Analysis and TYZ expansions for Kepler manifolds

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Classical Kepler varieties

hermitian vector space $Z = \mathbf{C}^{n+1}, \ rank = 2$

$$Z_1 := \{z = (z_0, \ldots, z_n) \in Z : \ z_0^2 + \ldots + z_n^2 = 0\} \text{ complex light cone}$$

Symplectic interpretation (cotangent bundle)

$$Z_1 \approx T^*(\mathbf{S}^n) \setminus \{0\} = \{(x, \xi) : \|x\| = 1, \ \xi \neq 0, \ (x|\xi) = 0\}, \ z = \frac{x + i\xi}{2}$$

contact manifold (cosphere bundle)

$$S_1 \approx S^*(\mathbf{S}^n) = \{(x, \xi) : \|x\| = 1, \ \|\xi\| = 1, \ (x|\xi) = 0\}$$

polar decomposition

$$Z_1 = \mathbf{R}^+ \cdot S_1$$

Determinantal varieties

$$Z = \mathbf{C}^{r\times s},\ rank = r \le s,\ 1 \le \ell \le r$$

$Z_\ell := \{z \in Z :\ rank(z) \le \ell\}$ vanishing of all $(\ell+1)\times(\ell+1)$ – minors

$\mathring{Z}_\ell = \{z \in Z :\ rank(z) = \ell\} = Z_\ell \setminus Z_{\ell-1}$ regular part

$Z_1 := \{z \in Z :\ rank(z) \le 1\}$ vanishing of all 2×2 – minors

$$\mathring{Z}_1 := \{z \in Z :\ rank(z) = 1\} = Z_1 \setminus \{0\}$$

$$S_1 = \{\xi \otimes \eta^* :\ \xi \in \mathbf{C}^r, \eta \in \mathbf{C}^s\ \|\xi\| = 1 = \|\eta\|\}$$

$$\mathring{Z}_1 = \mathbf{R}^+ \cdot S_1$$

Let Z complex vector space and $D \subset Z$ be a bounded domain. The group

$$G = Aut(D) = \{g : D \to D \text{ biholomorphic}\}$$

is a real Lie group by a deep theorem of H. Cartan. D is called **symmetric** iff G acts **transitively** on D, and around each point $o \in D$ there exists an involutive **symmetry** $s_o \in G$ fixing o. By Harish-Chandra, D can be realized as a convex circular domain, i.e. as the unit ball

$$D = \{z \in Z : \ \|z\| < 1\}$$

for an (essentially unique) norm called the **spectral norm**. Then the isotropy subgroup

$$K = \{g \in G : \ g(0) = 0\}$$

is a compact group consisting of linear transformations. For domains $D = G/K$ of rank $r > 1$ the boundary ∂D will not be smooth and D is only (weakly) pseudoconvex but not strongly pseudoconvex.

Let $Z = \mathbf{C}^{r\times s}$, endowed with the operator norm $\|z\| = \sup\, spec(zz^*)^{1/2}$. Then the **matrix unit ball**

$$D = \{z \in \mathbf{C}^{r\times s} :\ \|z\| < 1\} = \{z \in \mathbf{C}^{r\times s} :\ I - zz^* > 0\}$$

is a symmetric domain, under the **pseudo-unitary group**

$$G = U(r,s) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL(r+s) :\ \begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix} a^* & c^* \\ b^* & d^*\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right\},$$

which acts on D via **Moebius transformations**

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}(z) = (az+b)(cz+d)^{-1}.$$

Its maximal compact subgroup is

$$K = \left\{\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix} :\ a \in U(r),\ d \in U(s)\right\}.$$

with the linear action $z \mapsto azd^*$.

A basic theorem of M. Koecher characterizes hermitian symmetric domains in **Jordan algebraic** terms. Let Z be a complex vector space, endowed with a ternary composition $Z \times Z \times Z \to Z$, denoted by

$$(x, y, z) \mapsto \{x; y; z\},$$

which is bilinear symmetric in (x, z) and anti-linear in the inner variable. Putting $D(x, y)z := \{x; y; z\}$, Z is called a **hermitian Jordan triple** if the Jordan triple identity (a kind of Jacobi identity)

$$\Big[D(x, y), D(u, v)\Big] = D(\{x; y; u\}, v) - D(u, \{v; x; y\})$$

holds, and the hermitian form

$$(x, y) \mapsto trace D(x, y)$$

is a positive definite. **Every symmetric domain can be realized as the unit ball of a unique hermitian Jordan triple (and conversely)**. Moreover, K consists of all linear transformations preserving the Jordan triple product.

Classification of hermitian Jordan triples

- **matrix triple** $Z = \mathbf{C}^{r\times s}, \quad \{x;y;z\} = xy^*z + zy^*x,$

$$K = U(r)\times U(s):\ z\mapsto uzv,\ u\in U(r),\ v\in U(s)$$

  $rank = r \leq s,\ a = 2$ complex case, $b = s - r$

- $r = 1,\ Z = \mathbf{C}^{1\times d},\ \{x;y;z\} = (x|y)z + (z|y)x, \quad K = U(d)$
- **symmetric matrices** $a = 1$ (real case)
- **anti-symmetric matrices** $a = 4$ (quaternion case)
- **spin factor** $Z = \mathbf{C}^{n+1},\ \{xy^*z\} = (x\cdot\overline{y})\ z + (z\cdot\overline{y})\ x + (x\cdot z)\overline{y}$
  $r = 2,\ a = n-1,\ b = 0$

$$K = \mathbf{T}\cdot SO(n+1)$$

- **exceptional Jordan triples** of dimension 16 ($r = 2$) and 27 ($r = 3$), $a = 8$ (octonion case), $K = \mathbf{T}\cdot E_6$.

A real vector space X is a **Jordan algebra** iff X has a non-associative product $x, y \mapsto x \circ y = y \circ x$, satisfying the Jordan algebra identity

$$x^2 \circ (x \circ y) = x \circ (x^2 \circ y).$$

The **anti-commutator product**

$$x \circ y = (xy + yx)/2$$

of self-adjoint matrices satisfies the Jordan identity. By a fundamental result of Jordan/von Neumann/Wigner (1934), every (euclidean) Jordan algebra has a realization as self-adjoint matrices $X \approx \mathcal{H}_r(\mathbf{K})$ over $\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$ (quaternions), or $\mathbf{K} = \mathbf{O}$ (octonions) if $r \leq 3$. For $r = 2$ we obtain formal 2×2-matrices

$$\mathcal{H}_2(\mathbf{K}) = \{\begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} : \ \alpha, \delta \in \mathbf{R}, \ b \in \mathbf{K} := \mathbf{R}^{2+a}\}.$$

Thus X is characterized by the **rank** r and the **multiplicity** $a = \dim_{\mathbf{R}} \ \mathbf{K}$.

For a euclidean Jordan algebra X, the complexification $Z := X^{\mathbf{C}}$ becomes a (complex) Jordan triple via

$$\{u;v;w\} = (u \circ v^*) \circ w + (w \circ v^*) \circ u - v^* \circ (u \circ w).$$

For the spin factor we obtain

$$\{u;v;w\} = (u|v)w + (w|v)u - \overline{v}(u|\overline{w}).$$

The Jordan triples arising this way are called of **tube type**. For example, the matrix triple $\mathbf{C}^{r \times s}$ is of tube type if and only if $r = s$.

Let Z be an irreducible hermitian Jordan triple of rank r, and $\ell \leq r$. Define the **Kepler variety**

$$Z_\ell = \{z \in Z : \ rank(z) \leq \ell\}.$$

Its regular part (Kepler manifold)

$$\mathring{Z}_\ell = \{z \in Z_\ell : \ rank(z) = \ell\}$$

is a $K^{\mathbf{C}}$-homogeneous manifold, not compact or symmetric. (Matsuki dual of open G-orbits in flag manifolds).
For the spin factors (r=2) we recover classical Kepler variety. For matrix spaces, we obtain the determinantal varieties.
Theorem: The Kepler variety Z_ℓ is a normal variety having only rational singularities.
Remark: Result classical for spin factor ($r = 2$) and proved by Kaup-Zaitsev for non-exceptional types. Our proof is uniform, based on Kempf's collapsing vector bundle theorem.

An element $u \in Z$ is called a **tripotent** if $\{u; u; u\} = 2u$. For $Z = \mathbf{C}^{r \times s}$ the tripotents satisfy $uu^*u = u$ and are called partial isometries. The set S_ℓ of all tripotents of fixed rank $\ell \leq r$ is a compact K-homogeneous manifold. In particular, S_1 consists of all minimal (rank 1) tripotents. For the spin factor, S_1 is the cosphere bundle. For $Z = \mathbf{C}^{1 \times d}$

$$S_1 = \{u \in Z : \ (u|u) = 1\} = \mathbf{S}^{2d-1}$$

is the full boundary.

Every tripotent u induces a **Peirce decomposition** $Z = Z_u^{(2)} \oplus Z_u^{(1)} \oplus Z_u^{(0)}$, where

$$Z_u^{(j)} = \{z \in Z : \ \{u; u; z\} = 2jz\}$$

is an eigenspace of $D(u, u)$. Moreover, the Peirce 2-space $Z_u^{(2)}$ becomes a Jordan algebra with unit element u and multiplication $\{x; u; y\}$.

The **Jordan Grassmann manifold**

$$M_\ell = S_\ell/\sim = \text{ set of all Peirce 2-spaces } U = Z_u \text{ of rank } \ell$$

is a compact hermitian symmetric space, both a K-orbit and a $K^{\mathbf{C}}$-orbit. (Center of open G-orbit). Define the **tautological vector bundle**

$$\mathcal{T}_\ell := \bigcup_{U \in M_\ell} U \to M_\ell.$$

Fix a base point $c \in S_\ell$, put $V := Z_c^{(2)}$ and let $L := \{k \in K^{\mathbf{C}} : \; kV = V\}$. Then

$$\mathcal{T}_\ell = K^{\mathbf{C}} \times_L V$$

becomes a homogeneous vector bundle, and

$$\mathcal{T}_\ell \to Z_\ell : \; U \ni z \mapsto z \in Z_\ell$$

is a collapsing map. By **Kempf's Theorem**, the range of a collapsing map of homogeneous vector bundles is a normal variety. Let $\mathring{U}$ denote the invertible elements of the Jordan algebra $U = Z_u^{(2)}$. Then

$$\bigcup_{U \in M_\ell} \mathring{U} \to Z_\ell$$

is a birational isomorphism. This proves the normality theorem.

Let (M,ω) be a compact Kähler n-manifold, with a quantizing line bundle $\mathcal{L} \to M$. Consider holomorphic sections $\Gamma(M,\mathcal{L}^\nu)$ of a (high) tensor power $\mathcal{L}^\nu$. **Kodaira (coherent state) embedding**

$$\kappa_\nu : M \to \mathbf{P}(\Gamma(M,\mathcal{L}^\nu)),\ z \mapsto [s_\nu^0(z),..,s_\nu^{d_\nu}(z)]$$

$$\Omega_\nu = \frac{i}{2}\partial\overline{\partial}\log\sum_{i=0}^{d_\nu}|Z^i|^2 \text{ Fubini-Study metric on } \mathbf{P}^{d_\nu}$$

$$\kappa_\nu^*(\Omega_\nu) = \nu\omega + \frac{i}{2}\partial\overline{\partial}\log T_\nu$$

$$T_\nu(z) = \sum_{i=0}^{d_\nu} h_z(s_\nu^i(z), s_\nu^i(z)) \text{ Kempf distortion function}$$

TYZ (Tian-Yau-Zelditch) expansion

$$\left|\frac{T_\nu(z)}{\nu^n} - \sum_{j=0}^{N-1}\frac{a_j(z)}{\nu^j}\right| \leq \frac{c_N}{\nu^N} \text{ as } \nu \to \infty$$

For non-compact Kähler manifolds (such as Z_ℓ) we need infinite-dimensional Hilbert spaces of holomorphic functions.

Every (euclidean) Jordan algebra X has a **symmetric cone**

$$\Omega = \{x^2 : \ x \in X \text{ invertible}\}.$$

In fact, another theorem of Koecher-Vinberg characterizes euclidean Jordan algebras in terms of symmetric (i.e. self-adjoint homogeneous) cones. For matrices $X = \mathcal{H}_r(\mathbf{K}), \ \Omega$ consists of all positive definite matrices. For the real spin factor, we obtain the forward light cone.
For a tripotent $u \in S_\ell$ the Peirce 2-space $Z_u^{(2)}$ is the complexification of a euclidean Jordan algebra with symmetric cone

$$\Omega_u := \{x = \{u; x; u\} \in Z_u^{(2)} : \ x = \{y; u; y\}, \ y \text{ invertible}\}.$$

Proposition: The Kepler manifold has a polar decomposition

$$\mathring{Z}_\ell = \bigcup_{u \in S_\ell} \Omega_u.$$

For $\ell = 1$ this reduces to $\mathring{Z}_1 = \mathbf{R}_+ \cdot S_1$, since $Z_u^{(2)} = \mathbf{C} \cdot u$ and $\Omega_u = \mathbf{R}_+ \cdot u$ is an open half-line.

Choose a base point $c = e_1 + \ldots + e_\ell \in S_\ell$. Every positive density function $\rho(t)$ on symmetric cone Ω_c induces a K-invariant **radial measure**

$$\int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ f(z) = \int_{\Omega_c} dt\ \rho(t) \int_K dk\ f(k\sqrt{t}).$$

For $\ell = 1$ this simplifies to

$$\int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ f(z) = \int_0^\infty dt\ \rho(t) \int_{S_1} du\ f(u\sqrt{t}).$$

Define a **Hilbert space of holomorphic functions**

$$\mathcal{H}_\rho := \{f : \mathring{Z}_\ell \to \mathbf{C} \text{ holomorphic} : \int_{\mathring{Z}_\ell} d\tilde{\rho}(z)\ |f(z)|^2 < +\infty\}.$$

In the trivial case, where $\ell = r$ is maximal and Z is a Jordan algebra, negative powers of the (Jordan) determinant have no holomorphic extension beyond $\mathring{Z}$. Apart from this we have

Extension Theorem: Every holomorphic function on $\mathring{Z}_\ell$ has a unique extension to the closure Z_ℓ

Proof: Singular set

$$Z_\ell \setminus \mathring{Z}_\ell = \bigcup_{j<\ell} Z_j$$

has codimension ≥ 2. By the Normality Theorem (and using Lojasiewicz), the **second Riemann extension theorem** holds for holomorphic functions on $\mathring{Z}_\ell$.

Under the natural action of K the **holomorphic polynomials** $\mathcal{P}(Z)$ on a hermitian Jordan triple Z have a **Hua-Schmid-Kostant decomposition**

$$\mathcal{P}(Z) = \sum_{\boldsymbol{m}\in\mathbf{N}_+^r} \mathcal{P}_{\boldsymbol{m}}(Z)$$

where $\boldsymbol{m} = m_1 \geq m_2 \geq \ldots \geq m_r \geq 0$ ranges over all **integer partitions** (Young diagrams) and each $\mathcal{P}_{\boldsymbol{m}}(Z)$ is an irreducible K-module, whose highest weight vector $N_{\boldsymbol{m}}$ can be uniformly described in terms of Jordan theoretic minors. The **dimension** $d_{\boldsymbol{m}} := \dim \mathcal{P}_{\boldsymbol{m}}(Z)$ is explicitly known. Let

$$(\phi|\psi) := \frac{1}{\pi^d} \int_Z dz\ e^{-(z|z)}\ \overline{\phi(z)}\ \psi(z)$$

denote the **Fischer-Fock inner product**. Then there is an expansion

$$e^{(z|w)} = \sum_{\boldsymbol{m}} E^{\boldsymbol{m}}(z,w)$$

into finite-dimensional kernel functions $E^{\boldsymbol{m}}(z,w)$ of $\mathcal{P}_{\boldsymbol{m}}(Z)$. For $r = 1,\ E^m(z,w) = \frac{(z|w)^m}{m!}$.

Using the characteristic multiplicity a of Z, we define the **Pochhammer symbol**

$$(\nu)_{\boldsymbol{m}} := \prod_{j=1}^{r} (\nu - \frac{a}{2}(j-1))_{m_j}.$$

Since the underlying measure $\tilde{\rho}$ is K-invariant, the extension theorem implies that

$$\mathcal{H}_\rho = \sum \mathcal{P}_{m_1 \ldots, m_\ell, 0 \ldots, 0}(Z)$$

is a Hilbert sum involving only (non-negative) partitions of length $\leq \ell$, since the other components vanish on Z_ℓ.

Theorem: The Hilbert space $\mathcal{H}_\rho$ has the **reproducing kernel**

$$\mathcal{K}^\rho(z,w) := \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{d_{\boldsymbol{m}}}{\int\limits_{\Omega_c} d\rho(t) E^{\boldsymbol{m}}(t,e)} \; E^{\boldsymbol{m}}(z,w)$$

$$= \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\int\limits_{\Omega_c} d\rho(t) N_{\boldsymbol{m}}(t)} \; \frac{d_{\boldsymbol{m}}}{d^c_{\boldsymbol{m}}} \; E^{\boldsymbol{m}}(z,w).$$

Every euclidean Jordan algebra X has a **Jordan algebra determinant** $N(x)$ which is a r-homogeneous polynomial defined via Cramer's rule

$$x^{-1} = \frac{\nabla_x N}{N(x)}.$$

For real or complex matrices, this is the usual determinant. For even anti-symmetric matrices, it is the Pfaffian. In the rank 2 case, $N(x_0, x) = x_0^2 - (x|x)$ is the Lorentz metric.
Let $\partial_N = N(\frac{\partial}{\partial x})$ be the associated constant coefficient differential operator on X. Then $N(x)\partial_N$ is a kind of Euler operator, of order r.
Applying these concepts to the Jordan algebra $Z_c^{(2)}$ of rank ℓ, we define a **universal differential operator**

$$\mathcal{D}_\ell := N^{\frac{a}{2}(\ell-r)}\partial_N^b\ N^{\frac{a}{2}(r-\ell-1)+b+1}\ \left(\partial_N\ N^{\frac{a}{2}}\ \partial_N^{a-1}\right)^{r-\ell}\ N^{\frac{a}{2}(r-\ell+1)-1}$$

of order $\ell((r-\ell)a+b)$ on the symmetric cone Ω_c. For $\ell = 1$, $N(t) = t$ and $\mathcal{D}_1$ becomes a polynomial differential operator of order $(r-1)a+b$ on the half-line.

Our main result is that the reproducing kernel $\mathcal{K}_\rho(z,w)$ can be expressed in closed form, by

$$\mathcal{K}_\rho(\sqrt{t},\sqrt{t}) = (\mathcal{D}_\ell F_\rho)(t)$$

for all $t \in \Omega_c$, where F_ρ is a special function of **hypergeometric type.** Since hypergeometric functions have good asymptotic expansions (quite difficult for $\ell > 1$), this will lead to the desired TYZ-expansions. Generalizing the 1-dimensional case, we define the **hypergeometric series** of type (p,q)

$$_pF_q\binom{\alpha_1,\ldots,\alpha_p}{\beta_1,\ldots,\beta_q}(z,w) := \sum_{\boldsymbol{m}} \frac{(\alpha_1)_{\boldsymbol{m}}\ldots(\alpha_p)_{\boldsymbol{m}}}{(\beta_1)_{\boldsymbol{m}}\ldots(\beta_q)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z,w),$$

using the Fischer-Fock reproducing kernels $E^{\boldsymbol{m}}(z,w)$. For $z,w \in Z_\ell$, only partitions of length $\le \ell$ occur.

Consider first a Fock space situation. For $\alpha > 0$ consider the **pluri-subharmonic function** $\phi = (z|z)^\alpha$ on Z_ℓ. Let $\omega := \partial\overline{\partial}\phi$ denote the associated Kähler form. Then we have the polar decomposition

$$\int_{\mathring{Z}_\ell} \frac{|\omega^n|}{n!}(z)\ e^{-\nu\phi(z)} f(z) = \int_{\Omega_c} dt\ N_c(t)^{a(r-\ell)+b}\ (t|c)^{n(\alpha-1)}\ e^{-\nu(t|c)^\alpha},$$

where $N_c(t)$ is the rank ℓ determinant function on Ω_c. Let $\alpha = 1$.
Theorem: The Hilbert space $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has the reproducing kernel

$$\mathcal{K}_\nu(t, e) = \mathcal{D}_\ell\ {}_1F_1\binom{d_\ell/\ell}{n/\ell}(\nu t)$$

for the confluent hypergeometric function

$${}_1F_1\binom{\sigma}{\tau}(z, w) = \sum_{\boldsymbol{m}} \frac{(\sigma)_{\boldsymbol{m}}}{(\tau)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z, w).$$

In order to look at a Bergman type Hilbert space, let

$$(u|v) = \frac{1}{p} tr_Z \ D(u,v)$$

be the normalized K-invariant inner product, where p is the so-called genus of D. Define **Bergman operators**

$$B(u,v)z = z - 2\{u;v;z\} + \{u;\{v;z;v\};u\}$$

for $u,v,z \in Z$. Then the G-invariant **Bergman metric** is given by

$$(u|v)_z = (B(z,z)^{-1}u|v).$$

For the matrix case $Z = \mathbf{C}^{r\times s}$, we have

$$B(u,v)z = z - uv^*z - zv^*u + u(vz^*v)^*u = (1-uv^*)z(1-v^*u).$$

and $p = r + s$. Therefore

$$(u|v) = \frac{1}{r+s} tr_Z \ D(u,v) = trace \ uv^*$$

and the Bergman metric at $z \in D$ is

$$(u,v)_z = trace(1-zz^*)^{-1} \ u(1-z^*z)^{-1} \ v^*.$$

In terms of the Bergman operator, the **Bergman kernel function** $K : D \times \overline{D} \to \mathbf{C}$ of D can be expressed as

$$K(z,w) = \det\, B(z,w)^{-1}.$$

It is a fundamental fact that there exists a sesqui-polynomial called the **Jordan triple determinant** $\Delta : Z \times \overline{Z} \to \mathbf{C}$ such that the Bergman kernel function has the form

$$K(z,w) = \det\ B(z,w)^{-1} = \Delta(z,w)^{-p}$$

for all $z, w \in D$. In the matrix case $Z = \mathbf{C}^{r\times s}$ we have

$$\det\ B(z,w) = \det(1_r - zw^*)^{r+s}.$$

It follows that

$$\Delta(z,w) = \det(1 - zw^*).$$

The **Kepler ball** is the intersection

$$\{z \in \mathring{Z}_\ell : \ \|z\|_\infty < 1\} = \mathring{Z}_\ell \cap D$$

of $\mathring{Z}_\ell$ with the bounded symmetric domain $D := \{z \in Z : \ \|z\|_\infty < 1\}$. Consider the pluri-subharmonic function

$$\phi(z) = \log \Delta(z,z)^{-p} = \log \det B(z,z)^{-1}$$

on the Kepler ball, given by the Bergman kernel. As before, the associated radial measure $e^{-\nu\phi}|\omega^n|/n!$ has a nice polar decomposition, this time involving the unit interval $\Omega_c \cap (c - \Omega_c)$.
Theorem: For the Kepler ball, $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has the reproducing kernel

$$\mathcal{K}_\nu(t,e) = \mathcal{D}_\ell \ {}_2F_1\binom{d_\ell/\ell;\nu}{n_\ell/\ell}(t)$$

involving a Gauss hypergeometric function ${}_2F_1$ on Ω_c.

Pattern

Fock space on $Z = \mathbf{C}^d,\ \mathcal{K}_\nu = e^{(z|w)} = {}_0F_0(z,w)$

Bergman space on D, $\mathcal{K}_\nu = \Delta(z,w)^{-\nu} = {}_1F_0(z,w)$ Faraut-Koranyi formula

Kepler manifold $\mathring{Z}_\ell,\ \mathcal{K}_\nu = \mathcal{D}_\ell\ {}_1F_1$

Kepler ball $D \cap \mathring{Z}_\ell,\ \mathcal{K}_\nu = \mathcal{D}_\ell\ {}_2F_1$

Asymptotic expansion: $\alpha = 1, \ \ell$ **arbitrary**

Theorem: Consider the Fock-type situation, $\alpha = 1, \ \ell$ arbitrary. There exist polynomials $p_j(x)$, with $p_0 = const$, such that

$$\frac{1}{\nu^n}\mathcal{K}_\nu(\sqrt{x},\sqrt{x}) \approx \ \frac{e^{\nu \ tr(x)}}{N_c(x)^{n/\ell}} \sum_{j=0}^{n} \frac{p_j(x)}{\nu^j},$$

as $|x| \to +\infty$, uniformly for x in compact subsets of Ω_c.

Proof: For $Re(\mu) > \frac{d}{r} - 1$ and $Re(\gamma) > \frac{d}{r} - 1$, there is an asymptotic expansion

$${}_1F_1\binom{\mu}{\gamma}(z) \approx \frac{\Gamma_r(\gamma)}{\Gamma_r(\mu)} \frac{e^{tr(z)}}{N(z)^{\gamma-\mu}} \ {}_2F_0\left(\frac{d}{r} - \mu; \gamma - \mu\right)(z^{-1})$$

as $|z| \to +\infty$ while $\frac{z}{|z|}$ stays in a compact subset of Ω_c.

This expansion can be differentiated termwise any number of times.

Remark: Similar expansion for Kepler ball, based on asymptotics of ${}_2F_1$.

Asymptotic expansion: $\ell = 1,\ \alpha$ arbitrary
Theorem: Radial measure $d\rho(t) = \alpha e^{-\nu t^\alpha} t^{\alpha(p-1)-1}\, dt$ on half-line. Then, as $\nu \to +\infty$

$$\frac{e^{-\nu(z|z)^\alpha}}{\nu^{p-1}} \mathcal{K}_\nu(z,z) = \sum_{j=0}^{p-2} \frac{b_j}{\nu^j (z|z)^{j\alpha}} + O(e^{-\eta\nu(z|z)^\alpha}),\ \ \eta > 0$$

Proof: The measure $d\rho$ has moments

$$\alpha \int_0^\infty dt\ e^{-\nu t^\alpha}\ \ t^{\alpha(p-1)+m-1} = \frac{\Gamma(p-1+\frac{m}{\alpha})}{\nu^{p-1+\frac{m}{\alpha}}}.$$

Therefore $\mathcal{K}_\nu = \mathcal{D}_1 F_\rho$ with

$$F_\rho(t) = \nu^{p-1} \sum_{m=0}^\infty \frac{(\nu^{1/\alpha} t)^m}{\Gamma(p-1+\frac{m}{\alpha})} = \nu^{p-1} E_{1/\alpha,p-1}(\nu^{1/\alpha} t),$$

for the **Mittag-Leffler function**

$$E_{A,B}(t) = \sum_{m=0}^\infty \frac{t^m}{\Gamma(Am+B)}.$$

As $t \to +\infty$, it is known that

$$E_{A,B}(t) = \frac{1}{A} t^{(1-B)/A} e^{t^{1/A}} + O(e^{(1-\eta)t^{1/A}})$$

with some $\eta > 0$. This remains valid for derivatives of any order. For any polynomial P in one variable

$$\partial_t^k t^{\gamma\alpha+c} P(t^\alpha) e^{t^\alpha} = t^{\gamma\alpha+c-k} Q(t^\alpha) e^{t^\alpha}$$

with another polynomial Q, where $\deg Q = \deg P + k$. It follows that

$$(tD_j + D_j t) t^{\gamma\alpha} P(t^\alpha) e^{t^\alpha} = t^{\gamma\alpha} Q(t^\alpha) e^{t^\alpha},$$

with $\deg Q = \deg P + a$. Hence for $\mathcal{D} = \grave{D} \prod_{j=2}^{r} (tD_j + D_j t)$

$$\grave{D} \prod_{j=2}^{r} (tD_j + D_j t) t^{\gamma\alpha} e^{t^\alpha} = t^{\gamma\alpha} Q(t^\alpha) e^{t^\alpha},$$

where Q has degree $(r-1)a + b = p - 2$ and leading coefficient $2^{r-1}\alpha^{p-2}$.

Invariant volume forms and measures

$\mathring{Z}_\ell$ is homogeneous under $K^{\mathbf{C}}$ or a suitable transitive subgroup. When does an invariant holomorphic volume form or invariant measure exist? Only for domains of tube type $b = 0$.

- **Theorem:** Let Z be of tube type and $0 < \ell < r$. Then there exists an invariant holomorphic n-form Ω on $\mathring{Z}_\ell$ if and only if $p - a\ell = 2 + a(r - \ell - 1)$ is even.
- Among all tube type domains, $p - a\ell$ is odd only for the symmetric matrices $Z = \mathbf{C}^{r\times r}_{sym}$ with $r - \ell$ even.
- **Theorem:** An invariant measure μ on $\mathring{Z}_\ell$ exists in all cases (for $b = 0$), and has polar decomposition

$$\int_{\mathring{Z}_\ell} d\mu(z)\ f(z) = const \int_{\Omega_c} \frac{dt}{N_c(t)^{d_\ell/\ell}}\ N_c(t)^{ar/2} \int_K dk\ f(k\sqrt{t}).$$

Special case, $r = 2, \ell = 1$

Englis et al, Gramchev-Loi: Lie ball $r = 2$

$$T^*(\mathbf{S}^n) \setminus \{0\} = \{(x,\xi) : \|x\| = 1,\ (x|\xi) = 0,\ \xi \neq 0\}$$

$\mathbf{C}^{n+1}$ hermitian Jordan triple, rank 2

$\{uv^*w\} = (u|v)w + (w|v)u + (u|\overline{w})\overline{v}$ Jordan triple product

$N(z) = (z|\overline{z})$ Jordan algebra determinant

$T^*(\mathbf{S}^n) \setminus \{0\} = \{z \in \mathbf{C}^{n+1} : N(z) = 0\}$ rank 1 elements

$$\frac{T_\nu(z)}{\nu^n} = 1 + \frac{(n-1)(n-2)}{2|\nu z|} + \sum_{j=2}^{n-2} \frac{2a_j}{|\nu z|^j} + R_\nu(|z|)$$

exponentially small error term

Theorem:

For $t \in \Omega_c$ define the function

$$F_\rho(t) = \sum_{\boldsymbol{m} \in \mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\int\limits_{\Omega_c} d\rho \; N_{\boldsymbol{m}}} \; E^{\boldsymbol{m}}(t, c).$$

Every euclidean Jordan algebra X has a **positive cone**

$$\Omega = \{x^2 : \; x \in X \setminus \{0\}\},$$

corresponding to the positive definite matrices $\mathcal{H}_r^+(\mathbf{K})$. The **Jordan algebra determinant** $N : X \to \mathbf{R}$ is a r-homogeneous polynomial defined via Cramer's rule

$$x^{-1} = \frac{\nabla_x N}{N(x)}.$$

For the rank 2 case,

$$N \begin{pmatrix} \alpha & b \\ b^* & \delta \end{pmatrix} = \alpha\delta - (b|b)$$

is the Lorentz metric on $\boldsymbol{\rho}^{1,\,1+a}$. The complexification $Z := X^{\mathbf{C}}$ becomes a (complex) Jordan triple (of tube type)

$$\{uv^*w\} = (u \circ v^*) \circ w + (w \circ v^*) \circ u - v^* \circ (u \circ w).$$

For the spin factor we obtain

$$\phi = (z|z)^\alpha,\ \alpha > 0 \text{ pluri-subharmonic function on } Z_\ell$$

$$\omega := \partial\overline{\partial}\phi \text{ Kähler form}$$

$$\int_{\mathring{Z}_\ell} \frac{|\omega^n|}{n!}(z)\ e^{-\nu\phi(z)} f(z) = \int_{\Omega_c} dt\ N_c(t)^{a(r-\ell)+b}\ (t|c)^{n(\alpha-1)}\ e^{-\nu(t|c)^\alpha}.$$

Theorem: $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has kernel $\mathcal{K}_\nu(\sqrt{t},\sqrt{t}) = \mathcal{D}_\ell F_\nu(t)$

$$F_\nu(t) = const \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\Gamma_{\Omega_c}(\boldsymbol{m}+\frac{n}{\ell})}\ \frac{\Gamma(n+|\boldsymbol{m}|)}{\Gamma(n+\frac{|\boldsymbol{m}|}{\alpha})}\ E^{\boldsymbol{m}}(\nu^{1/\alpha}t,c)$$

confluent hypergeometric function

$${}_1F_1(z,w) = \sum_{\boldsymbol{m}} \frac{(\sigma)_{\boldsymbol{m}}}{(\tau)_{\boldsymbol{m}}} E^{\boldsymbol{m}}(z,w)$$

For $\alpha = 1$

$$F_\nu(t) = {}_1F_1\binom{d_\ell/\ell}{n/\ell}(\nu t)$$

Theorem: $\mathcal{H}_\nu = H^2(e^{-\nu\phi}|\omega^n|/n!)$ has kernel $\mathcal{K}_\nu(\sqrt{t},\sqrt{t}) = \mathcal{D}_\ell F_\nu(t)$

$$F_\nu(t) = const \sum_{\boldsymbol{m}\in\mathbf{N}_+^\ell} \frac{(d_\ell/\ell)_{\boldsymbol{m}}}{\Gamma_{\Omega_c}(\boldsymbol{m}+\frac{n}{\ell})}\ \frac{\Gamma(n+|\boldsymbol{m}|)}{\Gamma(n+\frac{|\boldsymbol{m}|}{\alpha})}\ E^{\boldsymbol{m}}(\nu^{1/\alpha}t,c)$$

coherent state embedding

$$\sigma_\nu : U \to \mathcal{L}^\nu \setminus 0 \text{ local trivializing section}$$

$$\nu\omega(z) = -\frac{i}{2}\partial\overline{\partial}\log h_\nu(\sigma_\nu(z), \sigma_\nu(z)) \text{ Ricci form}$$

$$\mathbf{P}(\mathcal{O}(M, \mathcal{L}^\nu)),\ (s_\nu^0, .., s_\nu^{d_\nu}) \text{ ONB of sections}$$

$$\mathbf{P}^{d_\nu} : [Z^0, .., Z^{d_\nu}] \text{ homogeneous coordinates}$$

junk

be the corresponding with closure

$$\overline{D} = \{z \in \mathbf{C}^{r\times s} : \ I - zz^* \geq 0\}.$$

In this case we take

$$K = U(r) \times U(s) \ni (u, v)$$

and

$$S = \{z \in \mathbf{C}^{r\times s} : \ zz^* = I\}$$

consists of all (injective) isometries. In the special case $r = s$ we obtain $S = U(r)$. In this case the domain D is said to be of 'tube type'.
compact linear Lie group

$$d := \dim Z = r\left(1 + \frac{a}{2}(r-1) + b\right), \ r = rank(Z)$$

characteristic multiplicities: a, b
tube type: $b = 0$

Hankel operators

$$d\rho(t) = N_c(t)^{a(r-\ell)+b}\,dt,\ d\tilde{\rho}\ \text{Lebesgue surface measure on}\ \mathring{Z}_\ell$$

Kepler ball

$$D_\ell := \{z \in Z_\ell :\ \|z\| < 1\}$$

bounded symmetric domain, spectral norm

$$H_{\overline{g}} f := (I - P)(\overline{g} f)\ \text{Hankel operator, antiholomorphic symbol function}$$

Theorem: For $p \geq 1$ the following are equivalent

- There exists nonconstant $g \in H^2_\rho$ with $H_{\overline{g}} \in \mathcal{S}^p$ (Schatten class)
- There exists a nonzero partition $\boldsymbol{m} \in \mathbf{N}^r_+$ such that $H_{\overline{g}} \in \mathcal{S}^p$ for all $g \in \mathcal{P}_{\boldsymbol{m}}$
- $\ell = 1$ and $p > 2 \dim Z_1$
- $H_{\overline{g}} \in \mathcal{S}^p$ for any polynomial g

Analogous to BBCZ-Theorem.

$$Z_1 = \{z \in Z : N(z) = 0\} \text{ algebraic variety}$$

$$\rho(z) = (z|z)^\alpha - 1$$

$$S_1 = \{z \in Z : N(z) = 0,\ \rho(z) = 0\} \text{ strongly pseudoconvex boundary}$$

$$\dim S_1 = 1 + a(r-1) + b$$

$$\vartheta = \frac{\partial\rho - \overline{\partial}\rho}{2i} = \alpha(z|z)^{\alpha-1}\frac{(dz|z) - (z|dz)}{2i} \text{ contact 1-form}$$

$$du = \vartheta \wedge (d\vartheta)^{n-1}\ K\text{-invariant measure } (2n-1\text{-form}) \text{ on } S_1$$

$$\pi : \mathbf{R} \times S_1 \to S_1 \text{ symplectic cone, } s = e^t$$

$$\omega = d(e^t \cdot \pi^*\vartheta) = e^t(dt \wedge \pi^*\vartheta + \pi^* d\vartheta) \text{ symplectic form}$$

$$\omega^n = e^{nt} dt \wedge \pi^*\vartheta \wedge (\pi^* d\vartheta)^{n-1} = e^{nt} dt \wedge du = s^{n-1} ds \wedge du$$

$$\int_{Z_1} \omega^n(\zeta) f(\zeta) = \int_0^\infty ds\, s^{n-1} \int_{S_1} du\, f(u\sqrt{s})$$

$$z = su,\ u = \frac{x + i\xi}{2} \in S_1 \text{ rank1 tripotents}$$

$$(u|u) = 1 \Rightarrow (z|z) = s^2$$

$$\omega(z) = \frac{i}{2}\partial\overline{\partial}\sqrt{(z|z)} = \frac{i}{4}(z|z)^{-3/2}((z|z)(dz|dz) - \frac{(dz|z) \wedge (z|dz)}{2})$$

$$(dz|dz) = dz^i \wedge d\overline{z}^i,\ (dz|z) = \overline{z}^i\, dz^i,\ (z|dz) = z^j\, d\overline{z}^j$$

$$(dz|dz) \wedge \alpha = 0$$

$$\omega^n = \frac{((dz|z) \wedge (z|dz))^n}{(z|z)^{3n/2}}$$

$$K^1(z, w) = \sum_{m=0}^{\infty} \frac{\Gamma(m + a)}{\Gamma(2m + a)} E_m(z, w)$$

$$\frac{K^\nu(z, z)}{\nu^{d-1}} = \sum_{m=0}^{\infty} \frac{\Gamma(m + a)}{\Gamma(2m + a)} E_j(\nu z, \nu w)$$

Jordan Kepler manifolds

Z irreducible hermitian Jordan triple, rank r

$\{uv^*w\}$ Jordan triple product

$N_k(z)$ Jordan minors

$S_1^{\mathbf{C}} = \{z \in Z : N_k(z) = 0 \, \forall k > 1\}$ rank 1 elements

S_1 rank 1 tripotents

$z = su,\ s > 0,\ u \in S_1$ polar decomposition

$$(u|u) = 1 \Rightarrow (z|z) = s^2$$

$$\int_{S_1^{\mathbf{C}}} d\Omega(\zeta)\, e^{-\nu\sqrt{(\zeta|\zeta)}} f(\zeta) = \int_0^\infty ds\, e^{-\nu s} s^{n-1} \int_K dk\, f(ks)$$

$$(\psi|\psi) = \int_{S_1^{\mathbf{C}}} d\Omega(\zeta)\, e^{-\nu\sqrt{(\zeta|\zeta)}} |\psi(\zeta)|^2 = \int_0^\infty ds\, e^{-\nu s} s^{n-1} \int_{S_1} du |\psi(su)|^2$$

$$\psi \in \mathcal{P}_m(Z) \Rightarrow (\psi|\psi) = \int\limits_0^\infty ds\, e^{-\nu s} s^{n-1} s^{2m} \int\limits_{S_1} du\, |\psi(u)|^2$$

$$\int\limits_K dk\, E_m(z, ks) E_n(ks, w) = \frac{\delta_{m,n}}{(d/r)_m} \frac{(a/2)_m}{(ra/2)_m} \phi_1^m(s^2)\, E_m(z, w)$$

$$= \frac{\delta_{m,n}}{(d/r)_m} \frac{(a/2)_m}{(ra/2)_m} s^{2m}\, E_m(z, w)$$

$$\int\limits_{S_1^{\mathbf{C}}} d\Omega(\zeta)\, e^{-\nu\sqrt{(\zeta|\zeta)}}\, E_m(z, \zeta) E_n(\zeta, w) = \int\limits_0^\infty ds\, e^{-\nu s} s^{n-1} \int\limits_K dk\, E_m(z, ks) E_n(ks, w$$

$$= \frac{\delta_{m,n}}{(d/r)_m} \frac{(a/2)_m}{(ra/2)_m}\, E_m(z, w) \int\limits_0^\infty ds\, e^{-\nu s} s^{2m+n-1}$$

$$= 2 \frac{\delta_{m,n}}{(d/r)_m} \frac{(a/2)_m}{(ra/2)_m} \frac{\Gamma(2m+n)}{\nu^{2m+n}}\, E_m(z, w)$$

$$K^\nu(z,w) = \sum_{m=0}^\infty \frac{\nu^{2m+n}(d/r)_m(ra/2)_m}{\Gamma(2m+n)(a/2)_m} E_m(z,w)$$

$$= \nu^n \sum_{m=0}^\infty \frac{(d/r)_m(ra/2)_m}{\Gamma(2m+n)(a/2)_m} E_m(\nu z,\nu w)$$

$$= \nu^n \sum_{m=0}^\infty \frac{\Gamma(m+1+\frac{a}{2}(r-1))\Gamma(m+\frac{a}{2}r)}{\Gamma(2m+1+a(r-1))\Gamma(m+\frac{a}{2})} E_m(\nu z,\nu w)$$

$$= \nu^n \sum_{m=0}^\infty \frac{\Gamma(m+\frac{a}{2}r)}{\Gamma(m+\frac{1}{2}+\frac{a}{2}(r-1))\Gamma(m+\frac{a}{2})} E_m(2\nu z,2\nu w)$$

$r=2,\ d=n+1,\ a=n-1$

$$K^\nu(z,w) = \nu^n \sum_{m=0}^\infty \frac{\Gamma(m+a)}{\Gamma(m+\frac{a+1}{2})\Gamma(m+\frac{a}{2})} E_m(2\nu z,2\nu w)$$

$$= \nu^n \sum_{m=0}^\infty \frac{\Gamma(m+a)}{\Gamma(2m+a)} E_m(2\nu z,2\nu w)$$

Complex Geometry of Bounded Symmetric Domains In the following consider a complex vector space $Z = \mathbf{C}^d$, endowed with a norm $\|\cdot\|$, and let

$$D = \{z \in Z : \ \|z\| < 1\}$$

be its unit ball. In general, the boundary

$$\partial D = \{z \in Z : \ \|z\| = 1\}$$

will not be smooth. Consider the set

$$S = \partial_{\text{ex}} D \subset \partial D$$

of all extreme points and let

$$U(Z) = \{g \in GL(Z) : \ \|gz\| = \|z\|\}$$

be the linear isometry group. Then $U(Z) \subset GL(Z)$ is a closed subgroup, therefore $U(Z)$ is a compact Lie group. Clearly, the natural $U(Z)$-action satisfies

$$U(Z) \cdot D = D, \quad U(Z) \cdot \partial D = \partial D, \quad U(Z) \cdot S = S.$$

Example: For $Z = \mathbf{C}^d$, define the scalar product

$$(z|w) = zw^* = \sum_i z_i \, \overline{w}_i$$

and let $\|z\| = (z|z)^{1/2}$ be the Hilbert norm. Then

$$D = \{z \in \mathbf{C}^d : \ (z|z) < 1\}$$

is the Hilbert unit ball. In this case $K = U(d)$ is the unitary group and

$$S = \partial D = \mathbf{S}^{2d-1}$$

is the odd sphere. For $d = 1$ we have the unit disk $D = \{z \in \mathbf{C}; \ |z| < 1\}$ and the unit circle $S = \{z \in \mathbf{C} : \ |z| = 1\} = \mathbf{T}$.

We now introduce holomorphic functions. By definition, a (possibly vector-valued) function $f : D \to \mathbf{C}^m$ is **holomorphic** if it has a power series expansion

$$f(z) = \sum_{\alpha \in \mathbf{N}^d} c_\alpha \; z_1^{\alpha_1} \cdots z_d^{\alpha_d}$$

(compact convergence) with coefficients $c_\alpha \in \mathbf{C}^m$. By collecting monomials of equal (total) degree, we obtain the expansion

$$f(z) = \sum_{k \ge 0} \sum_{|\alpha| = k} c_\alpha \, z_1^{\alpha_1} \cdots z_d^{\alpha_d} = \sum_{k \ge 0} p_k(z)$$

into a series of k-homogeneous polynomials. Let $\mathcal{P}_k(Z, \mathbf{C}^m)$ be the space of all k-homogeneous polynomials $p_k : Z \to \mathbf{C}^m$ satisfying $p_k(\lambda z) = \lambda^k \, p_k(z) \quad \forall \; \lambda \in \mathbf{C}$. Since D is bounded and circular around $0 \in D$, the space

$$\mathcal{P}(Z, \mathbf{C}^m) = \sum_{k \ge 0} \mathcal{P}_k(Z, \mathbf{C}^m)$$

of all polynomials is dense in the space of all holomorphic functions $\mathcal{O}(D, \mathbf{C}^m)$ under compact convergence.

In the rank 1 case of row vectors $Z = \mathbf{C}^d = \mathbf{C}^{1\times d}$ D is the Hilbert unit ball, and we obtain as a special case

$$G = SU(1,d) \supset K = U(d).$$

For dimension $d = 1$, we have $Z = \mathbf{C}$ and D is the unit disk. In this case, we may identify

$$G = SU(1,1) \supset K = U(1).$$

Via a Cayley transformation

$$z \mapsto w = \frac{z+i}{1-iz}$$

onto the upper half-plane $\{w \in \mathbf{C} : \ Im(w) > 0\}$ we obtain another realization

$$G \approx SL(2,\mathbf{R}) \supset K = SO(2).$$

This is fundamental to applications in number theory since the 'Shilov' boundary $\mathbf{R} \cup \infty$ of the upper half-plane is a completion of $\mathbf{Q}$.

It is a fundamental fact that hermitian symmetric domains have an algebraic description in terms of the so-called **Jordan algebras** and **Jordan triples**. In order to explain this connection, consider the Lie algebra $\mathfrak{g}$ of the real Lie group $G = Aut(D)$, which can be realized as follows: Consider a 1-parameter group $t \mapsto g_t \in G$, and define, for $f \in \mathcal{O}(D, \mathbf{C}^m)$, the **infinitesimal generator**

$$\frac{\partial f(g_t(z))}{\partial t} = (Xf)(z) = h(z)\ f'(z) = (h(z)\ \frac{\partial}{\partial z})f(z),$$

with the derivative operator acting from the right. Here $h : D \to Z = \mathbf{C}^d$ is a **holomorphic vector field**, with commutator bracket

$$[h\ \frac{\partial}{\partial z},\ k\ \frac{\partial}{\partial z}] = (h\ \frac{\partial k}{\partial z} - k\ \frac{\partial h}{\partial z})\ \frac{\partial}{\partial z}.$$

The **adjoint action** of G on $\mathfrak{g}$ is defined by

$$Ad(g)(h\ \frac{\partial}{\partial z}) = h(g(z))\ g'(z)\ \frac{\partial}{\partial z},$$

Let $s_0(z) = -z$ be the **symmetry** at the origin $0 \in D$. We obtain the **Cartan decomposition** $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the $\pm$-eigenspaces

$$\mathfrak{k} = \{X \in \mathfrak{g} : \ Ad(s_0)X = X\}, \quad \mathfrak{p} = \{X \in \mathfrak{g} : \ Ad(s_0)X = -X\}$$

Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, we obtain a **Lie triple system**

$$[\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k} \supset [\mathfrak{p},\mathfrak{p}], \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p} \supset [\mathfrak{p},\mathfrak{k}].$$

The **Jordan triple product** $Z \times \overline{Z} \times Z \to Z$, denoted by $u, v, w \mapsto \{uv^*w\} = \{wv^*u\}$, satisfies

$$\mathfrak{p} = \{(v - \{zv^*z\}) \ \frac{\partial}{\partial z} : \ v \in Z\},$$

$$\mathfrak{k} = \{h(z) \ \frac{\partial}{\partial z} linear : \ h\{zv^*z\} = 2\{hz, v^*, z\} + \{z(hv)^*z\}\}.$$

Define $(u \,\square\, v^*)z = \{uv^*z\}$. Since $[[\mathfrak{p},\mathfrak{p}],\mathfrak{p}] \subset [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, we obtain the **Jordan triple identity**

$$[u \,\square\, v^*, \ x \,\square\, y^*] = \{uv^*x\} \,\square\, y^* - x \,\square\, \{yu^*v\}^*.$$

In the matrix case $Z = \mathbf{C}^{r\times s}$, consider a 1-parameter group

$$\begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}(z) = (a_t z + b_t)(c_t z + d_t)^{-1} \in U(r,s)$$

of Moebius transformations. Its differential is computed as

$$\frac{d}{dt}\Big|_{t=0}(a_t z + b_t)(c_t z + d_t)^{-1} = \dot{a}z + \dot{b} - z\dot{c}z - z\dot{d}$$

with $\begin{pmatrix} \dot{a} & \dot{b} \\ \dot{c} & \dot{d} \end{pmatrix} \in \mathfrak{u}(r,s)$. The eigenspace decomposition is given by

$$\mathfrak{k} = \left\{(\dot{a}z - z\dot{d})\frac{\partial}{\partial z} : \ \dot{a} \in \mathfrak{u}(r),\ \dot{d} \in \mathfrak{u}(s)\right\}, \quad \mathfrak{p} = \left\{(\dot{b} - z\dot{b}^* z)\frac{\partial}{\partial z} : \ \dot{b} \in \mathbf{C}^{r\times s}\right\}.$$

Therefore we obtain

$$\{zv^*z\} = zv^*z, \quad \{uv^*w\} = \frac{1}{2}(uv^*w + wv^*u), \quad u\Box v^* = \frac{1}{2}(L_{uv^*} + R_{v^*u}).$$

Let Z be an irreducible Jordan triple of rank r, with multiplicities a, b. Then $d/r = 1 + \frac{a}{2}(r-1) + b$ and $p = 2 + a(r-1) + b$. The **tube type** case corresponds to $b = 0$ or, equivalently, to $\frac{p}{2} = \frac{d}{r}$. Fix a parameter $\nu > p - 1 = 1 + a(r-1) + b$. Let Γ_Ω denote the **Koecher-Gindikin Gamma**-function of the symmetric cone Ω. Then

$$dM_\nu(z) = \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - d/r)}\ \Delta(z,z)^{\nu-p}\ \frac{dz}{\pi^d}$$

is a probability measure on D. The **weighted Bergman space** of holomorphic functions on D is defined as

$$H^2_\nu(D) = \{\psi \in \mathcal{O}(D, \mathbf{C}) : \int_D dM_\nu(z)\ |\psi(z)|^2 < \infty\}.$$

Define an irreducible projective unitary representation of G on $H^2_\nu(D)$

$$(g^{-\nu}\psi)(z) = \det\ g'(z)^{\nu/p}\ \psi(g(z)).$$

These representations give the **scalar holomorphic discrete series** of G.

$H^2_\nu(D)$ has the reproducing kernel

$$K_\nu(z,w) = \Delta(z,w)^{-\nu} = \det\ B(z,w)^{-\nu/p}.$$

Thus the **orthogonal projection** $P_\nu : L^2(D, dM_\nu) \to H^2_\nu(D)$ is

$$(P_\nu\psi)(z) = \int_D dM_\nu(w)\ \Delta(z,w)^{-\nu}\ \psi(w).$$

The inner product is

$$(\psi_1|\psi_2)_\nu = \frac{\Gamma_\Omega(\nu)}{\Gamma_\Omega(\nu - d/r)} \int_D \frac{dz}{\pi^d}\ \Delta(z,z)^{\nu-p}\ \overline{\psi_1(z)}\ \psi_2(z).$$

For $\nu = p$, $dM_p(z)$ is a multiple of the Lebesgue measure dz and

$$H^2_p(D) = \{\psi \in \mathcal{O}(D, \mathbf{C}) : \int_D dz\ |\psi(z)|^2 < \infty\}$$

is the standard (unweighted) Bergman space.

Consider the algebra $\mathcal{P}(Z)$ of all polynomials $p : Z \to \mathbf{C}$ and the natural action $(k^{-1}\ p)(z) := p(kz)$ of K. The **Hua-Schmid-Kostant decomposition** asserts that

$$\mathcal{P}(Z) = \sum_{m} \mathcal{P}_{\boldsymbol{m}}(Z)$$

is a direct sum of inequivalent irreducible K-modules, labelled by all integer **partitions**

$$\boldsymbol{m} = (m_1, \ldots, m_r) \in \mathbf{N}^r$$

with $m_1 \geq \ldots \geq m_r \geq 0$. Moreover, the highest weight vector in $\mathcal{P}_{\boldsymbol{m}}(Z)$ has the form

$$N_{\boldsymbol{m}}(z) = N_1(z)^{m_1 - m_2}\ N_2(z)^{m_2 - m_3} \cdots N_r(z)^{m_r},$$

where N_k is the Jordan theoretic minor of rank k. For the simplest partition $\boldsymbol{m} = (1, 0, \ldots, 0)$, $\mathcal{P}_{\boldsymbol{m}}(Z)$ is the dual space of linear forms on Z. In this case $N_{\boldsymbol{m}}(z) = (e_1|z)$.

The **Wallach set**

$$W(D) = \{\nu \in \mathbf{C} : \ (\Delta(z_i, z_j)^{-\nu})_{1 \le i,j \le n} \gg 0 \ \forall \ z_1, \ldots, z_n \in D\}$$

consists of all parameters ν, beyond $\nu > p - 1$, such that the kernel $K_\nu(z, w)$ is still positive. The Hilbert space completion of $\mathcal{P}(Z)$ is the **Segal-Bargmann-Fock space**

$$H^2(Z) = \{\psi \in \mathcal{O}(Z) : \ \int_Z \frac{dz}{\pi^d} \ e^{-(z|z)} \, |\psi(z)|^2 < \infty\}$$

of entire functions, with reproducing kernel expanded in a series

$$e^{(z|w)} = \sum_{\boldsymbol{m}} K_{\boldsymbol{m}}(z, w),$$

where $K_{\boldsymbol{m}}(z, w)$ denotes the reproducing kernel of the finite-dimensional space $\mathcal{P}_{\boldsymbol{m}}(Z) \subset H^2(Z)$.

The **Faraut-Koranyi binomial formula** is

$$\Delta(z,w)^{-\nu} = \sum_{\boldsymbol{m}} (\nu)_{\boldsymbol{m}} \, K_{\boldsymbol{m}}(z,w),$$

where

$$(\nu)_{\boldsymbol{m}} = \frac{\Gamma_\Omega(\nu + \boldsymbol{m})}{\Gamma_\Omega(\nu)} = \prod_{j=1}^{r} (\nu - \frac{a}{2}(j-1))_{m_j}$$

denotes the Pochhammer symbol. Here we use Gindikin's formula

$$\Gamma_\Omega(s) = (2\pi)^{(r-d)/2} \prod_{j=1}^{r} \Gamma(s_j - \frac{a}{2}(j-1)).$$

By checking positivity of the coefficients $(\nu)_{\boldsymbol{m}}$ for all partitions $\boldsymbol{m}$ it follows that

$$W(D) = \{\frac{a}{2}\ell : \ 0 \le \ell < r\} \cup \{\nu > \frac{a}{2}(r-1)\}$$

consists of r discrete equidistant points and a continuous part.

The **Shilov boundary** $S = K/L$, endowed with the K-invariant probability measure, corresponds to the Wallach parameter $\nu = \frac{d}{r} = 1 + \frac{a}{2}(r-1) + b$, giving rise to the **Hardy space**

$$H^2(S) = \{\psi \in L^2(S) : \ \psi \text{ holomorphic on } D\}.$$

In the matrix case $Z = \mathbf{C}^{r \times s}$ we have $a = 2$, and hence $d/r = s, \ p = r + s$. The associated **Szegö projection** $P : L^2(S) \to H^2(S)$ has the form

$$(P\psi)(z) = \int_S du \ \Delta(z,u)^{-d/r} \ \psi(u).$$

Now let $f \in \mathcal{C}(S)$ be a continuous 'symbol' function. The **Hardy-Toeplitz operator** T_f is the bounded operator

$$T_f \psi = P(f\psi)$$

acting on $H^2(S)$. Here $f \in L^\infty(S)$ would be possible, but the fine structure of Toeplitz operators is known only for continuous symbols.

Define the **Toeplitz C^*-algebra**

$$\mathcal{T}(S) = C^*(T_f : \ f \in \mathcal{C}(S))$$

on the Hardy space. A similar concept applies to the weighted Bergman spaces and even to the discrete Wallach points (G. Zhang).

Theorem The Toeplitz C^*-algebra contains all compact operators $\mathcal{K}(H^2(S))$ and is therefore irreducible.

Proof Consider the group C^*-algebra $C^*(K) = \widehat{L_1(K)}$ and the commutative C^*-algebra $\mathcal{C}(K)$. Then we may form the **cocrossed-product** $C^*(K) \otimes_\delta \mathcal{C}(K)$ for a suitable coaction δ. By Katayama duality there is a natural C^*-isomorphism

$$C^*(K) \otimes_\delta \mathcal{C}(K) \approx \mathcal{K}(L^2(K)).$$

Now embed $H^2(S) \subset L^2(S) \subset L^2(K)$ by averaging over the subgroup $L \subset K$. In this way $\mathcal{T}(S)$ contains a full corner $\pi(C^*(K) \otimes_\delta \mathcal{C}(K))\pi$.

For the unit disk $D \subset \mathbf{C}$ and its Hardy space

$$H^2(\mathbf{T}) = \{\sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

the Toeplitz C^*-algebra $\mathcal{T}(\mathbf{T})$ has the irreducible representation on $H^2(\mathbf{T})$ as in the general case, but also the **characters**

$$\sigma_c : \mathcal{T}(\mathbf{T}) \to \mathbf{C}, \quad \sigma_c(T_f) = f(c)$$

for any $c \in \mathbf{T}$. One shows that these are all irreducible representations so that there is a C^*-algebra extension

$$\mathcal{K}(H^2(\mathbf{T})) \subset \mathcal{T}(\mathbf{T}) \xrightarrow{\sigma} \mathcal{C}(\mathbf{T})$$

given by the symbol map $\sigma(A)(c) := \sigma_c(A)$ for all $A \in \mathcal{T}(\mathbf{T})$.

For a Jordan triple Z an element $c \in Z$ is called a **tripotent** if $\{cc^*c\} = c$. For matrices these are the partial isometries. Every tripotent c induces a **Peirce decomposition**

$$Z = Z_c^2 \oplus Z_c^1 \oplus Z_c^0.$$

into eigenspaces of $c \,\square\, c^*$. Each Peirce subspace is again a Jordan triple, the Peirce 2-space is even a Jordan algebra with unit element c. One can show that

$$\|u + w\| = \max(\|u\|, \|w\|)$$

whenever $u \in Z_c^2,\ w \in Z_c^0$. Therefore, if $c \neq 0$, the set

$$c + D_c^0 := \{c + w : \ w \in D \cap Z_c^0\}$$

belongs to the boundary ∂D. These are the so-called **boundary components** of D. One can show that they are pairwise disjoint and cover the whole boundary. Thus we have a disjoint union

$$\partial D = \dot{\bigcup}(c + D_c^0).$$

oveer all non-zero tripotent c. The tripotent $c = 0$ gives the interior D.

The **rank** of a tripotent c is the rank of Z_c^2. For $0 \le j \le r$ the set

$$S_j := \{c \in Z \text{ tripotent}: \text{ rank } c = j\}$$

is a connected compact K-homogeneous manifold. Equivalently, these are the connected components of the set of all tripotents. Moreover

$$\partial_j D = \dot{\bigcup}_{c \in S_j} (c + D_c^0)$$

is a G-orbit contained in the boundary. Thus

$$\partial D = \dot{\bigcup}_{1 \le j \le r} \partial_j D$$

is a disjoint union of G-orbits, called the **boundary strata**. If $j = r$ then

$$\partial_r D = S_r = S$$

is the Shilov boundary (realized as a G-orbit $S = G/P$) and the corresponding boundary components are the extreme points, since $Z_c^0 = (0)$. For rank $r = 1$ (unit ball) the boundary is smooth.

For each tripotent c consider the Peirce 0-space Z_c^0. Then

$$S_c^0 := S_j \cap Z_c^0 = S(D_c^0)$$

is the Shilov boundary of the symmetric domain D_c^0 of rank $r - j$. Note that $c + S_c^0 \subset S_r = S$.

Theorem There exists an irreducible representation σ_c of $\mathcal{T}(S)$ on the 'little' Hardy space $H^2(S_c^0)$ satisfying

$$\sigma_c(T_S(f)) = T_{S_c^0}(f_c)$$

for all $f \in \mathcal{C}(S)$ where

$$f_c(s) := f(c + s)$$

is the boundary evaluation of f. Moreover, these representations are pairwise inequivalent.

Proof Either using the representations of co-crossed products, or by a limit procedure (peaking functions).

For maximal tripotents $c \in S_r$ we obtain the characters above, since $H^2(pt) = \mathbf{C}$.

The main theorem asserts that the irreducible representations σ_c, for any tripotent c, exhaust the full spectrum of $\mathcal{T}(S)$. The main work is to show that

$$\bigcap_{c \neq 0} Ker(\sigma_c) = \mathcal{K}(H^2(S))$$

consists of all compact operators. The original proof was based on the Choi-Effros theorem on completely positive cross-sections. A proof using Katayama duality is also possible. As a consequence there is a **composition series**

$$\mathcal{K}(H^2(S)) = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \ldots \subset \mathcal{I}_{r-1} \subset \mathcal{I}_r = \mathcal{T}(S)$$

of C^*-ideals $\mathcal{I}_j$ such that the subquotients

$$\mathcal{I}_j / \mathcal{I}_{j-1} = \mathcal{C}(S_j) \otimes \mathcal{K}_j$$

are essentially commutative. Here $\mathcal{I}_{-1} := (0)$. In general, $\mathcal{I}_{r-1}$ is the commutator ideal, so $\mathcal{K}_r = \mathbf{C}$. The first non-trivial ideal $\mathcal{I}_1$ satisfies

$$\mathcal{I}_1 / \mathcal{K} \approx \mathcal{C}(S_1) \otimes \mathcal{K}_1.$$

For the **unit ball** $D \subset \mathbf{C}^d$ we have $S = \partial D = \mathbf{S}^{2d-1}$ and

$$\mathcal{T}(\mathbf{S}^{2d-1})/\mathcal{K} \approx \mathcal{C}(\mathbf{S}^{2d-1})$$

is Coburn's Theorem. For the **Lie ball** in $\mathbf{C}^d$, of rank $r = 2$

$$S_2 = \mathbf{T}\ \mathbf{S}^{d-1}$$

is the so-called **Lie sphere**, and

$$S_1 = \mathbf{S}^*(\mathbf{S}^{d-1}) = \{\frac{x + i\xi}{2} : \ \|x\| = \|\xi\| = 1,\ (x|\xi) = 0\}$$

is the **cosphere bundle**. For the Lie ball $\mathcal{K}_1 = \mathcal{K}(H^2(\mathbf{T}))$ and

$$H^2(S) = H^2(\mathbf{T}) \otimes L^2(\mathbf{S}^{d-1})$$

via the identification $(\phi \otimes \psi)(\vartheta x) = \phi(\vartheta)\ \psi(x)$ for $\vartheta \in \mathbf{T},\ x \in \mathbf{S}^{d-1}$. Then

$$\mathcal{I}_1 = \mathcal{T}(\mathbf{T}) \otimes CZ(\mathbf{S}^{d-1})$$

for the **Calderon-Zygmund operators** on the sphere $\mathbf{S}^{d-1}$. In general, $\mathcal{I}_1$ consists of (certain) generalized Toeplitz operators on S_1 in the sense of Boutet-de-Monvel/Guillemin.

$T \in \mathcal{K}(H)$ compact operator, positive part $|T| = (T^*T)^{1/2}$

$$s_j(T) = \lambda_j(|T|),\ s_1 \geq s_2 \geq s_3 \geq \dots$$

singular values, counted with multiplicity. Put $S_k(T) = \sum_{j=1}^{k} s_j(T)$.

$$\sum_j^{\infty} s_j(T) = \lim_k S_k(T) < \infty : \quad T \in \mathcal{L}^1(H) \text{ trace class}$$

$$T \geq 0 : \quad tr\ T = \sum_j s_j(T) = \lim_k S_k(T) \text{ trace}$$

$$s_j(T) = O(\frac{1}{j}),\ \sup_k \frac{S_k(T)}{\log(1+k)} < +\infty : \quad T \in \mathcal{L}^{1,\infty} \text{ Dixmier class}$$

$$T \geq 0 \text{ measurable} : \quad tr_\omega T = \lim_k \frac{S_k(T)}{\log(1+k)} \text{ Dixmier trace}$$

independent of ultrafilter ω. Extend by linearity $tr_\omega : \mathcal{L}^{1,\infty} \to \mathbf{C}$.

M compact Riemannian manifold, dimension n, cosphere bundle

$$\mathbf{S}^*(M) = \{(x, \xi) \in T^*(M) : |\xi| = 1\}$$

$$Q \in \Psi_{-n}(M)$$

pseudo-differential operator of order $-n$, symbol $\sigma_{-n}(Q)$. Then Connes has shown

$$tr_\omega(Q) = \frac{1}{n!(2\pi)^n} \int_{\mathbf{S}^*(M)} dVol\ \sigma_{-n}(Q)$$

independent of choice of Riemannian metric, since $\sigma_{-n}(Q)$ is homogeneous

$\Omega \subset \mathbf{C}^n$ strongly pseudoconvex domain, $\partial\Omega$ smooth boundary, $L^2(\Omega)$ for Lebesgue measure

$$H^2(\Omega) = \{\phi \in L^2(\Omega) : \phi \text{ holomorphic}\} \text{ Bergman space}$$

$$P : L^2(\Omega) \to H^2(\Omega) \text{ orthogonal projection}$$

$$T_f\,\phi = P(f\phi) \text{ Bergman-Toeplitz operator, } f \in \mathcal{C}^\infty(\overline{\Omega})$$

$n = 1$, $\Omega = \mathbf{D}$ unit disk, $f, g \in \mathcal{O}(\overline{\mathbf{D}})$ holomorphic

$$tr[T_f^*, T_g] = \int_{\mathbf{D}} dz\,\overline{f'(z)}g'(z)$$

For $n > 1, f_1, g_1, \ldots, f_n, g_n$ holomorphic

$$[T_{f_1}^*, T_{g_1}] \cdots [T_{f_n}^*, T_{g_n}] \notin \mathcal{L}^1(\Omega)$$

Let $\Omega = \mathbf{B}_n \subset \mathbf{C}^n$ unit ball

Helton-Howe $f_1, \ldots, f_{2n} \in \mathcal{C}^\infty(\overline{\mathbf{B}}_n)$, complete anti-symmetrization

$$[T_{f_1}, \ldots, T_{f_{2n}}] \in \mathcal{L}^1(H^2(\mathbf{B}_n))$$

$$tr\,[T_{f_1}, \ldots, T_{f_{2n}}] = \int_{\mathbf{B}_n} df_1 \wedge \ldots \wedge df_{2n}$$

Englis-Guo-Zhang $f_1, g_1, \ldots, f_n, g_n \in \mathcal{C}^\infty(\overline{\mathbf{B}}_n)$

$$[T_{f_1}, T_{g_1}] \cdots [T_{f_n}, T_{g_n}] \in \mathcal{L}^{1,\infty}(\Omega)$$

$$tr_\omega [T_{f_1}, T_{g_1}] \cdots [T_{f_n}, T_{g_n}] = \frac{1}{n!} \int_{\partial \mathbf{B}_n} d\sigma \ \prod_j \{f_j, g_j\}_b$$

boundary Poisson bracket, surface measure $d\sigma$

For symmetric domains, we have an orthogonal decomposition

$$H^2(S) = \sum_{\boldsymbol{m}} H^2_{\boldsymbol{m}}(S)$$

over all partitions λ, and the irreducible K-type

$$H^2_{\boldsymbol{m}}(S) = \{p|_S : p \in \mathcal{P}_{\boldsymbol{m}}(Z)\}$$

has the inner product

$$(p|q)_S = \frac{1}{(d/r)_{\boldsymbol{m}}}(p|q)_Z$$

for all $p, q \in \mathcal{P}_{\boldsymbol{m}}(Z)$. Here the Pochhammer symbol $(\nu)_{\boldsymbol{m}}$ is defined by

$$(\nu)_{\boldsymbol{m}} = \prod_{j=1}^{r}(\nu - \frac{a}{2}(j-1))_{m_j}$$

Let $e_1, \ldots, e_r$ be a frame of minimal tripotents of Z, with joint Peirce decomposition

$$Z = \sum_{0 \le i \le j \le r}^{\oplus} Z_{ij}.$$

Then $\mathcal{P}_{\boldsymbol{m}}(Z)$ has the highest weight vector

$$N_{\boldsymbol{m}}(z) := N_1(z)^{m_1 - m_2} N_2(z)^{m_2 - m_3} \cdots N_r(z)^{m_r}$$

where $N_1, \ldots, N_r$ are the Jordan theoretic minors. For $1 \le i < j \le r$ put

$$a := \dim Z_{ij}, \; b := \dim Z_{0j}$$

Then

$$\frac{d}{r} = 1 + \frac{a}{2}(r-1) + b.$$

The numerical invariant

$$p := 2 + a(r-1) + b$$

is called the genus of Z.

The following is joint work with Kai Wang. For partition $(m, 0 \ldots, 0)$ of length ≤ 1 we have

$$N_{m,0,\ldots,0} = N_1^m(z) = (z|e_1)^m.$$

Let Z_1 be the complex variety of all elements in Z of rank ≤ 1. Then

$$n := \dim_{\mathbf{C}} Z_1 = 1 + a(r-1) + b = p - 1.$$

The subspace

$$\mathcal{P}_1^{\perp}(Z) = \{p \in \mathcal{P}(Z) : p|_{Z_1} = 0\} = \sum_{m_2 > 0} \mathcal{P}_{\boldsymbol{m}}(Z).$$

is an ideal in $\mathcal{P}(Z)$, with quotient module

$$\mathcal{P}_1(Z) = \sum_{m=0}^{\infty} \mathcal{P}_{m,0,\ldots,0}(Z) \approx \mathcal{P}(Z)/\mathcal{P}_1^{\perp}(Z)$$

Asymptotically, we have

$$\dim \mathcal{P}_{m,0}(Z) \sim c \cdot m^n$$

for some constant $c > 0$ independent of m. In fact, for any partition $\boldsymbol{m}$ we have

$$\dim \mathcal{P}_{\boldsymbol{m}}(Z) = \prod_{j=1}^{r} \frac{(1 + b + \frac{a}{2}(r-j))_{m_j}}{(1 + \frac{a}{2}(r-j))_{m_j}} \cdot$$

$$\cdot \prod_{1 \le i < j \le r} \frac{m_i - m_j + \frac{a}{2}(j-i)}{\frac{a}{2}(j-i)} \frac{(\frac{a}{2}(j+1-i))_{m_i - m_j}}{(1 + \frac{a}{2}(j-i-1))_{m_i - m_j}}.$$

For $\boldsymbol{m} = (m, 0, \ldots, 0)$ and m large enough we obtain

$$\dim \mathcal{P}_{m,0}(Z) \sim c \cdot m^{b + a(r-1)} = c \cdot m^n$$

for some constant $c > 0$ independent of m.

Consider the Hilbert sum

$$H_1^2(S)^\perp := \sum_{m_2>0} H^2_{\boldsymbol{m}}(S) = \overline{\mathcal{P}_1^\perp(Z)},$$

$$H_1^2(S) = \overline{\mathcal{P}_1(Z)} = \sum_m H^2_{m,0,\dots,0}(S) = H^2(S)/H_1^2(S)^\perp$$

regarded as a quotient module. One can show that the orthogonal projection $P_1 : H^2(S) \to H_1^2(S)$ belongs to $\mathcal{T}(S)$. Define

$$S_f := P_1\, f\, P_1 = P_1\, T_f\, P_1$$

Theorem For real-analytic polynomials f, g in $z, \overline{z}$ we have

$$[S_f, S_g] \in \mathcal{L}^{n,\infty}$$

It follows that n-fold products satisfy

$$[S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}] \in \mathcal{L}^{1,\infty}.$$

The compact manifold S_1 all tripotents of rank 1 is the boundary of the **strongly pseudoconvex** variety

$$\Omega = D \cap Z_1 = \{z \in Z : rank(z) \le 1, \|z\| < 1\}$$

which has its only singularity at 0. Let

$$H^2(S_1) := \{\phi \in L^2(S_1) : \phi \text{ holomorphic on } \Omega\}$$

be the associated 'rank 1' Hardy space. Let $\pi : L^2(S_1) \to H^2(S_1)$ denote the Cauchy-Szego projection. For $f \in L^\infty(S_1)$ we have the Hardy-Toeplitz operator

$$\tau_f \phi = \pi(f\phi)$$

acting on $\phi \in H^2(S_1)$. Let

$$\mathcal{T}(S_1) = C^*(\tau_f : f \in \mathcal{C}(S_1))$$

denote the Toeplitz C^*-algebra generated by continuous symbol functions.

For $m \geq 0$ put

$$\Lambda_m^2 := \frac{(a/2)_m}{(ra/2)_m}.$$

Then the mapping $U : H_1^2(S) \to H^2(S_1)$, defined by

$$Up := \Lambda_m \cdot p|_{S_1}$$

for $p \in H^2_{m,0,\ldots,0}$, is a unitary isomorphism. For $f(z) = (z|u)$, with restriction $f|_{S_1}$, we have for $p \in \mathcal{P}_m(Z)$

$$US_fU^*p|_{S_1} = \frac{1}{\Lambda_m}U(P_1((z|u)p)) = \frac{\Lambda_{m+1}}{\Lambda_m}P_1(z|u)p)|_{S_1}$$

$$= \frac{\Lambda_{m+1}}{\Lambda_m}(\zeta|u)p|_{S_1} = \frac{\Lambda_{m+1}}{\Lambda_m}\tau_{f|_{S_1}}p|_{S_1}.$$

It follows that

$$US_fU^* - \tau_{f|_{S_1}} = \tau_{f|_{S_1}} \circ \Delta,$$

where the diagonal operator Δ on $H^2(S_1)$ defined by

$$\Delta q_m = (\frac{\Lambda_{m+1}}{\Lambda_m} - 1)q_m.$$

There is a harmonic extension operator

$$\mathcal{P} : \mathcal{C}^\infty(S) \to \mathcal{C}^\infty(D)$$

given by a **Poisson integral**

$$f(z) = \int_S ds\, \mathcal{P}(z,s) f(s) = \int_S ds \left(\frac{\Delta(z,z)^{d/r}}{\Delta(z,s)^{d/r} \Delta(s,z)^{d/r}} \right) f(s).$$

Theorem For real-analytic polynomials $f_1, g_1, \ldots, f_n, g_n$ consider the harmonic extension $\hat{f}_j, \hat{g}_j$ restricted to S_1. Then the Dixmier trace is given by

$$tr_\omega [S_{f_1}, S_{g_1}] \cdots [S_{f_n}, S_{g_n}] = const. \cdot \int_{S_1} \{\hat{f}_1, \hat{g}_1\}_b \cdots \{\hat{f}_n, \hat{g}_n\}_b$$

in terms of the boundary Poisson bracket of S_1.