Techniques of constructions of variations of mixed Hodge structures (GAFA to appear)

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Variation of $\mathbb{R}$-mixed Hodge structure

A *real variation of mixed Hodge structure* ($\mathbb{R}$-VMHS) over M is $(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*)$ so that:

1. $\mathbf{E}$ is a local system of fin-dim. $\mathbb{R}$-vector spaces.
2. $\mathbf{W}_*$ is an increasing filtration of the local system $\mathbf{E}$.
3. $\mathbf{F}^*$ is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
4. The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
5. For any $k \in \mathbb{Z}$, the local system $Gr_k^{\mathbf{W}}(\mathbf{E})$ with the filtration induced by $\mathbf{F}^*$ is an $\mathbb{R}$-VHS of weight k.

Simply, VMHS is a flat bundle over a complex manifold with certain complex geometric properties.

Understanding VMHS by Monodromy

VMHSs have been understood by the following type
""Theorem" "

""Theorem""

Let M be a cpt Kähler (or sm. $\mathbb{C}$-alg var). Taking the monodromy representations of VMHSs
$(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*) \mapsto \rho : \pi_1(M) \to GL(V)$, *we obtain a correspondence*
$\{VMHSs\}$
$\to \{$*"Mixed Hodge representations" of* $\pi_1(M)\}$.

E.g. Rigidity theorem(Steenbrink-Zucker 1985),
Unipotent case:(Hain-Zucker 1987),
Non-unipotent case: (Pridham 2016)

Does the Algebraic understanding of Monodromy gives the Geometric understanding VMHS?

Main result(Overview)

Theorem (K.)

Let (M, g) be a compact Kähler manifold. Starting from real variations of Hodge structures, we make $\mathbb{R}$-VMHSs $(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*)$ over M from "Mixed Hodge representations" of certain (infinite-dim.) Lie algebra so that geometric informations of $(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*)$ (connection form, filtration of holom. bdl,etc.,) are described explicitly by the Kähler Geometry.

$\mathbb{R}$-Hodge structure

An $\mathbb{R}$-*Hodge structure* of weight n on a $\mathbb{R}$-vector space V is a bigrading

$$V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$$

on the complexification $V_{\mathbb{C}} = V \otimes \mathbb{C}$ such that

$$\overline{V^{p,q}} = V^{q,p},$$

or equivalently, a finite decreasing filtration F^* on $V_{\mathbb{C}}$ such that

$$F^p(V_{\mathbb{C}}) \oplus \overline{F^{n+1-p}(V_{\mathbb{C}})} = V_{\mathbb{C}}$$

for each p.

$\mathbb{R}$-Hodge structure on Kähler manifolds

Let (M, g) be a Kähler manifold, $\mathcal{H}^n(M)$ the vec. sp. of real harmonic forms. Then :

- $\mathcal{H}^n(M) \otimes \mathbb{C} = \bigoplus_{p+q=n} \mathcal{H}^{p,q}(M)$
- $\overline{\mathcal{H}^{p,q}(M)} = \mathcal{H}^{q,p}(M)$

where $\mathcal{H}^{p,q}(M)$ is the vec. sp. of complex harmonic forms of type (p, q).
Remark: By $\mathcal{H}^n(M) \cong H^n_{dR}(M, \mathbb{R})$, $\mathcal{H}^{p,q}(M) \cong H^{p,q}_{Dol}(M)$,
The above $\mathbb{R}$-Hodge structure does not depend on Kähler metric g.

Cohomological discription

$(A^*(M), d)$: de Rham cplx. of a Kähler mfd.
$(A^n(M)\otimes\mathbb{C} = \bigoplus_{p+q=n} A^{p,q}(M), d = \partial + \bar\partial)$
: the Dolbeault (double) cplx.
The filtration: $F^r(A^*(M)\otimes\mathbb{C}) = \bigoplus_{p\ge r} A^{p,q}(M)$.
Then the spectral sequence $E_r^{*,*}$ degenerates at E_1.
$(E_1^{p,q} \cong H^{p,q}_{Dol}(M))$
The filtration F^* on $H^n_{dR}(M,\mathbb{C})$ induced by the above filtration is the $\mathbb{R}$-Hodge structure as the previous description.

Polarization

A polarization of an $\mathbb{R}$-Hodge structure of weight n is a $(-1)^n$-symmetric bilinear form $S : V \times V \to \mathbb{R}$ so that:

1. The decomposition $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ is orthogonal for the sesquilinear form $S : V_{\mathbb{C}} \times \overline{V_{\mathbb{C}}} \to \mathbb{C}$.
2. $h : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}$ defined as $h(u, v) = S(Cu, \bar{v})$ is a positive-definite hermitian form where C is the Weil operator $C_{|V^{p,q}} = (\sqrt{-1})^{p-q}$.

Polarization on Kähler manifolds

On a Kähler manifold M, for the $\mathbb{R}$-Hodge structure on $\mathcal{H}^1(M)$, the canonical polarization (Hodge-Riemann bilinear form) is given by

$$\mathcal{H}^1(M) \times \mathcal{H}^1(M) \ni (\alpha, \beta) \mapsto \int \alpha \wedge \beta \wedge \omega^{\dim_{\mathbb{C}} . M - 1} \in \mathbb{R}$$

This depends on the Kähler metric g (ω is the Kähler form associated with g).

Importance of polarization

For a $\mathbb{R}$-Hodge structure F^* on a real vec. sp. V with a polarization S, consider the group $T = \mathrm{Aut}(V, S)$.
($T = Sp_{2k}(\mathbb{R})$ or $T = O(k, l)$)
For any $t \in T$, change the $\mathbb{R}$-Hodge structure tF^*.
This gives the transitive action of T on the classifying space $\mathcal{D}(V)$ of polarized $\mathbb{R}$-Hodge structures on V with the fixed Hodge numbers.
So $\mathcal{D}(V)$ is a homogeneous space of a (semi-simple) Lie group. (But not symmetric space).
Moreover $\mathcal{D}(V)$ is a complex manifold.
(Precisely, $\mathcal{D}(V)$ is an open submfd. in a flag mfd. of $T_{\mathbb{C}}$)

For studying Hodge structures as Complex Geometry
$\Rightarrow$ "Variation of Hodge structure"

Real variation of Hodge structure

Let M be a complex manifold.
A *real variation of Hodge structure* ($\mathbb{R}$-VHS) of weight $n \in \mathbb{Z}$ over M is a pair $(\mathbf{E}, \mathbf{F}^*)$ so that:

1. $\mathbf{E}$ is a local system of finite-dimensional $\mathbb{R}$-vector spaces.
2. $\mathbf{F}^*$ is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
3. The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
4. For any $x \in M$, (E_x, F_x^*) is a $\mathbb{R}$-Hodge structure of weight n.

Polarization of $\mathbb{R}$-VHS

A polarization of an $\mathbb{R}$-VHS is a non-degenerate pairing $\mathbf{S} : \mathbf{E} \times \mathbf{E} \to \mathbb{R}$ so that for any $x \in M$ $\mathbf{S}_x$ is a polarization of the $\mathbb{R}$-hodge structure $(\mathbf{E}_x, \mathbf{F}_x^*)$.

$\mathbb{R}$-VHS from Kähler manifolds

$\{X_b\}_{b\in B}$: holomorphic family (deformation) of compact Kähler manifolds over a complex manifold B.
The vector bundle $\mathbf{H}^n = \bigcup_{b\in B} H^n_{dB}(X_b, \mathbb{R})$ with the canonical flat (Gauss-Manin) connection.
The filtration $\mathbf{F}^p_b = F^p(H^n_{dR}(X_b, \mathbb{C}))$.
$(\mathbf{H}^n, \mathbf{F}^*)$ is a $\mathbb{R}$-VHS. (Griffiths)

Useful property of $\mathbb{R}$-VHS

For an $\mathbb{R}$-VHS $(\mathbf{E}, \mathbf{F}^*)$ of weight n, we have the decomposition $\mathbf{E}_{\mathbb{C}} = \oplus_{p+q=n} \mathbf{E}^{p,q}$
of $\mathcal{C}^\infty$ vector bundles so that $\mathbf{F}^r = \oplus_{p \geq r} \mathbf{E}^{p,q}$.
By the Griffiths transversality, for the de Rham cpx. $(A^*(M, \mathbf{E}_{\mathbb{C}}), D)$ with values in flat bdl $\mathbf{E}_{\mathbb{C}}$,
$D : A^{a,b}(\mathbf{E}^{c,d}) \to A^{a+1,b}(\mathbf{E}^{c,d}) \oplus A^{a,b+1}(\mathbf{E}^{c,d}) \oplus A^{a+1,b}(\mathbf{E}^{c-1,d+1}) \oplus A^{a,b+1}(\mathbf{E}^{c+1,d-1})$
Define $A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q} = \bigoplus_{a+c=P, b+d=Q} A^{a,b}(\mathbf{E}^{c,d})$.
Like the Dolbeault complex. we have the double complex

$$(\bigoplus A^*(M, \mathbf{E}_{\mathbb{C}})^{P,Q}, D', D'')$$

Hodge structure for polarized $\mathbb{R}$-VHS

If $(\mathbf{E}, \mathbf{F}^*)$ admits a polarization $\mathbf{S}$, then as $H^*(M, \mathbb{R})$ (resp. $\mathcal{H}^*(M)$), the cohomology $H^m(M, \mathbf{E})$ (harmonic forms $\mathcal{H}^m(M, \mathbf{E})$) with values in the flat bdl. $\mathbf{E}$ admits a canonical $\mathbb{R}$-Hodge structure of weight $m + n$ induced by the double complex.

Monodromy of polarized $\mathbb{R}$-VHS

For a $\mathbb{R}$-VHS $(\mathbf{E}, \mathbf{F}^*)$ with a polarization $\mathbf{S}$, the monodoromy rep. $\rho : \pi_1(M) \to GL(V)$ of the flat bdl. $\mathbf{E} = \pi_1(M,x)\backslash(\tilde{M} \times V)$ satisfies $\rho(\pi_1(M,x)) \subset T = \mathrm{Aut}(V, \mathbf{S}_x)$. Moreover ρ is semi-simple.

We have the equivriant holomorphic "horizontal" map $\tilde{M} \ni x \mapsto (\mathbf{E}_x, \mathbf{F}_x) \in \mathcal{D}(V)$

Mixed Hodge structure

An $\mathbb{R}$-*mixed Hodge structure* on an $\mathbb{R}$-vector space V is a pair (W_*, F^*) such that:

1. W_* is an increasing filtration on V.
2. F^* is a decreasing filtration on $V_{\mathbb{C}}$ such that the filtration on $Gr_n^W V_{\mathbb{C}}$ induced by F^* is an $\mathbb{R}$-Hodge structure of weight n.

We call W_* the weight filtration and F^* the Hodge filtration.

Useful decomposition of Mixed Hdg str

For a $\mathbb{R}$-mixed Hdg. str. on V, there exists a canonical decomposition (Deligne splitting) $V_{\mathbb{C}} = \bigoplus V^{p,q}$ so that

$$\overline{V^{p,q}} = V^{q,p} \mod \bigoplus_{r+s<p+q} V^{r,s},$$

$$W_i(V_{\mathbb{C}}) = \bigoplus_{p+q\le i} V^{p,q} \quad \text{and} \quad F^i(V_{\mathbb{C}}) = \bigoplus_{p\ge i} V^{p,q}.$$

Conversely this type dec. implies $\mathbb{R}$-mxd. Hdg str.
If $\overline{V^{p,q}} = V^{q,p}$, then we say that (W_*, F^*) is $\mathbb{R}$-split (that is the direct sum of (pure) $\mathbb{R}$-Hdg str).

Mixed Hodge structures appear on

- The cohomology of (noncompact) $\mathbb{C}$-algebraic varieties. (Deligne)
- Limits of polarized variations of Hodge structures. (Schmid, Steenbrink,etc.)
- Malcev completion $\pi_1(M) \otimes \mathbb{R}$ of the fund. grp. of $\mathbb{C}$-alg varieties M. (Morgan, Hain)
  or Real homotopy $\pi_*(M) \otimes \mathbb{R}$ of simply conn. $\mathbb{C}$-alg varieties M.

Variation of mxd Hdg str

A *real variation of mixed Hodge structure* ($\mathbb{R}$-VMHS) over M is $(\mathbf{E}, \mathbf{F}^*, \mathbf{W}_*)$ so that:

1. $\mathbf{E}$ is a local system of fin-dim. $\mathbb{R}$-vector spaces.
2. $\mathbf{W}_*$ is an increasing filtration of the local system $\mathbf{E}$.
3. $\mathbf{F}^*$ is a decreasing filtration of the holomorphic vector bundle $\mathbf{E} \otimes_{\mathbb{R}} \mathcal{O}_M$.
4. The Griffiths transversality $D\mathbf{F}^r \subset A^1(M, \mathbf{F}^{r-1})$ holds where D is the flat connection associated with the local system $\mathbf{E}_{\mathbb{C}}$.
5. For any $k \in \mathbb{Z}$, the local system $Gr_k^{\mathbf{W}}(\mathbf{E})$ with the filtration induced by $\mathbf{F}^*$ is an $\mathbb{R}$-VHS of weight k.

Usui proved that:

1. A VMHS is given by the relative logarithmic de Rham complex of logarithmic deformations. (generalization of family of projective manifolds)
2. Classifying spaces $\mathcal{D}(V)$ of graded polarized mxd Hodge strs are complex homo sp and VMHSs give equivriant holomorphic "horizontal" maps $\tilde{M} \to \mathcal{D}(V)$

Unipotent $\mathbb{R}$-VMHS

A $\mathbb{R}$-VMHS $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ called is unipotent if the $\mathbb{R}$-VHS on each $Gr_k^{\mathbf{W}}(\mathbf{E})$ is constant.
This condition is equivalent to "the monodoromy rep. $\rho : \pi_1(M) \to GL(V)$ of $\mathbf{E} = \pi_1(M)\backslash(\tilde{M} \times V)$ is unipotent" (i.e. ρ can be represented by upper triangular unipotent matrices).
Notice that $\rho : \pi_1(M) \to GL(V)$ is unipotent
$\Leftrightarrow$ ρ is lift to a rep. of the Malcev completion $\pi_1(M) \otimes \mathbb{R}$.

Hain-Zucker Theorem

Theorem (Hain, Hain-Zucker)

Let M be a cpt Kähler (or sm. $\mathbb{C}$-alg var). Then:

1. $\exists$ *canonical* $\mathbb{R}$*-mxd Hdg str. on* $\pi_1(M) \otimes \mathbb{R}$. *(Hain)*
2. *Taking the monodromy rep.* $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*) \mapsto \rho : \pi_1(M) \to GL(V)$ *induces a* 1-1 *correspondence* $\{\mathbb{R}\text{-VMHSs}\}$ $\to \{\text{"}\mathbb{R}\text{-mxd Hdg representations of } \pi_1(M) \otimes \mathbb{R}\text{"}\}$.

Rem: The $\mathbb{R}$-mxd Hdg str. on $\pi_1(M) \otimes \mathbb{R}$ depends on base point and so the correspondence also depends on base point.

Quick review Sullivan 1-minimal model

For any DGA A^* with $\dim H^0(A^*) = 1$ (e.g. de Rham of conn mfd), there exists a unique "1-minimal" DGA $\mathcal{M}^*$ s.t. we have a DGA map $\phi : \mathcal{M}^* \to A^*$ which induces isomorphisms on 0th and first cohom. and an injection on second cohom.

Note :

- ϕ is not unique but all of such maps are homotopy equiv.
- $\mathcal{M}^*$ is the dual complex of a pro-nilpotent Lie algebra $\mathfrak{u}$.

Main construction(Unipotent case)

Let (M, g) be a cpt Kähler manifold.
Take the canonical Sullivan 1-minimal model
$\phi : \mathcal{M}^* \to A^*_{dR}(M)$.
(Explicitly described by the $\mathbb{C}$-str using dd^c-Lemma)
We can construct a canonical $\mathbb{R}$-mixed Hodge structure on $\mathcal{M}^*$.
(Explicitly described by the $\mathbb{C}$-str and the Kähler metric)
Rem: Mixed Hodge structures on $\mathcal{M}^*$ are found by Morgan(1978). But they are not unique.

Main theorem(Unipoten case)

Consider the dual Lie algebra $\mathfrak{u}$ of $\mathcal{M}^*$ equipped with the dual $\mathbb{R}$-mixed Hodge str..
We say that a rep. of $\mathfrak{u}$ is mixed Hodge if it is a morphism of $\mathbb{R}$-mxd Hdg str.

Theorem (K.)

We can construct an explicit $\mathbb{R}$-VMHS from a mixed Hodge representation of $\mathfrak{u}$.

Notes

- The greater part of the constructions are based on "Formality" arguments of Deligne–Griffiths–Morgan–Sullivan
- By the de Rham homotopy theory(Sullivan), $\mathfrak{u}$ can be identified with $\pi_1(M) \otimes \mathbb{R}$. But our $\mathbb{R}$-mxd Hdg str and VMHS construction does not depend on the base point and so this is different from Theorem (Hain-Zucker).
- Our construction is functorial for holomorphic submersions.

Detail of mixed Hodge str. on $\mathcal{M}^*$

1. We construct a "real" 1-minimal model $\phi : \mathcal{M}^* \to \ker d^c \subset A^*_{dR}(M)$ with a grading $\mathcal{M}^* = \bigoplus \mathcal{M}^*_k$. ($dd^c$-Lemma)
2. We construct a "complex" 1-minimal model $\varphi : \mathcal{N}^* \to \ker\partial \subset A^*_{dR}(M) \otimes \mathbb{C}$ with a bigrading $\mathcal{N}^* = \bigoplus (\mathcal{N}^*)^{P,Q}$. ($\partial\bar{\partial}$-Lemma)
3. We take an isomorphism $\mathcal{I} : \mathcal{M}^*_{\mathbb{C}} \to \mathcal{N}$ which is compatible with filtrations $W_k(\mathcal{M}^*) = \bigoplus_{i \le k} \mathcal{M}^*_i$ and $W_k(\mathcal{N}^*) = \bigoplus_{P+Q \le k} (\mathcal{N}^*)^{P,Q}$ and a homotopy $H : \mathcal{M}^*_{\mathbb{C}} \to A^*(M) \otimes \mathbb{C} \otimes [t, dt]$ from $\varphi \circ \mathcal{I}$ to ϕ.
4. By $F^r(\mathcal{M}^*_{\mathbb{C}}) = \mathcal{I}^{-1}(\bigoplus_{P \ge r} (\mathcal{N}^*)^{P,Q})$, we define an $\mathbb{R}$-mixed Hodge structure on $\mathcal{M}^*$.

Detail of $\mathbb{R}$-VMHS

For a mxd Hdg $\mathfrak{u}$-rep.$\Omega : \mathfrak{u} \to \mathrm{End}(V, W, F)$, we can construct the $\mathbb{R}$-VMHS $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ so that:

1. $\mathbf{E}$ is the trivial $\mathcal{C}^\infty$-vector bundle $M \times V$ with the flat connection $d + \phi(\Omega)$.
2. $\mathbf{W}$ is the filtration of the vector bundle $M \times V$ induced by the weight filtration W_* on V.
3. We can take a gauge transformation A of the vector bundle $M \times V_{\mathbb{C}}$ such that $\varphi \circ \mathcal{I}(\Omega) = A^{-1}dA + A^{-1}\phi(\Omega)A$.
4. $\mathbf{F}^* = A\mathbf{F}_0^*$ such that $\mathbf{F}_0^*$ is the filtration of the vector bundle $M \times V_{\mathbb{C}}$ induced by the Hodge filtration F^* on $V_{\mathbb{C}}$.

Main construction(Non-unipoten case)

Let (M, g) be a cpt Kähler manifold and $(\mathbf{E}_0, \mathbf{F}_0^*)$ a $\mathbb{R}$-VHS with a polarization $\mathbf{S}$.
Assume the monodromy rep. of $\mathbf{E}_0 = \pi_1(M, x)\backslash(\tilde{M} \times V_0)$ is Zariski-dense in $T = \mathrm{Aut}(V, S)$.
We consider the T-equivariant de Rham DGA $A^*(M, \mathcal{O}(T))$. (Deligne, Hain)
Take the canonical "T-equiv." Sullivan 1-minimal model $\phi : \mathcal{M}^* \to A^*(M, \mathcal{O}(T))$.
We can construct a canonical $\mathbb{R}$-mixed Hodge structure on $\mathcal{M}^*$ which is compatible with the T-action.

Main theorem(Non-unipoten case)

Consider the dual Lie algebra $\mathfrak{u}$ of $\mathcal{M}^*$ equipped with the dual $\mathbb{R}$-mixed Hodge str..
We say that a T-equiv. rep. of $\mathfrak{u}$ is mixed Hodge if it is a morphism of $\mathbb{R}$-mxd Hdg str.

Theorem (K.)

We can construct an explicit $\mathbb{R}$-VMHS $(\mathbf{E}, \mathbf{W}_*, \mathbf{F}^*)$ from a mixed Hodge T-equivariant representation of $\mathfrak{u}$.

Note: $\mathbb{R}$-VHS on each $Gr^k_{\mathbf{W}}(\mathbf{E})$ given by "Schur functors" of starting $\mathbb{R}$-VHS $\mathbf{E}_0$. (Recall Rep. Theory of $SO(N)$)

Examples (Kuranishi theory)

Let U be a simple T-module (considered as the trivial $\mathfrak{u}$-rep.)
Via the DGLA technique, we obtain a "versal deformation" (Schlessinger's hull) of T-equiv $\mathfrak{u}$-reps
$\Omega : \mathfrak{u} \to \mathrm{End}(U) \otimes R$.
R is the ring of "analytic functions on Kuranishi space"

Proposition (K.)

We can obtain a $\mathbb{R}$-mixed Hodge structure (maybe not unique) on $U \otimes R$ such that
$\Omega : \mathfrak{u} \to \mathrm{End}(U) \otimes R \to \mathrm{End}(U \otimes R)$ *is mixed Hodge.*
Thus we construct $\mathbb{R}$-VMHS from Ω.

Inspired by the work of Eyssidieux–Simpson (2011)