Holomorphic Curves in Compact Complex Parallelizable Manifold $\Gamma\backslash\mathrm{SL}(2,\mathbb{C})$

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Compact ccomplex parallelizable manifolds.

• (Definition) A compact complex manifold X^n is said to be parallelizable iff $T^{(1,0)}X \cong \mathcal{O}_X^{\oplus n}$ holds.

Let X be a compact complex parallelizable manifold. If $\{e_1, \ldots, e_n\}$ is a basis of $\mathrm{Lie}(\mathrm{Aut}_{\mathcal{O}}(X))_0$ then $[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k$ where c_{ij}^k are constant. Therefore $\exists$ a discrete cocompact $\Gamma \subset \mathrm{Aut}_{\mathcal{O}}(X)_0$ such that $X = \Gamma \backslash \mathrm{Aut}_{\mathcal{O}}(X)_0$.

• (Classical results)

[Wang (1954)] A compact Kähler manifold X^n is parallelizable iff $X = T^n$.

[Grauert-Remmert (1962)] $\forall H$ a hypersurface, $\exists ! \widetilde{H} \subset \mathrm{Alb}(X)$ a hypersurface s.th. $H = \alpha^{-1}(\widetilde{H})$ where $\alpha : X \to \mathrm{Alb}(X)$ is the Albanese map.

Huckleberry-Winkelmann's results on subvarieties of compact complex parallelizable manifold.

[Huckleberry-Winkelmann (1993)]

(1) Let G be a simple complex Lie group and $X = \Gamma \backslash G$ a compact complex parallelizablen manifold. Then for any Kähler subvarieties Z of X we have $\mathrm{codim} Z \geq \sqrt{\dim G}$.

(2) Let G be a semi-simple complex Lie group and $X = \Gamma \backslash G$ a compact complex parallelizable manifold. Then for any Kähler subvarieties Z of X we have $\dim Z \leq \frac{1}{3} \dim X$.

(3) Let Z be an irreducible subvariety of $X = \Gamma \backslash G$. Suppose that $0 < \kappa(Z) < \dim Z$. Then the pluricanonical map of Z is described group theoretically as $Z \to H \backslash Z$ where H is roughly identified with $\mathrm{Aut}_{\mathcal{O}}(Z)_0$.

(4) (Bloch-Ochiai type theorem) Let $f : \mathbb{C} \to X = \Gamma \backslash G$ be an entire holomorphic curve. Then the Zariski closure $Z = \overline{f(\mathbb{C})}$ is an orbit of a subgroup of G.

Huckleberry-Winkelmann's question

• (Question) How to construct subvarieties of the compact complex parallelizable manifold $X := \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ other than orbits ?

Let A be a maximal torus of $\mathrm{SL}(2,\mathbb{C})$ such that $\Gamma \cap A \cong \mathbb{Z}$ (i.e., $A\Gamma$ is closed in $\mathrm{SL}(2,\mathbb{C})$). Then any translation of the image of A in $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ is a compact torus embedded in X.

Therefore it is natural to ask the following question :

• (Sub-Question) Are there proper subvarieties of $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ other than orbits of a maximal torus ?

Note :
1. $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ is non-Kähler.
2. The Albanese map of $X = \Gamma\backslash\mathrm{SL}(2,\mathbb{C})$ is trivial.
Therefore only codimension two subvarieties of X (i.e., holomorphic curves) are of interest.

Main Theorem

We consider the case $G = \mathrm{SL}(2, \mathbb{C})$, $\Gamma \subset \mathrm{SL}(2, \mathbb{C})$ a cocompact lattice and

$$X = \Gamma \backslash \mathrm{SL}(2, \mathbb{C})$$

an associated compact complex parallelizable manifold.

- (Theorem) Let M be a compact Riemann surface and

$$f : M \to X$$

a non-constant holomorphic map. Then f decomposes into the composition $t \circ h \circ \alpha$, where
$\alpha : M \to \mathrm{Alb}(M)$ is the Albanese map,
$h : \mathrm{Alb}(X) \to X$ has its image in a maximal torus $T \hookrightarrow X$ defining an algebraic group homomorphism $h : \mathrm{Alb}(M) \to T$ and
$t : X \to X$ is a right translation by some element of $\mathrm{SL}(2, \mathbb{C})$.

Outline of Proof, I

- Set up : $G := \mathrm{SL}(2,\mathbb{C})$, $\Gamma \subset G$ a cocompact lattice.
- We consider a compact complex parallelizable manifold

$$X := \Gamma \backslash G \ .$$

$\pi : G \to X$ the canonical projection.
$\mathrm{Aut}_{\mathcal{O}}(X)_0 = G = \mathrm{SL}(2,\mathbb{C})$. $\pi_1(X) = \Gamma$.

- CASE 1 : Suppose that

$$f : M \to X$$

is a non-constant holomorphic map from a compact Riemann surface M s.th. $f(M)$ has genus ≥ 2.
May assume that f is of degree 1 onto its image.

- (proof by contradiction) Want to show it is impossible.

Outline of Proof, II

- $f : M \to X = \Gamma\backslash G$ a non-constant holomorphic map as above.
- $m := f_* : \pi_1(M) \to \Gamma = \pi_1(X)$.
- $\mathbb{D}_m := \operatorname{Ker} m\backslash\mathbb{D}$.

$\exists f_m : \mathbb{D}_m \to G$ a lift of $f : M \to X$.
This is canonically defined as soon as we fix a lift of one point.

$q : \mathbb{D}_m \to M$ the Galois covering with $\operatorname{Gal}(\mathbb{D}_m : M) = \operatorname{Ker} m\backslash\pi_1(M)$.

$\widetilde{m} : \operatorname{Ker} m\backslash\pi_1(M) \to \Gamma$ is injective.

- (Lemma 1) (1) $f : M \to X$ lifts to $f_m : \mathbb{D}_m \to G$ and $\pi \circ f_m = f \circ q$.

(2) To any $\sigma \in \operatorname{Gal}(\mathbb{D}_m : M)$ associates $m(\sigma) \in \operatorname{Gal}(G : X)$ and $f_m \circ \sigma = m(\sigma) \circ f_m$.

(3) $\mathbb{D}_m$ is non-compact and $|\operatorname{Gal}(\mathbb{D}_m : M)| = |\operatorname{Ker} m\backslash\pi_1(M)| = \infty$.

Outline of Proof, III

- We consider a compact hyperbolic 3-manifold

$$Z = \Gamma\backslash\mathbb{H}^3 \ .$$

- $\pi_1(Z) = \Gamma$.

May assume : $\mathrm{Isom}_0(Z) = \{1\}$.

- $\exists$ a natural fibration

$$\mathrm{SU}(2) \to X \xrightarrow{p} Z \ .$$

- $\varpi : \mathbb{H}^3 \to Z = \Gamma\backslash\mathbb{H}^3$: the canonical projection.

- $\mathbb{D}_m$ decompose into fundamental domains wrt. the action of $\mathrm{Gal}\,(\mathbb{D}_m : M) = \mathrm{Ker}\, m\backslash\ \pi_1(M)$. May assume $o \in \mathbb{D}_m$ and $\varpi(f_m(o)) = o \in \mathbb{B}^3$ ($\mathbb{B}^3$ and $\mathbb{H}^3$ identified canonically).

- $\mathbb{H}^3$ decomposes into fundamental domains wrt. the action of Γ.

- We compare two metrics on G one induced from the hyperbolic metric on $\mathbb{H}^3$ and the other induced from the Fubini-Study metric of $\mathbb{P}^4$.

Outline of Proof, IV

- We consider the composition

$$f_m : \mathbb{D}_m \to G = \mathrm{SL}(2,\mathbb{C}) \hookrightarrow \mathbb{Q}^3 \hookrightarrow \mathbb{P}^4 \ .$$

Let "length" mean the minimum length of strings of adjacent fundamental domains starting at reference fundamental domain. Set

$$\mathbb{D}_m(n)_{\mathrm{Ker}\, m \backslash \pi_1(M), \mathbb{D}_m} := \{ x \in \mathbb{D}_m \mid \mathrm{length}_{\mathrm{Ker}\, m \backslash \pi_1(M), \mathbb{D}_m}(x, o) = n \} \ ,$$

$$n' := \inf\{ \mathrm{length}_{\Gamma, \mathbb{H}^3}(\varpi f_m(x), o) \mid x \in \mathbb{D}_m(n)_{\mathrm{Ker}\, m \backslash \pi_1(M), \mathbb{D}_m} \} \ .$$

- (Condition C) $(\exists \alpha > \frac{1}{2})$ $(\exists C > 0)$ $(\forall n \gg 1)$ $(\alpha n - C \le n')$

- (Lemma 2 [Basic Comparison]) Assume (Condition C).
Then : $(\exists C' > 0)$ s.th.

$$(\, f_m^* g_{\mathrm{FS}}|_{R \ni x} \le \exp\{-(\mathrm{dist}_{g_{\mathrm{hyp}}, \mathbb{D}_m}(x, o) - C')\}\, ds^2_{\mathrm{hyp}}\,)$$

holds on $\mathbb{D}_m$, where R is a fundamental domain in $\mathbb{D}_m$ wrt. the action of $\mathrm{Gal}\,(\mathbb{D}_m : M)$.

Outline of Proof, V

- (Lemma 3) Assume (Condition C).

(1) $f_m^* g_{\rm FS}$ is uniformly equivalent to $ds^2_{\rm euc}$ induced from $\mathbb{D}_m \subset \mathbb{D} \subset \mathbb{R}^2$.

(2) $\exists C > 0$ s.th. $\mathrm{Area}_f(r) := \int_{\mathbb{D}_m(r)} f^* g_{\rm FS} \leq C$.

- (Lemma 4) Assume (Condition C). We mean by $[t]$ the circle $|z| = t$.

For the length of $f_m(\partial \mathbb{D}_m[r])$ we have the following. Set

$$L(t) := \mathrm{length}_{\rm FS}(f_m(\partial \mathbb{D}_m[t])) \subset \mathbb{Q}^3 \ .$$

Then

$$\lim_{t \to 1} f_m(\partial \mathbb{D}_m[t])$$

is rectifiable and

$$\lim_{t \to 1} L(t) < \infty \ .$$

Moreover

$$\lim_{t \to 1} f_m(\partial \mathbb{D}_m[t])$$

lies in $\mathbb{Q}^2 = \mathbb{Q}^3 \setminus G$.

Outline of Proof, VI

• (Geometric Measure Theory [Bishop's Theorem]) Let $\{A_n\}$ be a sequence of pure $(2k)$ dimensional $\mathbb{C}$-analytic sets in an open set U of $\mathbb{C}^n$ converging to a set $A \subset U$. Let the real $(2k)$ dimensional volumes of A_n be finite and bounded by some constant b. Then A is a $\mathbb{C}$-analytic set of U.

• (Lemma 5 [Application of Bishop's Theorem]) Assume (Condition C). We mean by (t) the intersection with the subdisk $|z| < t$. Set

$$Y := \lim_{t \to 1} f_m(\mathbb{D}_m(t)) = \overline{f_m(\mathbb{D}_m)} \subset \mathbb{Q}^3 \subset \mathbb{P}^4 \ .$$

Then

(1) $Y \cap \mathbb{Q}^2$ ($\mathbb{Q}^2 = \mathbb{Q}^3 \setminus G$) is a finite set.

(2) Y is an algebraic curve in $\mathbb{Q}^3 = \overline{G} \subset \mathbb{P}^4$.

(3) Y intersects $\mathbb{Q}^2$ transversally.

Outline of Proof, VII

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(Lemma 5)

$\Rightarrow \partial\mathbb{D}_m \cap \mathbb{D}$ ($\mathbb{D}_m$ being realized as a fundamental domain w.r.to the action of $\mathrm{Ker}\, m$ on $\mathbb{D}$) consists of finitely many arcs and each arc decomposes into infinitely many geodesic segments.

$\Rightarrow$ we can draw a picture of how $f(M) \subset X$ is developed in a fundamental domain F (contractible nodulo $\mathrm{SU}(2)$) in G wrt. the action of Γ as a Riemann surface with boundary. The boundary consists of polygons traversing several faces of F and circles each of which appears as a boundary in a face of F of a Riemann surface attached to a polygon.

Outline of Proof, VIII

If there exists a non-constant holomorphic map $f : M \to X$ which is degree 1 onto its image $f(M)$ whose genus is ≥ 2, then (Condition C) holds.

• (Lemma 6) (Condition C) holds, for $f : M \to X$ which is degree 1 onto its image $f(M)$ (the genus of $M \geq 2$).

Strategy of proof of (Lemma 6).

By a measure theoretic argument, we prove :

The following situation never occurs :

$\exists$ a Borel set Θ in the angle space $|z| = 1$ with $\lambda(\Theta) > 0$ s.th.
for $\alpha \leq \frac{1}{2}$ there is no $C > 0$ satisfying $\liminf_{n\to\infty}(n' - \alpha n) > -C$ for geodesic rays in $\mathbb{D}_m$ emanating from $o \in \mathbb{D}_m$ corresponding to the angle parameter in Θ.

Outline of Proof, IX

Besides technicalities, the proof of (Lemma 6) is based on the following simple observation :

Observation. For $c, k > 0$ we define
$\mathbb{D}_{k,c} = \{w = u + iv \,|\, u \in I,\ v > kc,\ \mathrm{diam}_{\mathrm{euc}}(I) = c\} \subset \mathbb{H}$.
Then $\mathrm{Area}_{\mathrm{hyp}}(\mathbb{D}_{k,c}) = \int_{u \in I} du \int_{v > kc} \frac{dv}{v^2} = k^{-1}$.
$F \subset \mathbb{H}$ a fundamental domain corr. to a compact Riemann surface of genus $g \geq 2$. Assume that the Euclidean projection of F on the u-axis is the interval I of Euclidean diameter $c > 0$.
Gauss-Bonnet $\Rightarrow$ $\mathrm{Area}_{\mathrm{hyp}}(F) = 2\pi(2g-2) \geq 4\pi$.
γ : the hyperbolic geodesic in $\mathbb{H}$ asymptotic to ∂I (2 points in $\mathbb{R}$).

- Claim. $F \cap \gamma \neq \emptyset$.

Proof : If $F \cap \gamma = \emptyset$ then $F \subset \mathbb{D}_{2^{-1},c}$. So $\mathrm{Area}_{\mathrm{hyp}}(F) < 2 < 4\pi$.
This is a contradiction.

Outline of Proof, X

We have seen :

If $f : M \to X$ is a non-constant holomorphic map s.th. the genus of $f(M)$ is ≥ 2 and f is degree 1 onto its image $f(M)$, then :

- $\overline{f_m(\mathbb{D}_m)} \subset \mathbb{Q}^3$ is an algebraic curve.
- $\overline{f_m(\mathbb{D}_m)}$ intersects $\mathbb{Q}^2 = \mathbb{Q}^3 \setminus G$ at finitely many points transversally with multiplicity 1.

However, in the present setting, the infinite group $\operatorname{Ker} m \backslash \pi_1(M)$ operates on $f_m(\mathbb{D}_m) = \overline{f_m(\mathbb{D}_m)} \backslash \{\text{finitely many points}\}$ and the quotient curve is the normalization of $f(M)$.
This is impossible.
Therefore CASE 1 never occurs.

Outline of Proof, XI

- CASE 2. Enough to consider the case $f(M)$ has genus 1.

2-1. Assume first that the genus of M is 1, i.e., $g(M) = 1$ and $f : M \to f(M)$ is of degree 1.

We replace $\mathbb{D}_m$ by $\mathbb{C}_m$ in a similar way. Then $\mathbb{C}_m$ is either $\mathbb{C}$ or $\mathbb{C}^*$.

1) If $\mathbb{C}_m = \mathbb{C}$ we get a contradiction.

2) So $\mathbb{C}_m = \mathbb{C}^*$ and $\overline{f_m(\mathbb{C}_m)}$ is contained in a right translation of a maximal torus T in X.

2-2. Assume second that the genus of M is ≥ 2.
In this case $f : M \to X$ decomposes into $\alpha : M \to \mathrm{Alb}(M)$, $h : \mathrm{Alb}(M) \to T$ an algebraic group homomorphism and t a right translation by an element of G.