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Hurwitz action and surface braids

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Today's talk

$S, S' \subset D^4$: surface braids

$$\begin{array}{ccc}
S & \cong & S' \\
\downarrow & \Updownarrow \text{ by Kamada} & \downarrow \\
b_S & \underset{\text{Hurwitz equivalent}}{\sim} & b_{S'} \\
\downarrow & & \downarrow \\
M(b_S) & \underset{\text{Matrix equivalent}}{\sim} & M(b_{S'})
\end{array}$$

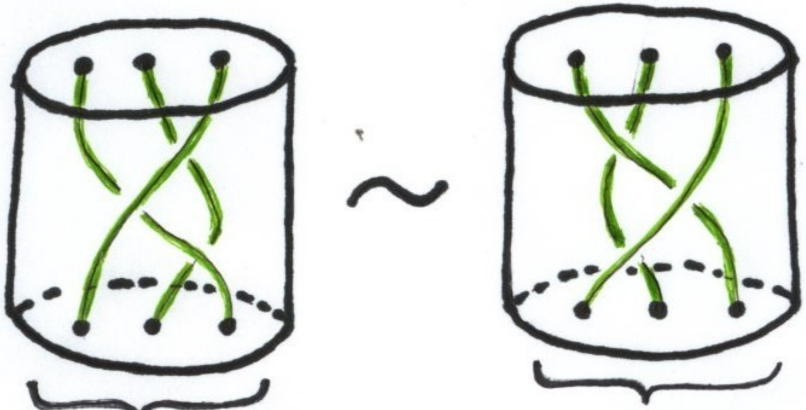
a tuple of simple braids : b_S

$b_{S'}$: a tuple of simple braids

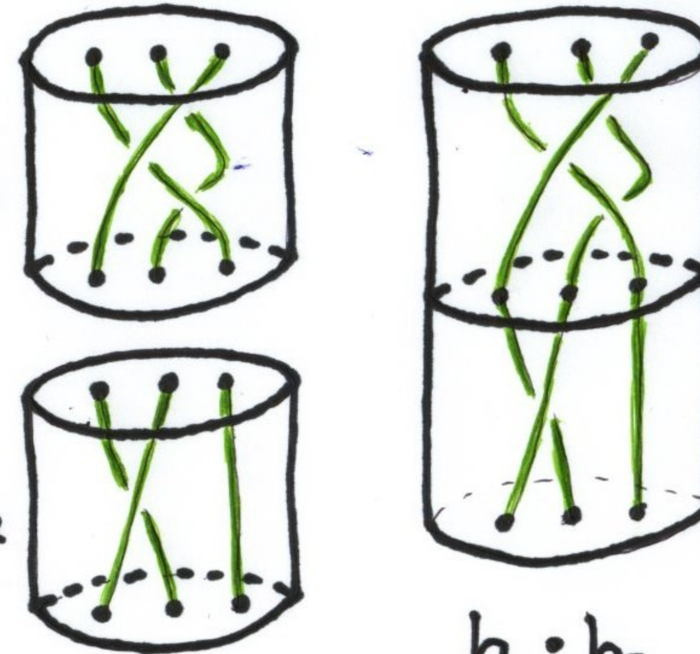
The matrix of b_S : $M(b_S)$

$M(b_{S'})$: The matrix of $b_{S'}$

§1 A braid of degree n



n strings $\qquad$ n strings

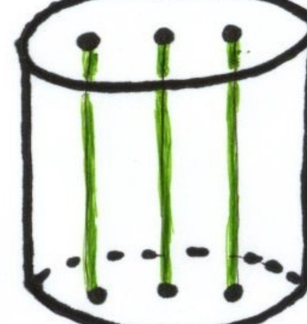


b_1, b_2, $b_1 \cdot b_2$

Braid group of degree n

$$B_n := \{\text{braids of degree } n\}/\sim$$

$$[b_1] \circ [b_2] := [b_1 \cdot b_2]$$



id

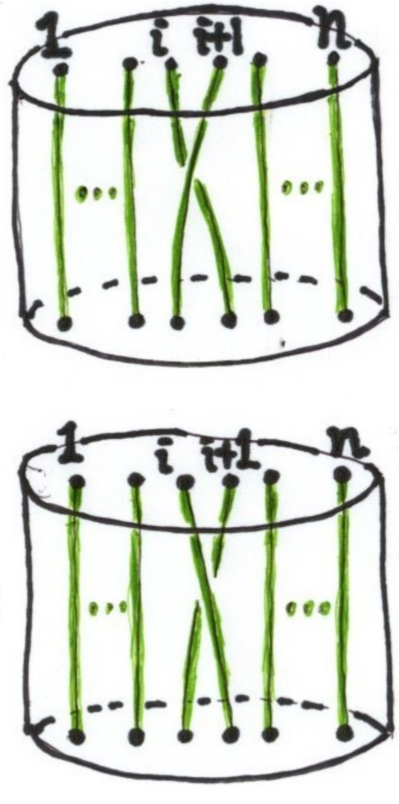
Thm [Artin]

$$B_n \cong \left\langle \sigma_1, \cdots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & (|i-j| \geq 2) \\ \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j & (|i-j| = 1) \end{array} \right\rangle$$

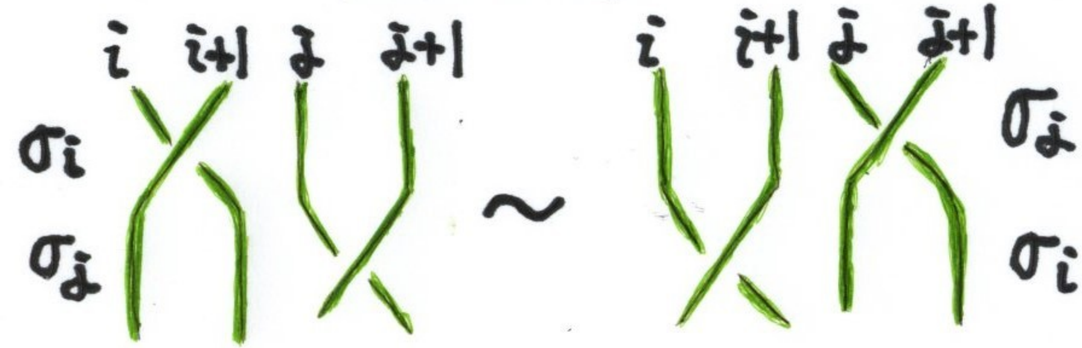
generators

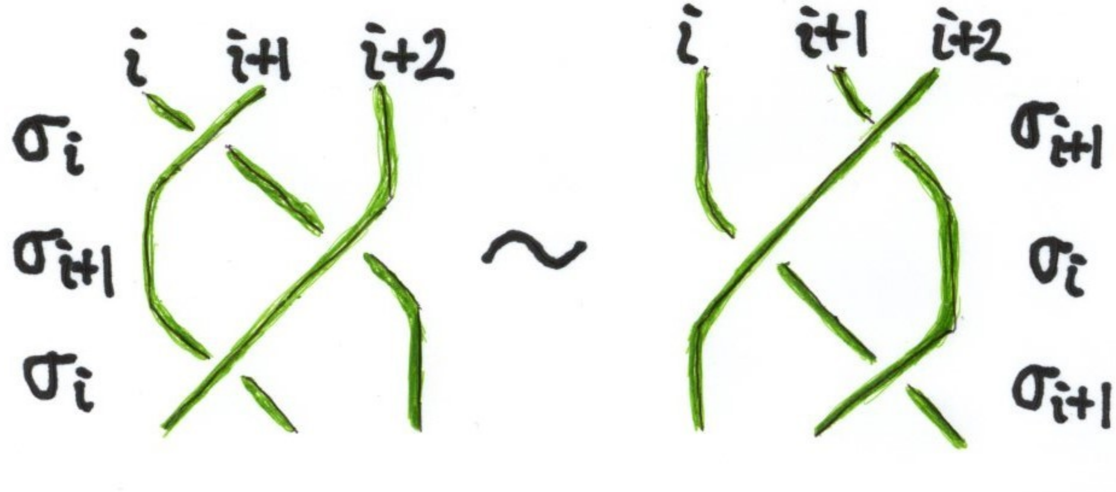
σ_i:

σ_i^{-1}:



relations





G : a group , B_n : braid group of degree n

Hurwitz action $G^n \curvearrowleft B_n$: a right action

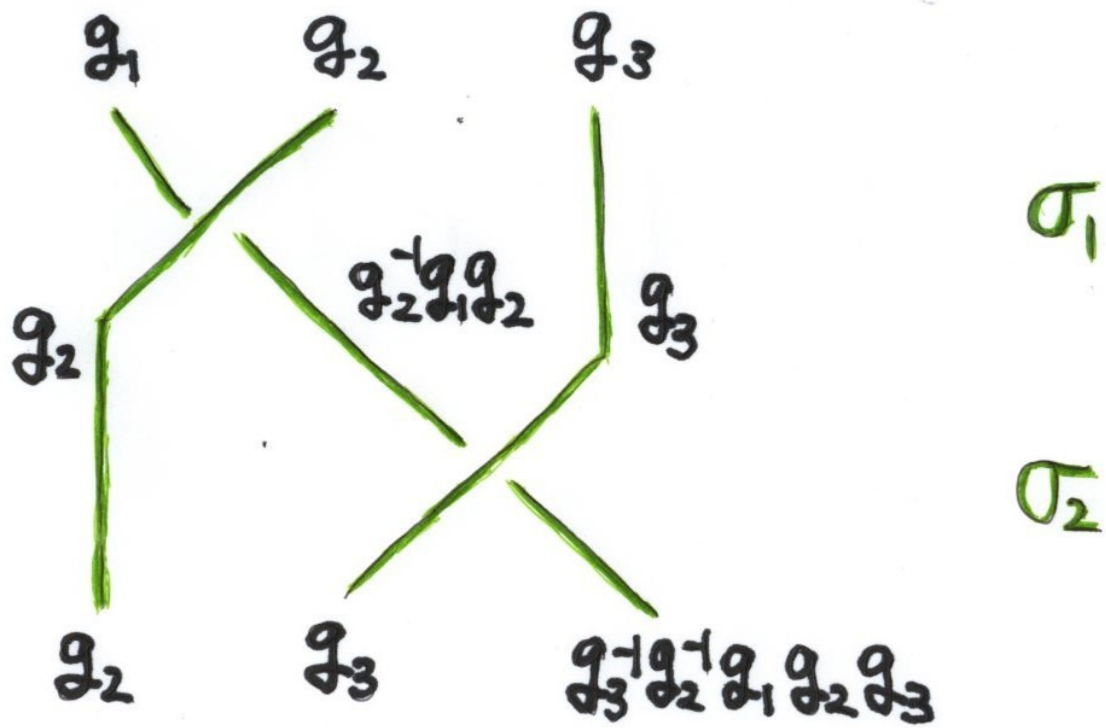
$$(g_1, \cdots, g_n) \cdot \sigma_i \overset{\text{def}}{:=} (g_1, \cdots, g_{i-1}, g_{i+1}, g_{i+1}^{-1} g_i g_{i+1}, g_{i+2}, \cdots, g_n)$$

$\mathfrak{g}, \mathfrak{g}' \in G^n$ are **Hurwitz equivalent** ($\mathfrak{g} \underset{\text{Hur.}}{\sim} \mathfrak{g}'$)

$\overset{\text{def}}{:\Leftrightarrow}$ $\exists \beta \in B_n$ s.t. $\mathfrak{g} \cdot \beta = \mathfrak{g}'$

<u>Ex</u> $(n=3)$ $\quad (g_1, g_2, g_3) \in G^3$

$$\Rightarrow (g_1, g_2, g_3) \cdot \sigma_1 \sigma_2 = (g_2, g_3, g_3^{-1} g_2^{-1} g_1 g_2 g_3)$$

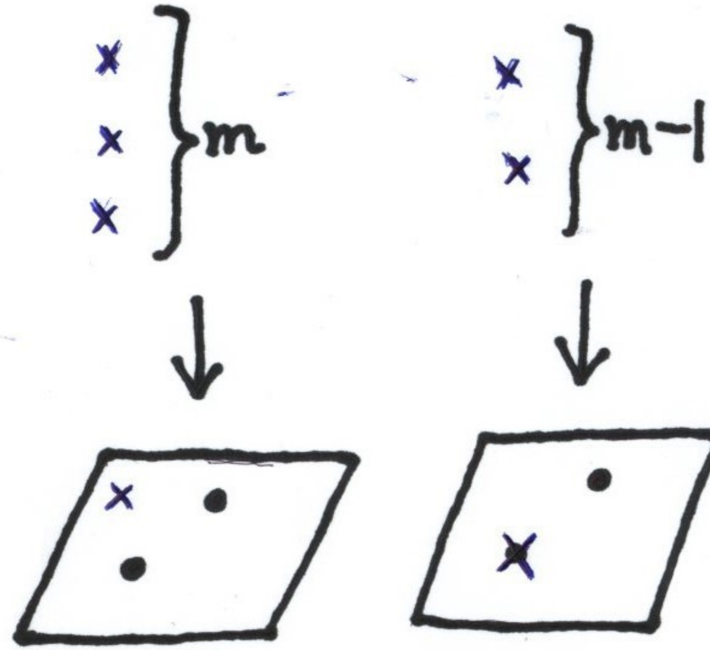


A surface braid of degree m

$$\begin{array}{ccc} D^2\times I^2 & \supset & S \\ {\scriptstyle p}\downarrow & & \downarrow {\scriptstyle p|S} \\ I^2 & & \end{array}$$

S: a simple

$p|S$: branched covering of degree m



regular point

branch point

$\partial S = Q_m \times \partial I^2$

($Q_m \subset \mathrm{Int}\, D^2$: a fixed m points)

Prop $\#\left\{\begin{array}{l}\text{branch pts of}\\ \text{a surface braid}\end{array}\right\}$ is an even.

A complete invariant of surface braids

Let $n\,(\geq 2)$ be an even number.

Theorem [S. Kamada]

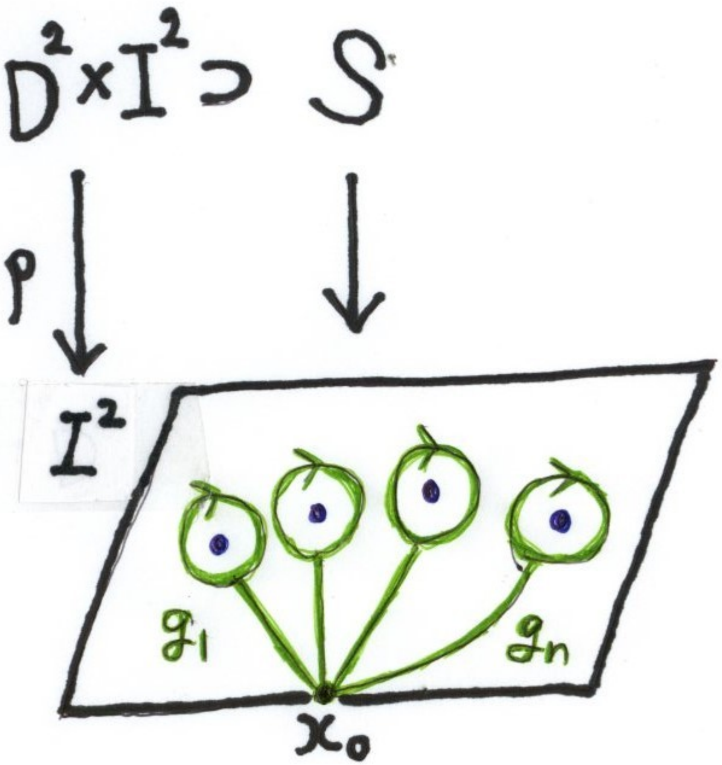
$$\left\{\begin{array}{l}\text{Surface braids of degree } m\\ \text{with } n \text{ branch points}\end{array}\right\}\Big/\sim \quad \overset{\exists\, 1:1}{\longleftrightarrow} \quad P^n(X_m)\Big/\underset{\text{Hur.}}{\sim}$$

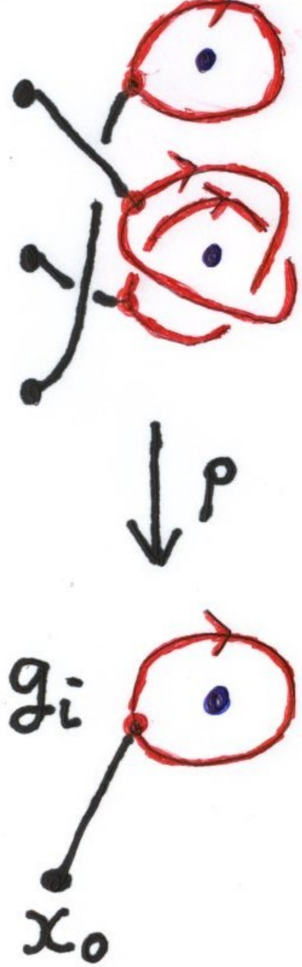
where $X_m := \left\{ \beta^{-1}\sigma_1^{\varepsilon}\beta \in B_m \;\middle|\; \beta \in B_m,\ \varepsilon \in \{\pm 1\} \right\}$

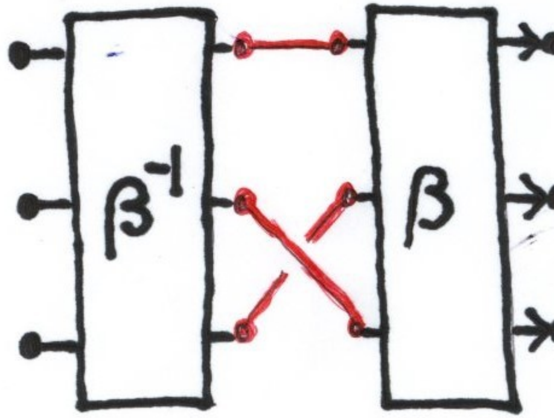
$(= \{\text{Simple braids of degree } m\})$

and $P^n(X_m) := \left\{ (b_1, \cdots, b_n) \in B_m^n \;\middle|\; \begin{array}{l} b_1, \cdots, b_n \in X_m \\ b_1 \cdots b_n = \mathrm{id} \end{array} \right\}$

S: surface braid of degree m with n branch pts







$\mathcal{G} = (g_1, \cdots, g_n)$:

Hurwitz generator system

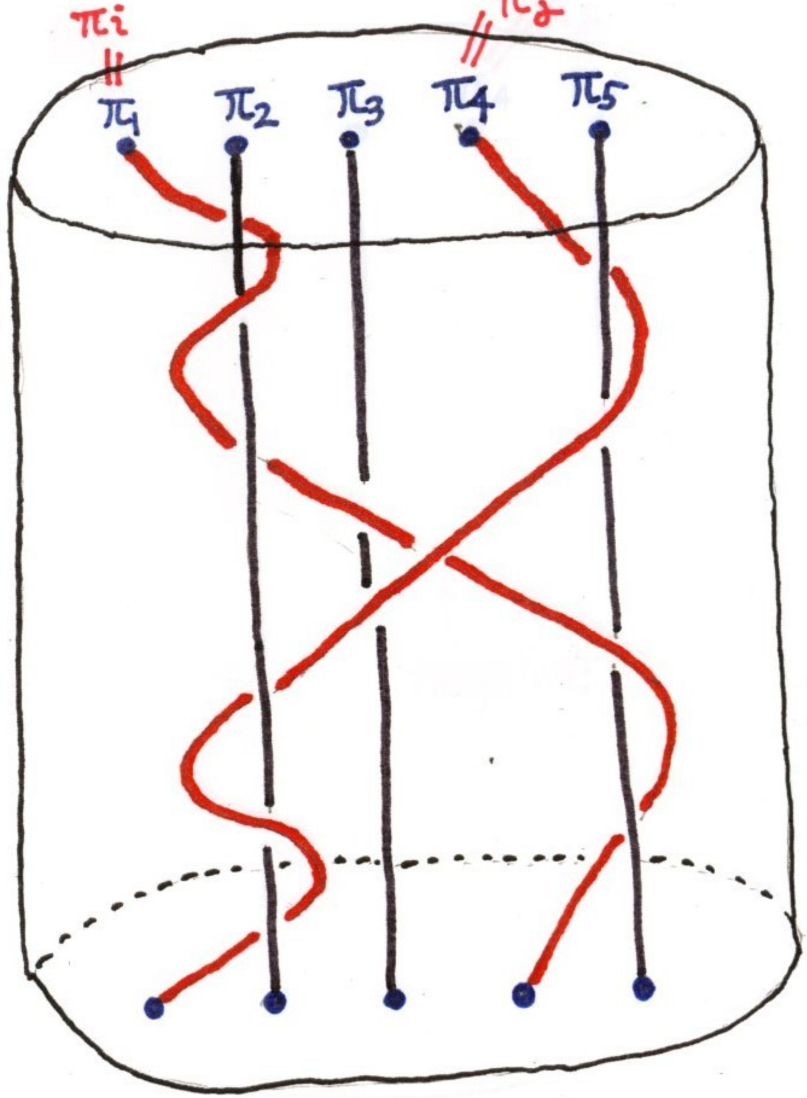
- $b_i \in X_m$

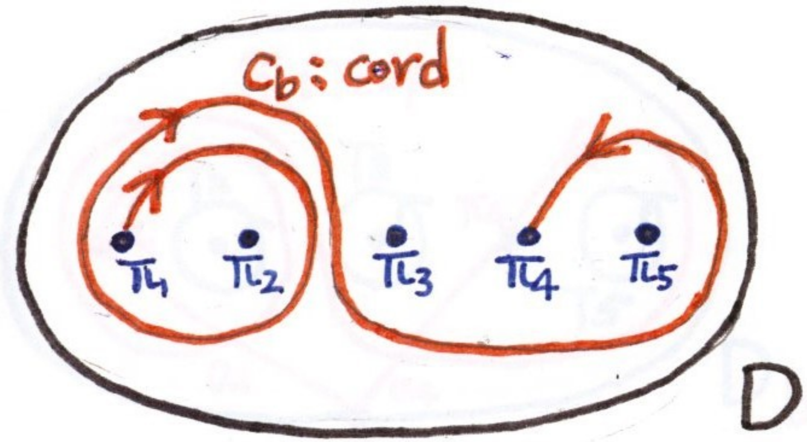
$$\underbrace{(b_1, \cdots, b_n)}$$

braid system of S associated with $\mathcal{G}$

- $b_1 \cdots b_n = \mathrm{id}$

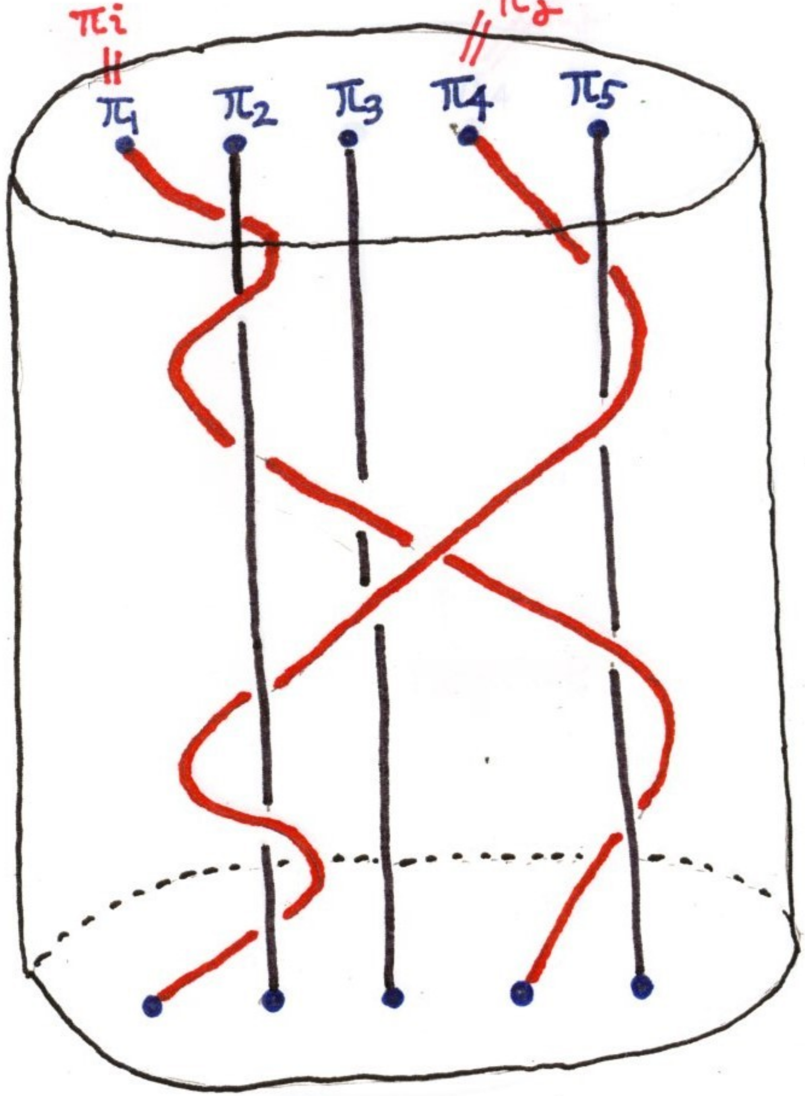
§3 $X_m \ni b \rightsquigarrow$ C_b: a cord of b



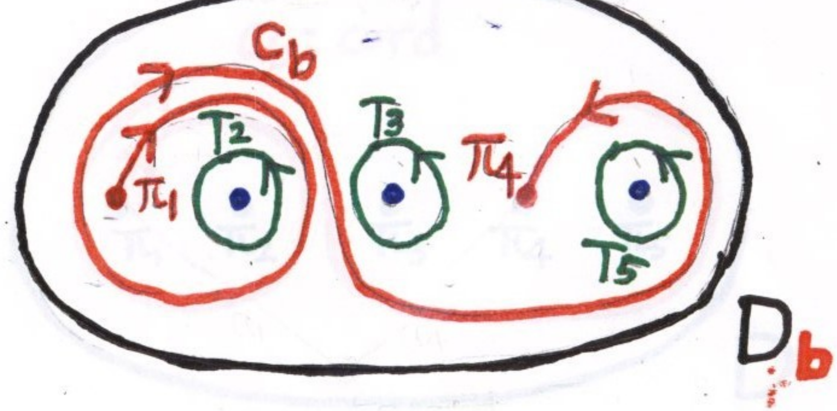


$$b=(\sigma_2\sigma_3^{-1}\sigma_4^{-2}\sigma_1^{-1})^{-1}\sigma_1\,\sigma_2\sigma_3^{-1}\sigma_4^{-2}\sigma_1^{-1}\in X_5$$

§3 $X_m \ni b \rightsquigarrow C_b$: a cord of b



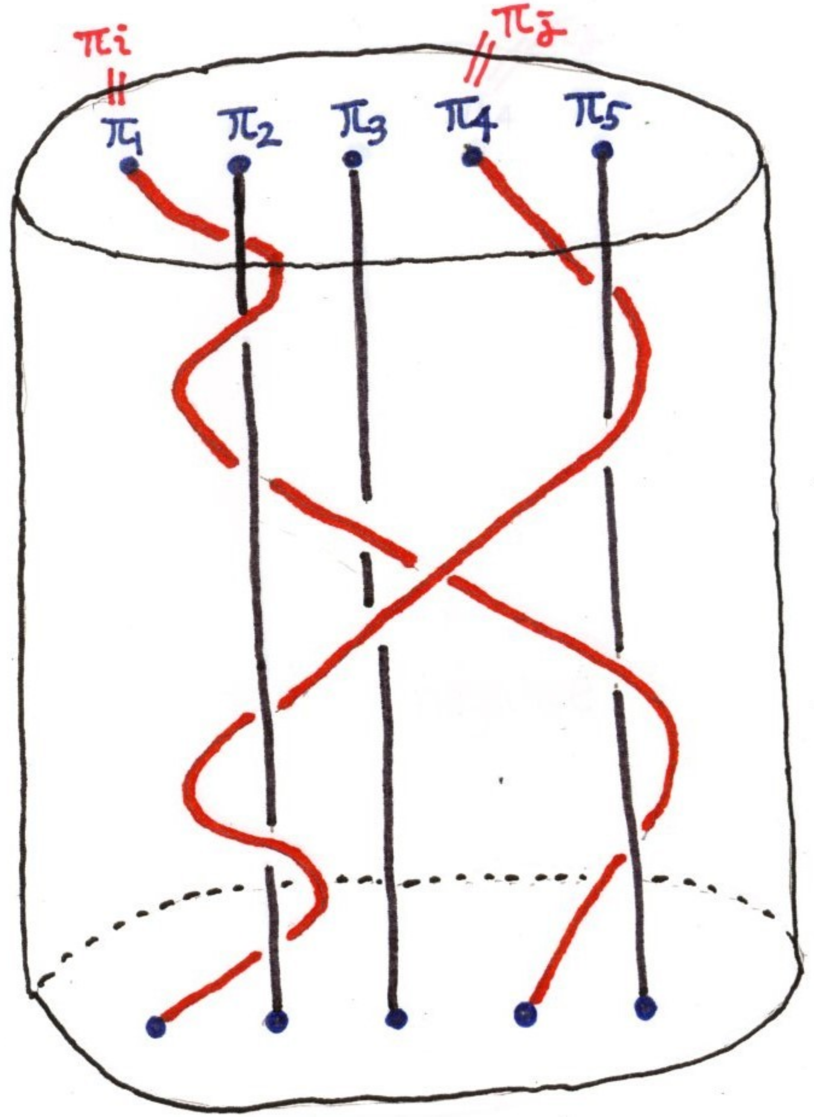
$$b = (\sigma_2 \sigma_3^{-1} \sigma_4^{-2} \sigma_1^{-1})^{-1} \sigma_1 \, \sigma_2 \sigma_3^{-1} \sigma_4^{-2} \sigma_1^{-1} \in X_5$$



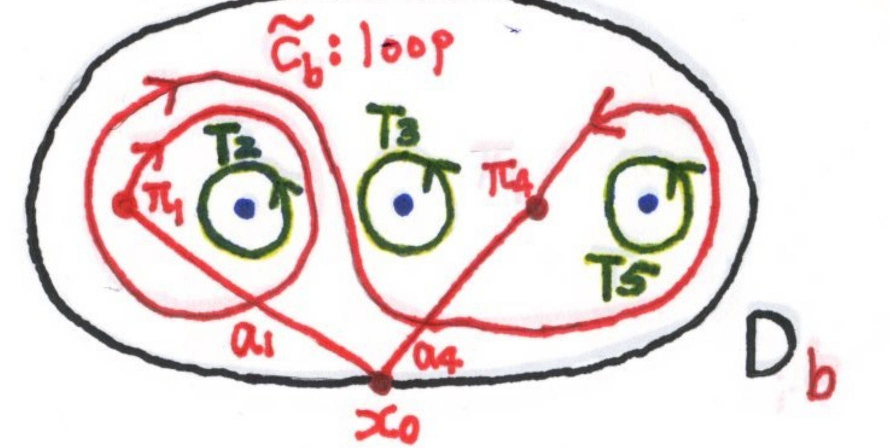
$$D_b := D \setminus \{\pi_2, \pi_3, \pi_5\}$$

$$H_1(D_b) \cong \mathbb{Z}\langle \tau_2 \rangle \oplus \mathbb{Z}\langle \tau_3 \rangle \oplus \mathbb{Z}\langle \tau_5 \rangle$$

§3 $X_m \ni b \longmapsto C_b$: a cord of b



$$b = (\sigma_2 \sigma_3^{-1} \sigma_4^{-2} \sigma_1^{-1})^{-1} \sigma_1 \, \sigma_2 \sigma_3^{-1} \sigma_4^{-2} \sigma_1^{-1} \in X_5$$



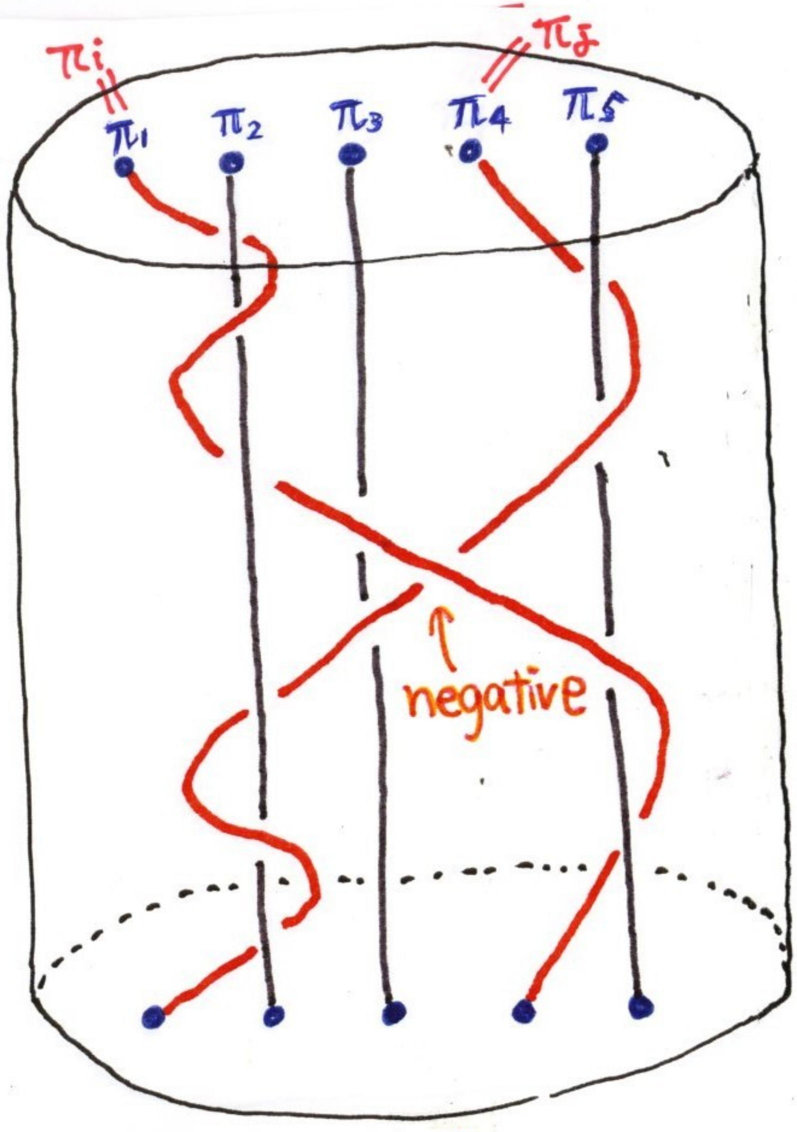
$$D_b := D \setminus \{\pi_2, \pi_3, \pi_5\}$$

$$H_1(D_b) \cong \mathbb{Z}\langle T_2 \rangle \oplus \mathbb{Z}\langle T_3 \rangle \oplus \mathbb{Z}\langle T_5 \rangle$$

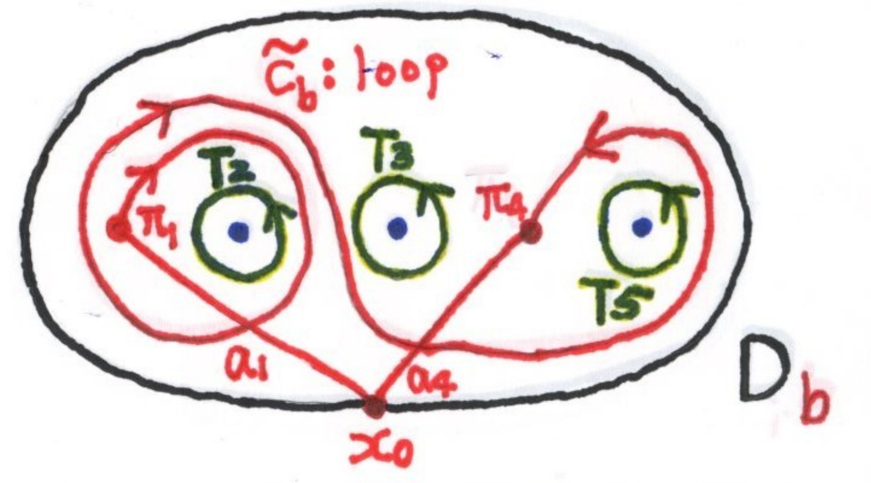
$$[\tilde{C}_b] = -2T_2 + T_5 \in H_1(D_b)$$

$$V(b) = \begin{array}{cl} \left[\begin{array}{c} +\infty \\ -2 \\ 0 \\ -\infty \\ 1 \end{array}\right] & \begin{array}{l} \cdots i\text{-th} \\ \\ \\ \cdots j\text{-th} \\ \\ \end{array} \end{array} \in (\mathbb{Z} \cup \{\pm\infty\})^5$$

§3 $X_m \ni b \longmapsto C_b$: a cord of b



$$b = (\sigma_2\sigma_3^{-1}\sigma_4^{-2}\sigma_1^{-1})^{-1}\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_4^{-2}\sigma_1^{-1} \in X_5$$



$$D_b := D \setminus \{\pi_2, \pi_3, \pi_5\}$$

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$$[\tilde{C}_b] = -2T_2 + T_5 \in H_1(D_b)$$

$$V(b) = \begin{array}{cl} \begin{bmatrix} -\infty \\ -2 \\ 0 \\ +\infty \\ 1 \end{bmatrix} & \begin{array}{l} \text{i-th} \\ \\ \\ \text{j-th} \\ \\ \end{array} \end{array} \in (\mathbb{Z} \cup \{\pm\infty\})^5$$

$$V : X_m \to (\mathbb{Z} \cup \{\pm\infty\})^m$$
$$\begin{array}{ccc} \Psi & & \Psi \\ b & \mapsto & V(b) \end{array} : \underline{\text{the vector of } b \in X_m}$$

$$V^n : X_m{}^n \to M(m, n ; \mathbb{Z} \cup \{\pm\infty\})$$
$$\begin{array}{ccc} \Psi & & \Psi \\ (b_1, \cdots, b_n) & \mapsto & [V(b_1) \cdots V(b_n)] \end{array}$$
$$: \underline{\text{the matrix of } (b_1, \cdots, b_n) \in X_m^n}$$

Prop 1 (Y)

$$
V(X_m) = \left\{ \begin{array}{c|c} x & \varepsilon\infty \\ y & -\varepsilon\infty \\ x,y \neq k & \alpha_k \end{array} \in (\mathbb{Z} \cup \{\pm\infty\})^m \;\middle|\; \begin{array}{l} 1 \leqq x < y \leqq m \\ \varepsilon \in \{\pm 1\} \\ \alpha_k \in \mathbb{Z} \ (k \neq x, y) \end{array} \right\}
$$

$$\parallel$$

$$V_m$$

Cor

$$
V^n(X_m^n) = \left\{ [v_1 \cdots v_n] \in M(m, n : \mathbb{Z} \cup \{\pm\infty\}) \;\middle|\; \begin{array}{l} v_i \in V_m \\ 1 \leqq i \leqq n \end{array} \right\}
$$

$$\parallel$$

$$V_m^n$$

Matrix move on V_m^n

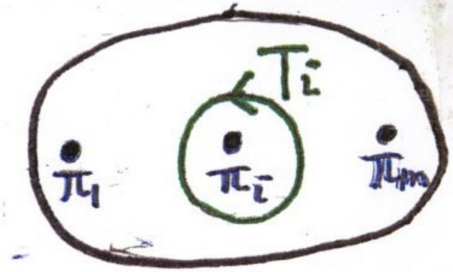
Claim 2

The Hurwitz move R_i on X_m^n induces a move R_i^* on V_m^n, through V^n.

$$\begin{tikzcd}
X_m^n \arrow[r, "R_i", "\text{Hurwitz move}"'] \arrow[d, "V^n"'] \arrow[dr, phantom, "\circlearrowleft"] & X_m^n \arrow[d, "V^n"] \\
V_m^n \arrow[r, "R_i^*"'] & V_m^n
\end{tikzcd} \qquad (1 \leqq i \leqq n-1)$$

We call it the matrix move

⟨Outline of proof of Claim⟩



Step1 $H := H_1(D \setminus \{\pi_1, \cdots, \pi_m\}) = \bigoplus_{i=1}^{m} \mathbb{Z}\langle T_K \rangle$

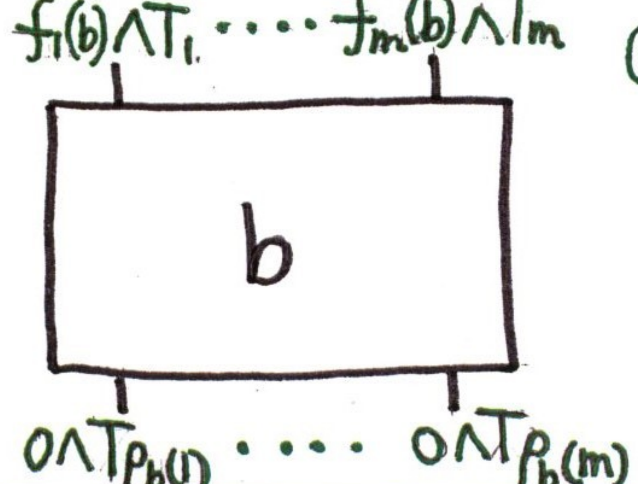
Lemma 2-1 (Kuno-Yaguchi)

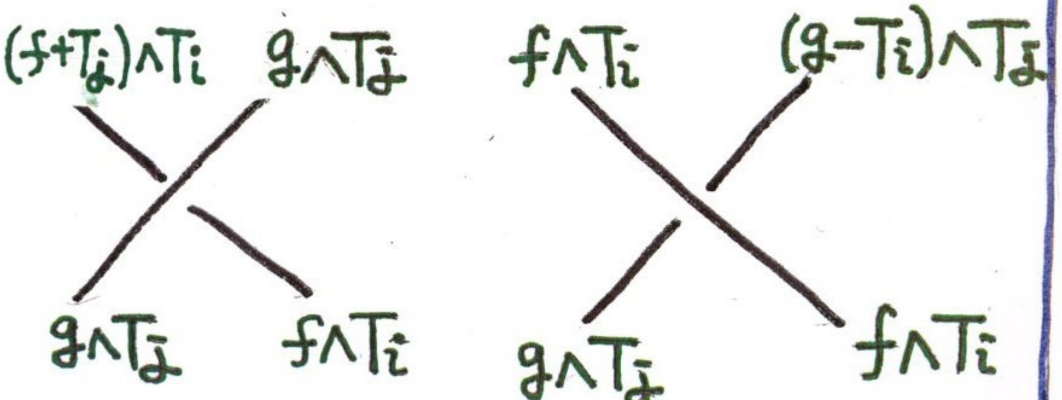
The 1-st Jhonson homomorphism of B_m

$$J : B_m \longrightarrow (\Lambda^2 H)^m \rtimes S_m$$

is given by $J(b) = ((f_1(b) \wedge T_1, \cdots, f_m(b) \wedge T_m), \rho_b)$,

where





Step 2

Lemma 2-2 (Y)

$$
\begin{tikzcd}
 & X_m \subset B_m \arrow[dl, "V"'] \arrow[d, "J"] \\
V_m \arrow[r, "\exists \varphi", "\text{bijective}"'] & J(X_m)
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
 & X_m^n \arrow[dl, "V^n"'] \arrow[d, "J^n"] \arrow[r, "R_i", "\text{Hurwitz move}"'] & X_m^n \arrow[d, "J^n"] \arrow[dr, "V^n"] & \\
V_m^n \arrow[r, "\varphi^n", "\text{bijective}"'] \arrow[rrr, bend right, "R_i^* : \text{matrix move}"'] & (J(X_m))^n \arrow[r, "R_i", "\text{Hurwitz move}"'] & (J(X_m))^n & V_m^n \arrow[l, "\varphi^n"', "\text{bijective}"]
\end{tikzcd}
$$

∎

$\varphi : B_m \longrightarrow G$: a group hom.

$$P^n_{(G,\varphi)} := \left\{ (g_1, \cdots, g_n) \in G^n \;\middle|\; \begin{array}{l} g_1, \cdots, g_n \in \varphi(X_m) \\ g_1 \cdots g_n = \mathrm{id} \end{array} \right\}$$

$$P^n(X_m) \big/ \underset{\text{Hur.}}{\sim} \;\rightarrow\; P^n_{(G,\varphi)} \big/ \underset{\text{Hur.}}{\sim} : [(b_1, \cdots, b_n)] \mapsto [(\varphi(b_1), \cdots, \varphi(b_n))]$$

$$\begin{tikzcd}
B_m \arrow[r, "J"] \arrow[dr] \arrow[ddr] \arrow[dddr, "\text{natural hom.}"'] & (\Lambda^2 H)^m \rtimes S_m \arrow[d] \\
& \mathbb{Z}^m \rtimes S_m \arrow[d] & \cdots \text{ Yaguchi (2011)} \\
& \mathbb{Z}_2^m \rtimes S_m \arrow[d] & \cdots \text{ Hasegawa (2006)} \\
& S_m & \cdots \text{ Hurwitz (1890)}
\end{tikzcd}$$