In search of Lagrangians with non-trivial Floer cohomology

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Paul Seidel, Fukaya categories and deformations, ICM 2002 Beijing.

> *Soon after their first appearance, Fukaya categories were brought to the attention of a wider audience through the homological mirror conjecture of Kontsevich. Since then Fukaya and his collaborators have undertaken the vast project of laying down the foundations, and as a result a fully general definition is available. The task that symplectic geometers are now facing is to make these categories into an effective tool, which in particular means developing more ways of doing computations in and with them.*

Towards Floer homology of a Lagrangian submanifold

1. A submanifold L in a symplectic manifold (X, ω) is *Lagrangian* if $\omega|_L = 0$ and $\dim L = \frac{1}{2} \dim X$.
2. Floer homology is an invariant of a Lagrangian. It is computed using counts of holomorphic disks in X with boundary on L.
3. Counts of disks produces an A_∞-algebra.

A_∞-algebra

1. A graded vector space is an A_∞ -algebra if there is a family of maps $(\mu^d)_{d\geq 0}$
$$\mu^d : A^{\otimes d} \to A[2-d]$$
that satisfy the A_∞ -relations
$$0 = \sum_{\substack{n,m\geq 0 \\ n+m\leq d}} (-1)^{n+\sum_{i=1}^{n}|a_i|}\mu^{d-m+1}(a_1,\dots,a_n,$$
$$\mu^m(a_{n+1},\dots,a_{n+m}),a_{n+m+1},\dots,a_d).$$

Holomorphic disks

1. We choose an ω-tame almost complex structure $J : TM \to TM$. For all non-zero vectors $v \in TX$,
$$(\mathit{Tameness}) \quad \omega(v, Jv) > 0.$$
2. We consider holomorphic disks. These are maps
$$u : ((\mathbb{D}^2, \partial\mathbb{D}^2), j) \to ((X, L), J)$$
satisfying $\overline{\partial}_J u = \frac{1}{2}(du + Jdu \circ j) = 0$.
3. We count maps with homological constraints on the boundary. Given cohomology classes $l_1, \dots, l_n \in H^*(L)$, let A_i be a cycle that is Poincare dual to a_i. Define a moduli space
$$\mathcal{M}_{\underline{l}} := \{(u, (z_1, \dots, z_n)) | u : (\mathbb{D}^2, \partial\mathbb{D}^2) \to (X, L) \text{ is holomorphic}, \\ z_i \in \partial\mathbb{D}^2, u(z_i) \in A_i\} / \operatorname{Aut}(\mathbb{D}^2).$$
Here, note that $\operatorname{Aut}(\mathbb{D}^2, j) = PSL(2, \mathbb{R})$.

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Quick definition : Morse cohomology of a manifold M

1. Choose a function $f : M \to \mathbb{R}$ with non-degenerate critical points, and a generic metric.
2. The *index* of a critical point is the dimension of its stable submanifold.
3. The j^{th} Morse cocycle is freely generated by the critical points of index j:
$$CM^j(f) := \mathbb{Z}\langle \mathrm{crit}_j(f) \rangle.$$
4. The coboundary operator is defined by a count of isolated Morse trajectories
$$\partial : CM^j(f) \to CM^{j+1}(f), \quad p \mapsto \sum_{u \in \mathcal{M}(p,q)} \epsilon(u) q,$$
where $\mathcal{M}(p,q)$ is the moduli space of gradient flow lines from p to q. $\dim \mathcal{M}(p,q) = \mathrm{ind}(q) - \mathrm{ind}(p) - 1$.
5. One can show that $\partial^2 = 0$ using the fact that for one-dimensional moduli spaces of flow lines, the end points are given by broken flow lines.

Quick definition : Morse cohomology of a manifold M

1. The cohomology

$$HM^*(f) := \ker \partial / \operatorname{Im} \partial$$

is independent of the choice of f and the metric.

2. Morse cocycles correspond to unstable submanifolds of critical points.

Treed holomorphic disks

1. The domain of a *treed holomorphic disk* is a treed disk. A *treed disk* consists of a
    1. surface part, which is a disk,
    2. and a tree part, consisting of n incoming and one outgoing semi-infinite segments attached to the boundary of the disk.
2. Let $f : L \to \mathbb{R}$ be a Morse function. On the surface part, a treed holomorphic disk is a holomorphic disk with boundary on the Lagrangian. The tree part is mapped to gradient flow lines on L.
3. For $\underline{l} \in (\mathrm{crit}(f))^{n+1}$, we define $\mathcal{M}_{\underline{l}}(X, L)$ to be the moduli space of treed holomorphic disks with gradient segments asymptotic to $(l_0, \ldots, l_n)$.

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A_∞ -algebra of a Lagrangian

1. Consider the graded vector space
$$CF(L) := \Lambda\langle \mathrm{crit}(f)\rangle,$$
where $\Lambda := \{a_i q^{c_i} : a_i \in \mathbb{C}, c_i \to \infty \text{ as } i \to \infty, c_i \geq 0\}$ is the *Novikov ring*.
2. A_∞ maps are
$$\mu^n(l_1, \ldots, l_n) := \sum_{l_0 \in \mathrm{crit}(f), u \in \mathcal{M}^0_{\underline{l}}} \epsilon(u) a_0 q^{E(u)} \rho(u).$$
Here $\mathcal{M}^0_{\underline{l}}$ is the zero-dimensional component of the moduli space of $\mathcal{M}_{\underline{l}}$.
3. (Fine print: We consider Lagrangians with a brane structure, which is a $\Lambda^\times$-line bundle. Here $\rho(u) \in \Lambda^\times$ is the holonomy of the line bundle about ∂u.)
4. Why are A_∞ relations satisfied? For any $\underline{l}$, the boundary of the one-dimensional components $\partial\mathcal{M}^1_{\underline{l}}$ make up the terms in the A_∞ relation.

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Some remarks

1. μ^0 is a count of disks with one output and no input. Therefore, it has no terms with zeroth power of q.
2. $\mu^1(p) = \partial_{Morse} p +$ terms higher order in q.
3. A_∞ -algebra of a Lagrangian is an extension of the A_∞ -structure arising from Morse theory.

Floer cohomology of the Lagrangian L

1. Assuming $(\mu^1)^2 = 0$, Floer cohomology is defined as

$$HF(L) := \ker(\mu^1)/\operatorname{Im}(\mu^1).$$

2. However the condition $(\mu^1)^2 = 0$ is not always satisfied. The first A_∞-relation is $\mu^2(\mu^0, a) - (-1)^{|a|}\mu^2(a, \mu^0) + (\mu^1)^2(a) = 0$.
3. A *strict unit* of an A_∞-algebra A is an element $e_A \in A_\infty$ such that

$$\mu^2(e_A, a) = a = (-1)^{|a|}\mu^2(a, e_A), \quad \mu^n(\dots, e_A, \dots) = 0\, n \neq 2.$$

4. The A_∞-algebra is *unobstructed* if $\mu^0 \in \Lambda e_A$. Floer cohomology $HF(L)$ can be defined for a unobstructed A_∞ -algebra.
5. A Lagrangian L is *Floer non-trivial* if the A_∞-algebra is unobstructed and the Floer cohomology is non-trivial.

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A geometric consequence of Floer non-triviality

1. A Lagrangian L in (X, ω) is *non-displaceable* if for any Hamiltonian diffeomorphism φ, $\varphi(L) \cap L \neq \emptyset$.
2. Question: Is there any non-displaceable loop in S^2?

A geometric consequence of Floer non-triviality

1. A Lagrangian L in (X, ω) is *non-displaceable* if for any Hamiltonian diffeomorphism φ, $\varphi(L) \cap L \neq \emptyset$.
2. Question: Is there any non-displaceable loop in S^2?
3. Answer: Yes, one that divides S^2 into halves of equal area.
4. Floer non-triviality implies non-displaceability.
5. Lagrangian Floer theory was initially developed by Floer to give a lower bound for the number of intersection points of L and $\phi(L)$.

Toric symplectic manifolds

1. A toric symplectic manifold is a symplectic manifold (X, ω) with a Hamiltonian action of a half-dimensional torus T. Further, the action is generated by a *moment map*

$$\mu : X \to \mathfrak{t}^\vee.$$

2. The image of the moment polytope is convex (by a result of Atiyah and Guillemin-Sternberg).
3. (Example) For the S^1-action on $\mathbb{P}^1$

$$[z_0 : z_1] \xrightarrow{\theta} [z_0 : e^{i\theta} z_1],$$

the moment map is

$$\mu([z_0 : z_1]) = \frac{|z_1|^2}{|z_0|^2 + |z_1|^2}.$$

The moment polytope is $\mu(\mathbb{P}^1) = [0, 1]$.
4. (Example) For the product action of $(S^1)^2$ on $(\mathbb{P}^1)^2$, the moment polytope is the square $[0, 1] \times [0, 1]$.

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Examples of symplectic toric manifolds

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   the moment map is

$$\mu([z_0 : z_1 : z_2]) = \frac{1}{|z_0|^2 + |z_1|^2 + |z_2|^2}(|z_1|^2, |z_2|^2).$$

   The moment polytope is
   $\mu(\mathbb{P}^2) = \{(\lambda_1, \lambda_2) \in (\mathbb{R}_{\geq 0})^2 : \lambda_1 + \lambda_2 \leq 1\}$.
2. The moment polytope of the $(S^1)^n$-action on $\mathbb{P}^n$ is $\{(\lambda_1, \dots, \lambda_n) \in (\mathbb{R}_{\geq 0})^n : \sum_i \lambda_i \leq 1\}$.
3. The fibers over the interior points of the moment polytope are Lagrangian tori.
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Examples of symplectic toric manifolds

1. The blow-up of a point p in (X^{2n}, ω) is a symplectic manifold given by
$$\mathrm{Bl}_p X := (X \backslash B_\epsilon(p)) / \sim,$$
where $\sim$ is the equivalence relation on $\partial B_\epsilon \sim S^{2n-1}$ given by the S^1-action.
2. Remark: The symplectic form on $\mathrm{Bl}_p X$ is in the class $[\omega] - \epsilon[E]$. The constant ϵ is the blow-up parameter.
3. Consider $\mathbb{CP}^2$ blown up at the point $[0:0:1]$ by a parameter ϵ. The moment polytope is
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Floer non-triviality for toric Lagrangians

1. Theorem (Fukaya, Oh, Ohta, Ono): For $\lambda \in \mathfrak{t}^\vee$, the Lagrangian $\mu^{-1}(\lambda)$ is Floer non-trivial exactly when μ^0_λ, as a function of λ, has a critical point.
2. Roughly speaking, μ^0_λ has a critical point when the disks contributing to the potential 'balance' each other.
3. Toric manifolds are git quotients of $\mathbb{C}^n$. By a result of Cho-Oh, disks in (X, L) are Blaschke products. In nice cases, the leading order terms in μ^0 are come from disks with a single intersection with toric divisors.
4. Examples:
   1. In $\mathbb{P}^1$, $\mu^{-1}(\frac{1}{2})$ is Floer non-trivial.
   2. In $\mathbb{P}^1 \times \mathbb{P}^1$, $\mu^{-1}(\frac{1}{2}, \frac{1}{2})$ is Floer non-trivial.
   3. In $\mathbb{P}^2$, $\mu^{-1}(\frac{1}{3}, \frac{1}{3})$ is Floer non-trivial.
   4. Similarly, in $\mathbb{P}^n$, we get a Floer non-trivial torus $\mu^{-1}(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$, with $n+1$ balancing disks. This torus is called the *Clifford torus*.

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Example: Point blow-up of projective space

The Floer non-trivial Lagrangians depend on the size of the blow-up.

1. Small blow up: In addition to the Clifford torus, there is a Floer non-trivial Lagrangian near the exceptional divisor.
2. Big blow-up: In this case, we view the space $\mathrm{Bl}_{point}\,\mathbb{P}^n$ as a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \oplus \mathbb{C})$ over $\mathbb{P}^{n-1}$ where the form in the fibers is small. There is a Floer non-trivial Lagrangian.
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Symplectic cuts

1. Suppose Z is an S^1-fibered separating hypersurface, and $X = X_+^\circ \cup_Z X_-^\circ$. Then,

$$X_\pm := X_\pm^\circ / \sim$$

   are symplectic manifolds, where $\sim$ is the S^1-equivalence relation on Z. $Y := Z/S^1$ is a symplectic divisor on $X_\pm$ with normal bundles of opposite sign.
2. The reverse operation of 'cut' is symplectic sum. X is a symplectic sum $X_+ \#_Y X_-$.
3. If Z corresponds to a hypersurface in the moment polytope, then the moment map descends to $X_\pm$. The moment polytopes are obtained by cutting the moment polytope of X.
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Idea of Charest-Woodward proof

1. In a symplectic sum $X_+ \#_Y X_-$, where $L \subset X_+$, the A_∞-algebra of a Lagrangian can be written as a combination of contributions from both pieces.
2. If one of the piece is toric Fano, and has a 'balanced' Lagrangian L, with none of the disks incident on the separating divisor, then L is Floer non-trivial in X.

Another example: line blow up

1. If the neighbourhood of the line has a toric structure, then there is a Floer non-trivial Lagrangian near the exceptional divisor.
2. The toric neighbourhood can be isolated via a symplectic cut.
3. In the toric piece, the Lagrangian is balanced in stages.
4. One can view this as a 'lift' of the equator circle in the blow-up locus.

Conjectures, results

1. Wishful conjecture: Suppose $Y \subset X$ is a symplectic submanifold, and the Lagrangian $L_Y \subset Y$ is Floer non-trivial. Then, L_Y lifts to a Floer non-trivial Lagrangian L in a small blow-up $\mathrm{Bl}_Y X$.
2. For this to work, we require the neighbourhood of L and L_Y to have a toric structure and balancing disks.
3. An example where the right conditions happen: $\overline{M}_{0,n}$ is an iterated blow-up of $\mathbb{P}^{n-3}$
4. Theorem (work in progress): The space $\overline{M}_{0,n}$ has Floer non-trivial Lagrangians corresponding to each of the blow-ups.

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A hint of the technical nightmares

1. Toric Fano case: Only disks with single divisor intersection contribute to the potential μ^0. Without a constraint on the output, the moduli space of disks has dimension $\dim(L)$. If we impose the condition that the boundary marked point maps to $\max(f)$, we get a zero-dimensional moduli space. The maximum point e is the unit of the A_∞ -algebra. Therefore, we have unobstructedness i.e. $\mu^0 \in \Lambda e$. In this case, FOOO result is straightforward.
2. We lose unobstructedness if the toric manifold has negative spheres. The moduli space for the disk + sphere homotopy type has lower dimension, and will have outputs to other critical points in L. The A_∞ algebra then has to be 'deformed' to regain unobstructedness.