Analysis and TYZ expansions for Kepler manifolds

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Classical Kepler varieties

hermitian vector space $Z = \mathbf{C}^{n+1},\ rank = 2$

$$Z_1 := \{z = (z_0, \ldots, z_n) \in Z :\ z_0^2 + \ldots + z_n^2 = 0\} \text{ complex light cone}$$

Symplectic interpretation (cotangent bundle)

$$Z_1 \approx T^*(\mathbf{S}^n) \setminus \{0\} = \{(x, \xi) : \|x\| = 1,\ \xi \neq 0,\ (x|\xi) = 0\},\ z = \frac{x + i\xi}{2}$$

contact manifold (cosphere bundle)

$$S_1 \approx S^*(\mathbf{S}^n) = \{(x, \xi) : \|x\| = 1,\ \|\xi\| = 1,\ (x|\xi) = 0\}$$

polar decomposition

$$Z_1 = \mathbf{R}^+ \cdot S_1$$

Determinantal varieties

$$Z = \mathbf{C}^{r\times s},\ rank = r \le s,\ 1 \le \ell \le r$$

$$Z_\ell := \{z \in Z:\ rank(z) \le \ell\} \quad \text{vanishing of all } (\ell+1)\times(\ell+1) - \text{minors}$$

$$\mathring{Z}_\ell = \{z \in Z:\ rank(z) = \ell\} = Z_\ell \setminus Z_{\ell-1} \quad \text{regular part}$$

$$Z_1 := \{z \in Z:\ rank(z) \le 1\} \quad \text{vanishing of all } 2\times 2 - \text{minors}$$

$$\mathring{Z}_1 := \{z \in Z:\ rank(z) = 1\} = Z_1 \setminus \{0\}$$

$$S_1 = \{\xi \otimes \eta^* :\ \xi \in \mathbf{C}^r, \eta \in \mathbf{C}^s\ \|\xi\| = 1 = \|\eta\|\}$$

$$\mathring{Z}_1 = \mathbf{R}^+ \cdot S_1$$

Let Z complex vector space and $D \subset Z$ be a bounded domain. The group

$$G = Aut(D) = \{g : D \to D \text{ biholomorphic}\}$$

is a real Lie group by a deep theorem of H. Cartan. D is called **symmetric** iff G acts **transitively** on D, and around each point $o \in D$ there exists an involutive **symmetry** $s_o \in G$ fixing o. By Harish-Chandra, D can be realized as a convex circular domain, i.e. as the unit ball

$$D = \{z \in Z : \ \|z\| < 1\}$$

for an (essentially unique) norm called the **spectral norm**. Then the isotropy subgroup

$$K = \{g \in G : \ g(0) = 0\}$$

is a compact group consisting of linear transformations. For domains $D = G/K$ of rank $r > 1$ the boundary ∂D will not be smooth and D is only (weakly) pseudoconvex but not strongly pseudoconvex.

Let $Z = \mathbf{C}^{r\times s}$, endowed with the operator norm $\|z\| = \sup\ spec(zz^*)^{1/2}$. Then the **matrix unit ball**

$$D = \{z \in \mathbf{C}^{r\times s} :\ \|z\| < 1\} = \{z \in \mathbf{C}^{r\times s} :\ I - zz^* > 0\}$$

is a symmetric domain, under the **pseudo-unitary group**

$$G = U(r,s) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(r+s) :\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\},$$

which acts on D via **Moebius transformations**

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z) = (az+b)(cz+d)^{-1}.$$

Its maximal compact subgroup is

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} :\ a \in U(r),\ d \in U(s) \right\}.$$

with the linear action $z \mapsto azd^*$.

A basic theorem of M. Koecher characterizes hermitian symmetric domains in **Jordan algebraic** terms. Let Z be a complex vector space, endowed with a ternary composition $Z \times Z \times Z \to Z$, denoted by

$$(x, y, z) \mapsto \{x; y; z\},$$

which is bilinear symmetric in (x, z) and anti-linear in the inner variable. Putting $D(x, y)z := \{x; y; z\}$, Z is called a **hermitian Jordan triple** if the Jordan triple identity (a kind of Jacobi identity)

$$\Big[D(x, y), D(u, v)\Big] = D(\{x; y; u\}, v) - D(u, \{v; x; y\})$$

holds, and the hermitian form

$$(x, y) \mapsto traceD(x, y)$$

is a positive definite. **Every symmetric domain can be realized as the unit ball of a unique hermitian Jordan triple (and conversely)**. Moreover, K consists of all linear transformations preserving the Jordan triple product.

Classification of hermitian Jordan triples

- **matrix triple** $Z = \mathbf{C}^{r\times s}, \quad \{x;y;z\} = xy^*z + zy^*x,$

$$K = U(r)\times U(s):\ z \mapsto uzv,\ u\in U(r),\ v\in U(s)$$

$rank = r \le s,\ a = 2$ complex case, $b = s - r$

- $r = 1,\ Z = \mathbf{C}^{1\times d},\ \{x;y;z\} = (x|y)z + (z|y)x, \quad K = U(d)$
- **symmetric matrices** $a = 1$ (real case)
- **anti-symmetric matrices** $a = 4$ (quaternion case)
- **spin factor** $Z = \mathbf{C}^{n+1},\ \{xy^*z\} = (x\cdot\overline{y})\ z + (z\cdot\overline{y})\ x + (x\cdot z)\overline{y}$
  $r = 2,\ a = n-1,\ b = 0$

$$K = \mathbf{T}\cdot SO(n+1)$$

- **exceptional Jordan triples** of dimension 16 $(r = 2)$ and 27 $(r = 3)$, $a = 8$ (octonion case), $K = \mathbf{T}\cdot E_6$.

A real vector space X is a **Jordan algebra** iff X has a non-associative product $x, y \mapsto x \circ y = y \circ x$, satisfying the Jordan algebra identity

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y).$$

The **anti-commutator product**

$$x \circ y = (xy + yx)/2$$

of self-adjoint matrices satisfies the Jordan identity. By a fundamental result of Jordan/von Neumann/Wigner (1934), every (euclidean) Jordan algebra has a realization as self-adjoint matrices $X \approx \mathcal{H}_r(\mathbf{K})$ over $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ (quaternions), or $\mathbf{K} = \mathbf{O}$ (octonions) if $r \leq 3$. For $r = 2$ we obtain formal 2×2-matrices

$$\mathcal{H}_2(\mathbf{K}) = \{\begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} : \ \alpha, \delta \in \mathbf{R}, \ b \in \mathbf{K} := \mathbf{R}^{2+a}\}.$$

Thus X is characterized by the **rank** r and the **multiplicity** $a = \dim_{\mathbf{R}} \ \mathbf{K}$.

For a euclidean Jordan algebra X, the complexification $Z := X^{\mathbf{C}}$ becomes a (complex) Jordan triple via

$$\{u;v;w\} = (u \circ v^*) \circ w + (w \circ v^*) \circ u - v^* \circ (u \circ w).$$

For the spin factor we obtain

$$\{u;v;w\} = (u|v)w + (w|v)u - \overline{v}(u|\overline{w}).$$

The Jordan triples arising this way are called of **tube type**. For example, the matrix triple $\mathbf{C}^{r\times s}$ is of tube type if and only if $r = s$.

Let Z be an irreducible hermitian Jordan triple of rank r, and $\ell \leq r$. Define the **Kepler variety**

$$Z_\ell = \{z \in Z : \ rank(z) \leq \ell\}.$$

Its regular part (Kepler manifold)

$$\mathring{Z}_\ell = \{z \in Z_\ell : \ rank(z) = \ell\}$$

is a $K^{\mathbf{C}}$-homogeneous manifold, not compact or symmetric. (Matsuki dual of open G-orbits in flag manifolds).
For the spin factors (r=2) we recover classical Kepler variety. For matrix spaces, we obtain the determinantal varieties.
Theorem: The Kepler variety Z_ℓ is a normal variety having only rational singularities.
Remark: Result classical for spin factor ($r = 2$) and proved by Kaup-Zaitsev for non-exceptional types. Our proof is uniform, based on Kempf's collapsing vector bundle theorem.

An element $u \in Z$ is called a **tripotent** if $\{u; u; u\} = 2u$. For $Z = \mathbf{C}^{r \times s}$ the tripotents satisfy $uu^* u = u$ and are called partial isometries. The set S_ℓ of all tripotents of fixed rank $\ell \leq r$ is a compact K-homogeneous manifold. In particular, S_1 consists of all minimal (rank 1) tripotents. For the spin factor, S_1 is the cosphere bundle. For $Z = \mathbf{C}^{1 \times d}$

$$S_1 = \{u \in Z : \ (u|u) = 1\} = \mathbf{S}^{2d-1}$$

is the full boundary.
Every tripotent u induces a **Peirce decomposition** $Z = Z_u^{(2)} \oplus Z_u^{(1)} \oplus Z_u^{(0)}$, where

$$Z_u^{(j)} = \{z \in Z : \ \{u; u; z\} = 2jz\}$$

is an eigenspace of $D(u, u)$. Moreover, the Peirce 2-space $Z_u^{(2)}$ becomes a Jordan algebra with unit element u and multiplication $\{x; u; y\}$.

The **Jordan Grassmann manifold**

$$M_\ell = S_\ell / \sim = \text{ set of all Peirce 2-spaces } U = Z_u \text{ of rank } \ell$$

is a compact hermitian symmetric space, both a K-orbit and a $K^{\mathbf{C}}$-orbit. (Center of open G-orbit). Define the **tautological vector bundle**

$$\mathcal{T}_\ell := \bigcup_{U \in M_\ell} U \to M_\ell.$$

Fix a base point $c \in S_\ell$, put $V := Z_c^{(2)}$ and let $L := \{k \in K^{\mathbf{C}} : \ kV = V\}$. Then

$$\mathcal{T}_\ell = K^{\mathbf{C}} \times_L V$$

becomes a homogeneous vector bundle, and

$$\mathcal{T}_\ell \to Z_\ell : \ U \ni z \mapsto z \in Z_\ell$$

is a collapsing map. By **Kempf's Theorem**, the range of a collapsing map of homogeneous vector bundles is a normal variety. Let $\mathring{U}$ denote the invertible elements of the Jordan algebra $U = Z_u^{(2)}$. Then

$$\bigcup_{U \in M_\ell} \mathring{U} \to Z_\ell$$

is a birational isomorphism. This proves the normality theorem.

Let (M,ω) be a compact Kähler n-manifold, with a quantizing line bundle $\mathcal{L} \to M$. Consider holomorphic sections $\Gamma(M,\mathcal{L}^\nu)$ of a (high) tensor power $\mathcal{L}^\nu$. **Kodaira (coherent state) embedding**

$$\kappa_\nu : M \to \mathbf{P}(\Gamma(M,\mathcal{L}^\nu)),\ z \mapsto [s_\nu^0(z),..,s_\nu^{d_\nu}(z)]$$

$$\Omega_\nu = \frac{i}{2}\partial\overline{\partial}\log\sum_{i=0}^{d_\nu}|Z^i|^2 \text{ Fubini-Study metric on } \mathbf{P}^{d_\nu}$$

$$\kappa_\nu^*(\Omega_\nu) = \nu\omega + \frac{i}{2}\partial\overline{\partial}\log T_\nu$$

$$T_\nu(z) = \sum_{i=0}^{d_\nu} h_z(s_\nu^i(z), s_\nu^i(z)) \text{ Kempf distortion function}$$

TYZ (Tian-Yau-Zelditch) expansion

$$\Big|\frac{T_\nu(z)}{\nu^n} - \sum_{j=0}^{N-1}\frac{a_j(z)}{\nu^j}\Big| \le \frac{c_N}{\nu^N} \text{ as } \nu \to \infty$$

For non-compact Kähler manifolds (such as Z_ℓ) we need infinite-dimensional Hilbert spaces of holomorphic functions.

Every (euclidean) Jordan algebra X has a **symmetric cone**

$$\Omega = \{x^2 : \ x \in X \text{ invertible}\}.$$

In fact, another theorem of Koecher-Vinberg characterizes euclidean Jordan algebras in terms of symmetric (i.e. self-adjoint homogeneous) cones. For matrices $X = \mathcal{H}_r(\mathbf{K}),\ \Omega$ consists of all positive definite matrices. For the real spin factor, we obtain the forward light cone.
For a tripotent $u \in S_\ell$ the Peirce 2-space $Z_u^{(2)}$ is the complexification of a euclidean Jordan algebra with symmetric cone

$$\Omega_u := \{x = \{u;x;u\} \in Z_u^{(2)} : \ x = \{y;u;y\},\ y \text{ invertible}\}.$$

Proposition: The Kepler manifold has a polar decomposition

$$\mathring{Z}_\ell = \bigcup_{u \in S_\ell} \Omega_u.$$

For $\ell = 1$ this reduces to $\mathring{Z}_1 = \mathbf{R}_+ \cdot S_1$, since $Z_u^{(2)} = \mathbf{C} \cdot u$ and $\Omega_u = \mathbf{R}_+ \cdot u$ is an open half-line.

Choose a base point $c = e_1 + \ldots + e_\ell \in S_\ell$. Every positive density function $\rho(t)$ on symmetric cone Ω_c induces a K-invariant **radial measure**

$$\int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ f(z) = \int_{\Omega_c} dt\ \rho(t) \int_K dk\ f(k\sqrt{t}).$$

For $\ell = 1$ this simplifies to

$$\int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ f(z) = \int_0^\infty dt\ \rho(t) \int_{S_1} du\ f(u\sqrt{t}).$$

Define a **Hilbert space of holomorphic functions**

$$\mathcal{H}_\rho := \{f : \mathring{Z}_\ell \to \mathbf{C} \text{ holomorphic} : \int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ |f(z)|^2 < +\infty\}.$$

In the trivial case, where $\ell = r$ is maximal and Z is a Jordan algebra, negative powers of the (Jordan) determinant have no holomorphic extension beyond $\mathring{Z}$. Apart from this we have
Extension Theorem: Every holomorphic function on $\mathring{Z}_\ell$ has a unique extension to the closure Z_ℓ
Proof: Singular set

$$Z_\ell \setminus \mathring{Z}_\ell = \bigcup_{j<\ell} Z_j$$

has codimension ≥ 2. By the Normality Theorem (and using Lojasiewicz), the **second Riemann extension theorem** holds for holomorphic functions on $\mathring{Z}_\ell$.

Under the natural action of K the **holomorphic polynomials** $\mathcal{P}(Z)$ on a hermitian Jordan triple Z have a **Hua-Schmid-Kostant decomposition**

$$\mathcal{P}(Z) = \sum_{\boldsymbol{m}\in\mathbf{N}_+^r} \mathcal{P}_{\boldsymbol{m}}(Z)$$

where $\boldsymbol{m} = m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$ ranges over all **integer partitions** (Young diagrams) and each $\mathcal{P}_{\boldsymbol{m}}(Z)$ is an irreducible K-module, whose highest weight vector $N_{\boldsymbol{m}}$ can be uniformly described in terms of Jordan theoretic minors. The **dimension** $d_{\boldsymbol{m}} := \dim \mathcal{P}_{\boldsymbol{m}}(Z)$ is explicitly known. Let

$$(\phi|\psi) := \frac{1}{\pi^d} \int_Z dz\ e^{-(z|z)}\ \overline{\phi(z)}\ \psi(z)$$

denote the **Fischer-Fock inner product**. Then there is an expansion

$$e^{(z|w)} = \sum_{\boldsymbol{m}} E^{\boldsymbol{m}}(z,w)$$

into finite-dimensional kernel functions $E^{\boldsymbol{m}}(z,w)$ of $\mathcal{P}_{\boldsymbol{m}}(Z)$. For $r = 1,\ E^m(z,w) = \frac{(z|w)^m}{m!}$.

Using the characteristic multiplicity a of Z, we define the **Pochhammer symbol**

$$(\nu)_{\boldsymbol{m}} := \prod_{j=1}^{r} (\nu - \frac{a}{2}(j-1))_{m_j}.$$

Since the underlying measure $\tilde{\rho}$ is K-invariant, the extension theorem implies that

$$\mathcal{H}_\rho = \sum \mathcal{P}_{m_1 \ldots, m_\ell, 0 \ldots, 0}(Z)$$

is a Hilbert sum involving only (non-negative) partitions of length $\leq \ell$, since the other components vanish on Z_ℓ.

Theorem: The Hilbert space $\mathcal{H}_\rho$ has the **reproducing kernel**

$$\mathcal{K}^\rho(z,w) := \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{d_{\boldsymbol{m}}}{\int\limits_{\Omega_c} d\rho(t) E^{\boldsymbol{m}}(t,e)} \; E^{\boldsymbol{m}}(z,w)$$

$$= \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\int\limits_{\Omega_c} d\rho(t) N_{\boldsymbol{m}}(t)} \; \frac{d_{\boldsymbol{m}}}{d_{\boldsymbol{m}}^c} \; E^{\boldsymbol{m}}(z,w).$$

Every euclidean Jordan algebra X has a **Jordan algebra determinant** $N(x)$ which is a r-homogeneous polynomial defined via Cramer's rule

$$x^{-1} = \frac{\nabla_x N}{N(x)}.$$

For real or complex matrices, this is the usual determinant. For even anti-symmetric matrices, it is the Pfaffian. In the rank 2 case, $N(x_0, x) = x_0^2 - (x|x)$ is the Lorentz metric.
Let $\partial_N = N(\frac{\partial}{\partial x})$ be the associated constant coefficient differential operator on X. Then $N(x)\partial_N$ is a kind of Euler operator, of order r.
Applying these concepts to the Jordan algebra $Z_c^{(2)}$ of rank ℓ, we define a **universal differential operator**

$$\mathcal{D}_\ell := N^{\frac{a}{2}(\ell-r)}\partial_N^b \ N^{\frac{a}{2}(r-\ell-1)+b+1} \left(\partial_N \ N^{\frac{a}{2}} \ \partial_N^{a-1}\right)^{r-\ell} \ N^{\frac{a}{2}(r-\ell+1)-1}$$

of order $\ell((r-\ell)a+b)$ on the symmetric cone Ω_c. For $\ell = 1$, $N(t) = t$ and $\mathcal{D}_1$ becomes a polynomial differential operator of order $(r-1)a+b$ on the half-line.

Our main result is that the reproducing kernel $\mathcal{K}_\rho(z,w)$ can be expressed in closed form, by

$$\mathcal{K}_\rho(\sqrt{t},\sqrt{t}) = (\mathcal{D}_\ell F_\rho)(t)$$

for all $t \in \Omega_c$, where F_ρ is a special function of **hypergeometric type.** Since hypergeometric functions have good asymptotic expansions (quite difficult for $\ell > 1$), this will lead to the desired TYZ-expansions. Generalizing the 1-dimensional case, we define the **hypergeometric series** of type (p,q)

$$ {}_pF_q\binom{\alpha_1,\ldots,\alpha_p}{\beta_1,\ldots,\beta_q}(z,w) := \sum_{\boldsymbol{m}} \frac{(\alpha_1)_{\boldsymbol{m}}\ldots(\alpha_p)_{\boldsymbol{m}}}{(\beta_1)_{\boldsymbol{m}}\ldots(\beta_q)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z,w),$$

using the Fischer-Fock reproducing kernels $E^{\boldsymbol{m}}(z,w)$. For $z,w \in Z_\ell$, only partitions of length $\le \ell$ occur.

Consider first a Fock space situation. For $\alpha > 0$ consider the **pluri-subharmonic function** $\phi = (z|z)^\alpha$ on Z_ℓ. Let $\omega := \partial\overline{\partial}\phi$ denote the associated Kähler form. Then we have the polar decomposition

$$\int\limits_{\mathring{Z}_\ell} \frac{|\omega^n|}{n!}(z)\ e^{-\nu\phi(z)} f(z) = \int\limits_{\Omega_c} dt\ N_c(t)^{a(r-\ell)+b}\ (t|c)^{n(\alpha-1)}\ e^{-\nu(t|c)^\alpha},$$

where $N_c(t)$ is the rank ℓ determinant function on Ω_c. Let $\alpha = 1$.
Theorem: The Hilbert space $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has the reproducing kernel

$$\mathcal{K}_\nu(t,e) = \mathcal{D}_\ell\ {}_1F_1\binom{d_\ell/\ell}{n/\ell}(\nu t)$$

for the confluent hypergeometric function

$${}_1F_1\binom{\sigma}{\tau}(z,w) = \sum_{\boldsymbol{m}} \frac{(\sigma)_{\boldsymbol{m}}}{(\tau)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z,w).$$

In order to look at a Bergman type Hilbert space, let

$$(u|v) = \frac{1}{p} tr_Z \; D(u,v)$$

be the normalized K-invariant inner product, where p is the so-called genus of D. Define **Bergman operators**

$$B(u,v)z = z - 2\{u;v;z\} + \{u;\{v;z;v\};u\}$$

for $u, v, z \in Z$. Then the G-invariant **Bergman metric** is given by

$$(u|v)_z = (B(z,z)^{-1}u|v).$$

For the matrix case $Z = \mathbf{C}^{r \times s}$, we have

$$B(u,v)z = z - uv^* z - zv^* u + u(vz^* v)^* u = (1 - uv^*)z(1 - v^* u).$$

and $p = r + s$. Therefore

$$(u|v) = \frac{1}{r+s} tr_Z \; D(u,v) = trace \; uv^*$$

and the Bergman metric at $z \in D$ is

$$(u,v)_z = trace(1 - zz^*)^{-1} \; u(1 - z^* z)^{-1} \; v^*.$$

In terms of the Bergman operator, the **Bergman kernel function** $K : D \times \overline{D} \to \mathbf{C}$ of D can be expressed as

$$K(z,w) = \det\ B(z,w)^{-1}.$$

It is a fundamental fact that there exists a sesqui-polynomial called the **Jordan triple determinant** $\Delta : Z \times \overline{Z} \to \mathbf{C}$ such that the Bergman kernel function has the form

$$K(z,w) = \det\ \ B(z,w)^{-1} = \Delta(z,w)^{-p}$$

for all $z, w \in D$. In the matrix case $Z = \mathbf{C}^{r\times s}$ we have

$$\det\ \ B(z,w) = \det(1_r - zw^*)^{r+s}.$$

It follows that

$$\Delta(z,w) = \det(1 - zw^*).$$

The **Kepler ball** is the intersection

$$\{z \in \mathring{Z}_\ell : \ \|z\|_\infty < 1\} = \mathring{Z}_\ell \cap D$$

of $\mathring{Z}_\ell$ with the bounded symmetric domain $D := \{z \in Z : \ \|z\|_\infty < 1\}$. Consider the pluri-subharmonic function

$$\phi(z) = \log \Delta(z,z)^{-p} = \log \det B(z,z)^{-1}$$

on the Kepler ball, given by the Bergman kernel. As before, the associated radial measure $e^{-\nu\phi}|\omega^n|/n!$ has a nice polar decomposition, this time involving the unit interval $\Omega_c \cap (c - \Omega_c)$.
Theorem: For the Kepler ball, $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has the reproducing kernel

$$\mathcal{K}_\nu(t,e) = \mathcal{D}_\ell \ {}_2F_1\binom{d_\ell/\ell; \nu}{n_\ell/\ell}(t)$$

involving a Gauss hypergeometric function ${}_2F_1$ on Ω_c.

Pattern

Fock space on $Z = \mathbf{C}^d,\ \mathcal{K}_\nu = e^{(z|w)} = {}_0F_0(z,w)$

Bergman space on D, $\mathcal{K}_\nu = \Delta(z,w)^{-\nu} = {}_1F_0(z,w)$ Faraut-Koranyi formula

Kepler manifold $\mathring{Z}_\ell,\ \mathcal{K}_\nu = \mathcal{D}_\ell\ {}_1F_1$

Kepler ball $D \cap \mathring{Z}_\ell,\ \mathcal{K}_\nu = \mathcal{D}_\ell\ {}_2F_1$

Asymptotic expansion: $\alpha = 1,\ \ell$ **arbitrary**

Theorem: Consider the Fock-type situation, $\alpha = 1,\ \ell$ arbitrary. There exist polynomials $p_j(x)$, with $p_0 = const$, such that

$$\frac{1}{\nu^n}\mathcal{K}_\nu(\sqrt{x},\sqrt{x}) \approx \frac{e^{\nu\ tr(x)}}{N_c(x)^{n/\ell}}\sum_{j=0}^{n}\frac{p_j(x)}{\nu^j},$$

as $|x| \to +\infty$, uniformly for x in compact subsets of Ω_c.

Proof: For $Re(\mu) > \frac{d}{r} - 1$ and $Re(\gamma) > \frac{d}{r} - 1$, there is an asymptotic expansion

$${}_1F_1\binom{\mu}{\gamma}(z) \approx \frac{\Gamma_r(\gamma)}{\Gamma_r(\mu)}\frac{e^{tr(z)}}{N(z)^{\gamma-\mu}}\ {}_2F_0\left(\frac{d}{r} - \mu; \gamma - \mu\right)(z^{-1})$$

as $|z| \to +\infty$ while $\frac{z}{|z|}$ stays in a compact subset of Ω_c.

This expansion can be differentiated termwise any number of times.

Remark: Similar expansion for Kepler ball, based on asymptotics of ${}_2F_1$.

Asymptotic expansion: $\ell = 1,\ \alpha$ arbitrary

Theorem: Radial measure $d\rho(t) = \alpha e^{-\nu t^\alpha} t^{\alpha(p-1)-1}\, dt$ on half-line. Then, as $\nu \to +\infty$

$$\frac{e^{-\nu(z|z)^\alpha}}{\nu^{p-1}} \mathcal{K}_\nu(z,z) = \sum_{j=0}^{p-2} \frac{b_j}{\nu^j (z|z)^{j\alpha}} + O(e^{-\eta\nu(z|z)^\alpha}),\ \eta > 0$$

Proof: The measure $d\rho$ has moments

$$\alpha \int_0^\infty dt\ e^{-\nu t^\alpha}\ t^{\alpha(p-1)+m-1} = \frac{\Gamma(p-1+\frac{m}{\alpha})}{\nu^{p-1+\frac{m}{\alpha}}}.$$

Therefore $\mathcal{K}_\nu = \mathcal{D}_1 F_\rho$ with

$$F_\rho(t) = \nu^{p-1} \sum_{m=0}^\infty \frac{(\nu^{1/\alpha} t)^m}{\Gamma(p-1+\frac{m}{\alpha})} = \nu^{p-1} E_{1/\alpha, p-1}(\nu^{1/\alpha} t),$$

for the **Mittag-Leffler function**

$$E_{A,B}(t) = \sum_{m=0}^\infty \frac{t^m}{\Gamma(Am+B)}.$$

As $t \to +\infty$, it is known that

$$E_{A,B}(t) = \frac{1}{A} t^{(1-B)/A} e^{t^{1/A}} + O(e^{(1-\eta)t^{1/A}})$$

with some $\eta > 0$. This remains valid for derivatives of any order. For any polynomial P in one variable

$$\partial_t^k t^{\gamma\alpha+c} P(t^\alpha) e^{t^\alpha} = t^{\gamma\alpha+c-k} Q(t^\alpha) e^{t^\alpha}$$

with another polynomial Q, where $\deg Q = \deg P + k$. It follows that

$$(tD_j + D_j t) t^{\gamma\alpha} P(t^\alpha) e^{t^\alpha} = t^{\gamma\alpha} Q(t^\alpha) e^{t^\alpha},$$

with $\deg Q = \deg P + a$. Hence for $\mathcal{D} = \grave{D} \prod_{j=2}^{r} (tD_j + D_j t)$

$$\grave{D} \prod_{j=2}^{r} (tD_j + D_j t) t^{\gamma\alpha} e^{t^\alpha} = t^{\gamma\alpha} Q(t^\alpha) e^{t^\alpha},$$

where Q has degree $(r-1)a + b = p - 2$ and leading coefficient $2^{r-1}\alpha^{p-2}$.

Invariant volume forms and measures

$\mathring{Z}_\ell$ is homogeneous under $K^{\mathbf{C}}$ or a suitable transitive subgroup. When does an invariant holomorphic volume form or invariant measure exist? Only for domains of tube type $b = 0$.

- **Theorem:** Let Z be of tube type and $0 < \ell < r$. Then there exists an invariant holomorphic n-form Ω on $\mathring{Z}_\ell$ if and only if $p - a\ell = 2 + a(r - \ell - 1)$ is even.
- Among all tube type domains, $p - a\ell$ is odd only for the symmetric matrices $Z = \mathbf{C}^{r\times r}_{sym}$ with $r - \ell$ even.
- **Theorem:** An invariant measure μ on $\mathring{Z}_\ell$ exists in all cases (for $b = 0$), and has polar decomposition

$$\int\limits_{\mathring{Z}_\ell} d\mu(z)\ f(z) = const \int\limits_{\Omega_c} \frac{dt}{N_c(t)^{d_\ell/\ell}}\ N_c(t)^{ar/2} \int\limits_K dk\ f(k\sqrt{t}).$$

Special case, $r = 2, \ell = 1$

Englis et al, Gramchev-Loi: Lie ball $r = 2$

$$T^*(\mathbf{S}^n) \setminus \{0\} = \{(x, \xi) : \|x\| = 1, \ (x|\xi) = 0, \ \xi \neq 0\}$$

$\mathbf{C}^{n+1}$ hermitian Jordan triple, rank 2

$\{uv^*w\} = (u|v)w + (w|v)u + (u|\overline{w})\overline{v}$ Jordan triple product

$N(z) = (z|\overline{z})$ Jordan algebra determinant

$T^*(\mathbf{S}^n) \setminus \{0\} = \{z \in \mathbf{C}^{n+1} : N(z) = 0\}$ rank 1 elements

$$\frac{T_\nu(z)}{\nu^n} = 1 + \frac{(n-1)(n-2)}{2|\nu z|} + \sum_{j=2}^{n-2} \frac{2a_j}{|\nu z|^j} + R_\nu(|z|)$$

exponentially small error term