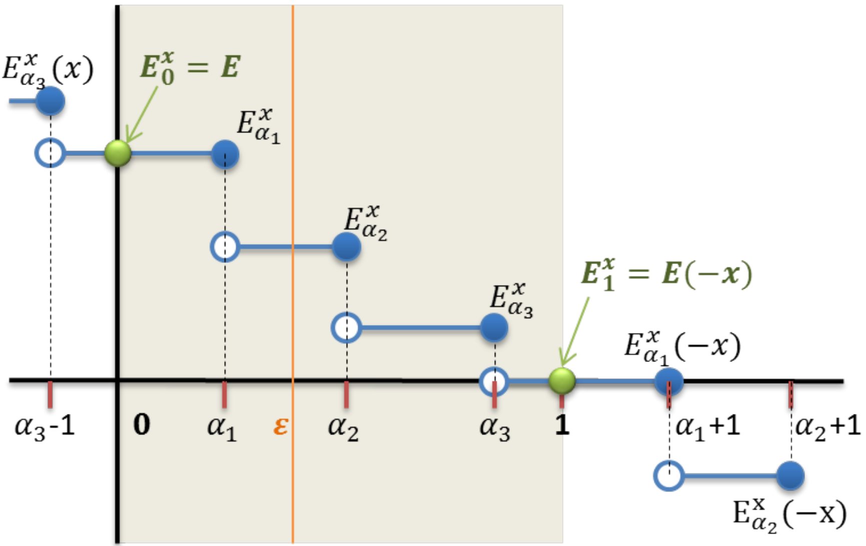

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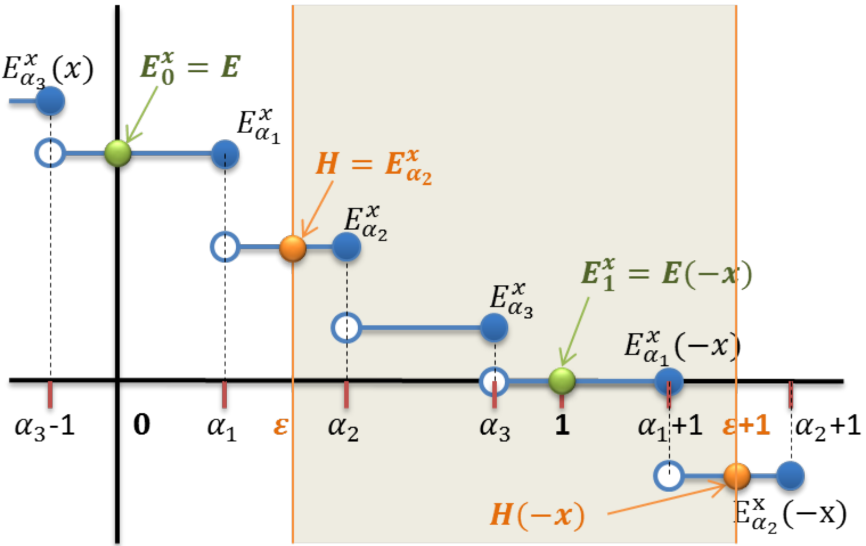

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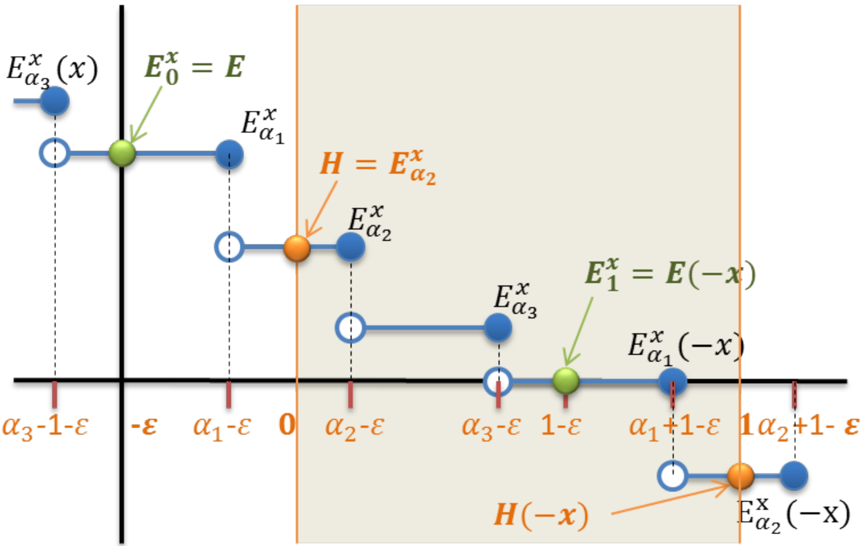

Properties of Hecke transformation

Let $(E, E_\bullet)$ be a rank r full flag parabolic vector bundle and let $x, y \in D$.

- If $(E, E_\bullet)$ is stable then $\mathcal{H}_x^l(E, E_\bullet)$ with the induced parabolic weights is stable, i.e. $\mathcal{H}_x^l$ **preserves stability**.
- Hecke transformations are (almost) **cyclic**

$$\mathcal{H}_x^r(E, E_\bullet) = (E, E_\bullet) \otimes \mathcal{O}_X(-x)$$

- Hecke transformations are **local** $\mathcal{H}_x \circ \mathcal{H}_y(E, E_\bullet) = \mathcal{H}_y \circ \mathcal{H}_x(E, E_\bullet)$

We can combine several Hecke transformations in different points into a single operation. Let $H = \sum_{x \in D} h_x \cdot x$ be a divisor supported on D, with $0 \leq H \leq (r-1)D$. Then define

$$\mathcal{H}_H(E, E_\bullet) = \mathcal{H}_{x_1}^{h_1} \circ \mathcal{H}_{x_2}^{h_2} \circ \cdots \circ \mathcal{H}_{x_n}^{h_n}(E, E_\bullet)$$

Interpretation in terms of tensor product

Using the parabolic tensor product, we can reinterpret Hecke transform. Let E^x_α and F^x_β be filtered vector bundles with systems of weights α and β respectively. Then define

$$(E \otimes F)^x_\gamma = \sum_{\alpha+\beta=\gamma} E^x_\alpha \otimes F^x_\beta$$

Taking F as the trivial vector bundle $\mathcal{O}_X$ with jump at ε_x for every $x \in D$ (i.e. $F^x_{\varepsilon_x} = \mathcal{O}_X$ and $F^x_{\varepsilon_x+\tau} = \mathcal{O}_X(-x)$ for small τ), then if E^x_α is a filtered vector bundle

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If $(E, E_\bullet)$ is stable then clearly $(E, E_\bullet) \otimes (\mathcal{O}_X, \mathcal{O}_{X,\bullet})$ is stable for all choices of the parabolic weights ε_x, if we take the weights induced by the tensor product.
In particular, $\mathcal{H}_H(E, E_\bullet)$ is stable when given the induced parabolic weights $\mathcal{H}_H(\alpha)$.

Summary of parabolic basic transformations

Transform. T	**Determinant $T(\xi)$**	**Weights $T(\alpha)$**
$\sigma^*(E, E_\bullet)$	$\sigma^*\xi$	$(\sigma^*\alpha)_i(x) = \alpha_i(\sigma(x))$
$(E, E_\bullet) \otimes L$	$\xi \otimes L^{\otimes r}$	$\alpha_i(x)$
$(E, E_\bullet)^\vee$	$\xi^{-1} \otimes \mathcal{O}_X(-rD)$	$(\alpha^\vee)_i(x) = 1 - \alpha_{r+1-i}(x)$
$\mathcal{H}_x(E, E_\bullet)$	$\xi \otimes \mathcal{O}_X(-x)$	If $\alpha_1(x) < \varepsilon < \alpha_2(x)$ $(\mathcal{H}_x(\alpha))_i(x) = \alpha_{i+1}(x) - \varepsilon$ $(\mathcal{H}_x(\alpha))_r(x) = \alpha_1(x) + 1 - \varepsilon$

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$$\left\{ \begin{array}{c} \alpha - \text{stable quasi-parabolic} \\ \text{v. b. with } \det \cong \xi \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} T(\alpha) - \text{stable quasi-parabolic} \\ \text{v. b. with } \det \cong T(\xi) \end{array} \right\}$$

Classification of automorphisms

Theorem (–, T. Gómez)

All the automorphisms $\Phi : \mathcal{M}(X,r,\alpha,\xi) \to \mathcal{M}(X,r,\alpha,\xi)$ are generated as compositions of basic transformations

- $(E, E_\bullet) \longmapsto \sigma^*(E, E_\bullet)$ for some automorphism $\sigma : X \to X$ such that $\sigma(D) = D$
- $(E, E_\bullet) \longmapsto (E, E_\bullet) \otimes L$ for some line bundle L
- $(E, E_\bullet) \longmapsto (E, E_\bullet)^\vee$
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Basic transformations induce 2-birational automorphisms

k-birational automorphisms

Definition

Let X and X' be two varieties. We say that X and X' are ***k*-birational** if there exist open subsets $U \subset X$ and $U' \subset X'$ and an isomorphism $\Phi : U \xrightarrow{\sim} U'$ such that

$$\mathrm{codim}(X \backslash U) \geq k$$

$$\mathrm{codim}(X' \backslash U') \geq k$$

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Examples

- X and X' are 1-birational if they are birational.
- The Picard group is invariant under 2-birational eq., but not under 1-birational eq.
- If X and X' are 2-birational normal algebraic varieties then $\Gamma(X) = \Gamma(X')$ (Hartog's theorem)

Classification of k-birational automorphisms

Theorem (Boden, Yokogawa '99)

If α is generic full flag, $\mathcal{M}(r, \alpha, \xi)$ is rational.

Corollary

$$\mathrm{Aut}_{1-\mathrm{bir}}(\mathcal{M}(r, \alpha, \xi)) = \mathrm{Bir}(\mathbb{P}^N) = \mathrm{Cremona}_N$$

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Theorem (–, T. Gómez)

Supp. X has genus $g \geq 8$, α is generic full flag. $\mathrm{Aut}_{2-\mathrm{bir}}(\mathcal{M}(X,r,\alpha,\xi))$ is the group generated by compositions of basic transformations

- $(E,E_\bullet) \longmapsto \sigma^*(E,E_\bullet)$ for $\sigma : X \xrightarrow{\sim} X$ such that $\sigma(D) = D$
- $(E,E_\bullet) \longmapsto (E,E_\bullet) \otimes L$ for some line bundle L
- $(E,E_\bullet) \longmapsto (E,E_\bullet)^\vee$
- $(E,E_\bullet) \longmapsto \mathcal{H}_H(E,E_\bullet)$ for some $0 \leq H \leq (r-1)D$

which preserve the determinant ξ.

Questions

1. **Classify the automorphisms of $\mathcal{M}(X, r, \alpha, \xi)$.**

   Comp. of basic transf.: pullback, tensorization, dualization, Hecke

2. **When are $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ isomorphic?**
3. **Describe isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$.**

Torelli Theorem

Let (X, D) and (X', D') be marked smooth complex projective curves of genus g and g' resp. Let α and α' be systems of weights over (X, D) and (X', D') resp. Let ξ and ξ' be line bundles over X and X' resp.

Torelli Theorem $r = 2$ (Balaji, del Baño, Biswas '01; Sebastian '10)

Sup. $g, g' \geq 2$, α, α' small of rank 2 and $\deg(\xi) = \deg(\xi') = 1$. Then

$$\mathcal{M}(X, 2, \alpha, \xi) \cong \mathcal{M}(X', 2, \alpha', \xi') \implies (X, D) \cong (X', D')$$

Torelli Theorem (–, T. Gómez)

Sup. $g, g' \geq 4$ and α and α' generic full flag of rank r. Then

$$\mathcal{M}(X, r, \alpha, \xi) \cong \mathcal{M}(X', r', \alpha', \xi') \implies (X, D) \cong (X', D') \text{ and } r = r'$$

.

k-birational Torelli theorem

If α is generic full flag, $\mathcal{M}(X, r, \alpha, \xi)$ is rational (Boden, Yokogawa '99), so we have

Proposition (Corollary of Boden, Yokogawa '99)

If α and α' are generic full flag, then $\mathcal{M}(X, r, \alpha, \xi) \stackrel{\text{bir}}{\cong} \mathcal{M}(X', r', \alpha', \xi')$ if and only if $\dim \mathcal{M}(X, r, \alpha, \xi) = \dim \mathcal{M}(X', r', \alpha', \xi')$.

Theorem (–, T. Gómez)

Let (X, D), (X', D') be marked smooth complex projective curves of genus at least 8. Let α and α' be generic full flag systems of weights of rank r and r' over (X, D) and (X', D') resp. Let ξ and ξ' be line bundles over (X, D) and (X', D') resp.

$$\mathcal{M}(X, r, \alpha, \xi) \stackrel{2-\text{bir}}{\cong} \mathcal{M}(X', r', \alpha', \xi') \iff r = r' \text{ and } (X, D) \cong (X', D')$$

Questions

1. **Classify the automorphisms of $\mathcal{M}(X, r, \alpha, \xi)$.**

   Comp. of basic transf.: pullback, tensorization, dualization, Hecke

2. **When are $\mathcal{M}(X, r, \alpha, \xi)$ and $\mathcal{M}(X', r', \alpha', \xi')$ isomorphic?**

$$\mathcal{M}(X, r, \alpha, \xi) \cong \mathcal{M}(X', r', \alpha', \xi') \implies r = r' \text{ and } (X, D) \cong (X', D')$$

$$\mathcal{M}(X, r, \alpha, \xi) \overset{2-\text{bir}}{\cong} \mathcal{M}(X', r', \alpha', \xi') \iff r = r' \text{ and } (X, D) \cong (X', D')$$

3. **Describe isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$.**

Classification of isomorphisms

Let (X, D), (X', D') be marked smooth complex proj. curves of genus g and g' resp. Let α, α' be generic full flag systems of weights over (X, D) and (X', D') of rank r and r' resp. ξ, ξ' line bundles over X and X' resp.

Theorem (–, T. Gómez)

Sup. $g, g' \geq 7$. $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$ isom. Then there is $\sigma : X \to X'$ isom. such that $\sigma(D) = D'$ and a composition of basic transformations T sending determinant ξ to $\sigma^*\xi'$ and α-st. parabolic v. b. to $\sigma^*\alpha'$-st. parabolic v. b. such that $\sigma^*\Phi(E, E_\bullet) = T(E, E_\bullet)$ $\forall (E, E_\bullet)$ α-st.

Theorem (–, T. Gómez)

Sup. $g, g' \geq 8$. $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{2-\mathrm{bir}} \mathcal{M}(X', r', \alpha', \xi')$ 2-bir. map. Then there is $\sigma : X \to X'$ isom. such that $\sigma(D) = D'$ and a composition of basic transformations T sending determinant ξ to $\sigma^*\xi'$such that $\sigma^*\Phi(E, E_\bullet) = T(E, E_\bullet)$ $\forall (E, E_\bullet)$ α-st.

Questions

1. **Classify the automorphisms of $\mathcal{M}(X, r, \alpha, \xi)$.**

   Comp. of basic transf.: pullback, tensorization, dualization, Hecke

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3. **Describe isomorphisms $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$.**

   There is $\sigma : (X, D) \xrightarrow{\sim} (X', D')$ and a composition of basic transformations T such that $T(\xi) = \sigma^* \xi'$, $T(\alpha) \sim \sigma^* \alpha'$ and

$$\sigma^* \Phi(E, E_\bullet) = T(E, E_\bullet)$$

Stability chamber analysis

Fix parabolic weights $\alpha \quad \longleftrightarrow \quad$ Fix stability chamber C_α

$\mathcal{M}(r, \alpha, \xi) \quad \longleftrightarrow \quad$ Moduli space of α-stable quasi-parabolic v. b.

Basic transformations T send α-stable quasi-parabolic v. b. to $T(\alpha)$-stable quasi-parabolic v. b.

Goal

- For each α, characterize compositions of basic transf. T such that

$$C_\alpha = C_{T(\alpha)}$$

- We need to characterize the geometrical barriers of C_α.

We will focus on high genus, so we can characterize numerically the chambers C_α.

Stability analysis

Idea

Given weights α and β, use

- Brill-Noether theory (Brambila-Paz, Grzegorczyk, Newstead '97)
- Stratification of the moduli of parabolic v. b. by Segre invariant (Boshle, Biswas '05)

To find α-stable, but β-unstable parabolic vector bundles.

If the genus is high enough numerical barriers transform into geometrical barriers $\implies$ $\exists$ numerical invariants identifying the chambers

Stability analysis

Definition: Chamber invariant

We say that $\{n_1(x), \dots, n_r(x)\} = \overline{n}$, with $n_i(x) \in \{0,1\}$ is **admissible** if $\sum_{i=1}^r n_i(x) = r'$. Let $d = \deg(\xi)$. For admissible $\overline{n}$ define the integer

$$M(r,\alpha,d,\overline{n}) = \left\lfloor \frac{r'd + r'\sum_{x\in D}\sum_{i=1}^r \alpha_i(x) - r\sum_{x\in D}\sum_{i=1}^r n_i(x)\alpha_i(x)}{r} \right\rfloor$$

Let $\overline{M}(r,\alpha,d) = (M(r,\alpha,d,\overline{n}))_{\overline{n} \text{ admissible}}$.

Theorem (–, T. Gómez)

If X has genus $g \geq 1 + (r-1)|D|$ then

$$C_\alpha = C_\beta \iff \overline{M}(r,\alpha,d) = \overline{M}(r,\beta,d)$$

Refined classification result

Let (X, D), (X', D') be marked smooth complex proj. curves of genus g and g' resp. Let α, α' be generic full flag systems of weights over (X, D) and (X', D') of rank r and r' resp. ξ, ξ' line bundles over X and X' resp.

Theorem (–, T. Gómez)

Sup. $g, g' \geq \max\{1 + (r-1)|D|, 7\}$. $\Phi : \mathcal{M}(X, r, \alpha, \xi) \xrightarrow{\sim} \mathcal{M}(X', r', \alpha', \xi')$ isom. Then $r = r'$ and there is $\sigma : X \to X'$ isom. such that $\sigma(D) = D'$ and a composition of basic transformations T such that

- $T(\xi) = \sigma^* \xi'$
- $\overline{M}(r, \sigma^*\alpha', \deg(\xi')) = \overline{M}(r, T(\alpha), \deg T(\xi))$
- For every $(E, E_\bullet) \in \mathcal{M}(X, r, \alpha, \xi)$, $\sigma^*\Phi(E, E_\bullet) \cong T(E, E_\bullet)$.

Corollary

$$\mathrm{Aut}(\mathcal{M}(r, \alpha, \xi)) = \left\langle \begin{array}{c} T \text{ comp. of} \\ \text{basic transf.} \end{array} \middle| \begin{array}{c} T(\xi) = \xi, \deg \xi = d \\ \overline{M}(r, \alpha, d) = \overline{M}(r, T(\alpha), d) \end{array} \right\rangle$$

Some applications

1. If the weights α and α' are concentrated (small parameters), $\mathcal{M}(X, r, \alpha, \xi) \cong \mathcal{M}(X', r', \alpha', \xi')$ if and only if $(X, D) \cong (X', D')$, $r = r'$ and $\deg(\xi) \equiv \pm \deg(\xi') (\mathrm{mod}\, r)$.
2. Analogously, in the compact case (working with the proof by Biswas, Gómez, Muñoz '13), $\mathcal{M}(X, r, \xi) \cong \mathcal{M}(X', r', \xi')$ if and only if $X \cong X'$, $r = r'$ and $\deg(\xi) \equiv \pm \deg(\xi') (\mathrm{mod}\, r)$.
3. Obtaining a Torelli theorem for generic rank r parabolic weights unlocks other Torelli theorems for high rank and generic weights
    - Moduli space of parabolic Higgs bundles.
    - Parabolic Hodge moduli space.
    - Parabolic Deligne-Hitchin moduli space.
4. Computation of the automorphism group of the moduli stack (work in progress).

Thank you for your attention