Anomalous transport in one-dimensional quantum systems

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Will discuss collaborative work with Christoph Karrasch (TU Braunschweig) and Joel E. Moore (UC Berkeley), pictured:

Figure: left to right: C. Karrasch, J. E. Moore

Introduction

Superdiffusion of energy in one-dimensional metals

Kardar-Parisi-Zhang physics in isotropic magnets

Normal transport

- First, what is normal transport? Three generic scenarios in **classical** many-body systems:
  1. Free particles, different velocities lead to dispersion: ballistic transport.
  2. Interacting particles in the hydrodynamic regime: ballistic transport with diffusive corrections.
  3. Microscopic Brownian motion: diffusive transport.
- Broadly speaking, we expect the same pattern in **quantum** many-body systems:
  1. Free quasiparticles: ballistic transport with subdiffusive corrections.
  2. Interacting quasiparticles in the hydrodynamic regime : ballistic transport with diffusive corrections.
  3. Chaotic systems with few conservation laws: diffusive transport.

Anomalous transport in 1D quantum systems

- Recently, some novel examples of anomalous transport in interacting, one-dimensional quantum systems have been discovered:
- Subdiffusion: Approaching the MBL transition from the ergodic side, charge transport appears to be subdiffusive, with a spreading exponent $x \sim t^{\alpha}$ and $0 < \alpha < 1/2$ (reviewed in *Agarwal, Altman, Demler, Gopalakrishnan, Huse, Knap, '17*)
- Superdiffusion:
  - In one-dimensional metals at low temperature, energy transport is superdiffusive and characterized by a non-universal superdiffusive exponent $x \sim t^{\alpha}$ and $2/3 < \alpha < 1$ (*VBB, Karrasch, Moore, '19*).
  - Isotropic quantum magnets exhibit "integrability protected" superdiffusion in the KPZ universality class, with $\alpha = 2/3$ (*Žnidarič, '11, Ljubotina, Žnidarič, Prosen, '17, Ljubotina, Žnidarič, Prosen, '19, De Nardis, Medenjak, Karrasch, Ilievski, '19, Dupont, Moore, '19, VBB, '19*).
- Today's talk will focus on the papers arXiv 1904.09287 and arXiv 1910.08266, addressing the superdiffusive examples.

Introduction

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One-dimensional metals as perturbed Luttinger liquids

- Ideal 1D metals are described by the free, Luttinger liquid theory, yielding divergent linear response transport coefficients (ballistic transport).
- Unperturbed Hamiltonian maps to free bosons,

$$H_0 = \frac{u}{2} \int_0^L dx \, \left(\Pi^2 + (\partial_x \phi)^2 \right) .$$

- Realistic 1D metals have finite transport coefficients, due to interactions or disorder (diffusive transport).
- Typical 1D interactions generate density-wave instabilities. In the metallic phase, these show up as irrelevant vertex operators:

$$\delta H \sim \cos \alpha\phi, \quad \alpha^2 > 8\pi.$$

Charge transport in perturbed Luttinger liquids

- Now consider a clean, interacting Luttinger liquid, with an irrelevant vertex operator perturbation:

$$H = \frac{u}{2}\int_0^L dx\ \left(\Pi^2 + (\partial_x\phi)^2\right) + h\int_0^L dx\ \cos\alpha\phi.$$

- When this is the most relevant interaction, the leading low-temperature dependence of the charge conductivity can be obtained analytically (*Oshikawa, Affleck, '02, Sirker, Pereira, Affleck, '11*).
- By the Kubo formula,

$$\sigma_c(q,\omega) = \frac{K}{2\pi} i\omega\langle\phi\phi\rangle_{ret}(\omega). \tag{1}$$

- Leading-order self-energy calculation yields an effective relaxation time

$$\tau_c \propto \lim_{\omega\to 0}\frac{\omega}{\mathrm{Im}[\Pi_h(q=0,\omega)]} \sim T^{3-\alpha^2/2\pi}. \tag{2}$$

Lattice realization of perturbed Luttinger liquid

- A microscopic realization of an interacting Luttinger liquid is spin-1/2 XXZ in an integrability-breaking staggered field:

$$H = \sum_{i=1}^{N} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z + (-1)^i h S_i^z. \qquad (3)$$

- Low-energy field theory has the form

$$H = \frac{u}{2}\int_0^L dx\ \left(\Pi^2 + (\partial_x\phi)^2\right) + ch\int_0^L dx\ \cos\left(2\sqrt{\pi K}\phi\right) + \ldots$$

  i.e. staggered field is the most relevant perturbation.
- A previous work (*Huang, Karrasch, Moore, '13*) numerically verified **non-integrability** for $h > 0$:
  1. For $h > 0$, level statistics flow from Poisson to Wigner-Dyson.
  2. Charge transport is diffusive, with linear-response conductivity matching the Sirker-Pereira-Affleck result

$$\sigma_c(T) \sim T^{3-2K}. \qquad (4)$$

Anomalous diffusion model for heat transport

- What about heat transport? Minimal assumption is Wiedemann-Franz - charge and heat transport controlled by the same relaxation time. Yields

$$\kappa(T) \sim T^{\lambda(K)}, \quad \lambda(K) = 4 - 2K. \tag{5}$$

- Unfortunately, accessing $\kappa(T)$ directly is beyond present numerical and analytical techniques, and WF need not hold.
- Even so, ansatz Eq. (5) has non-trivial, testable consequence for transport: energy density ρ_E should satisfy **fast diffusion equation**

$$\partial_t \rho_E = D\partial_x^2 \left(\rho_E^m\right), \quad m = \frac{1+\lambda}{2}. \tag{6}$$

- Fundamental solutions are superdiffusive "Barenblatt-Pattle profiles", with anomalous scaling

$$x \sim t^{\alpha}, \quad \alpha = \frac{2}{\lambda + 3}. \tag{7}$$

Testing the theoretical model

- To test this, we simulated XXZ in staggered field, with localized thermal wavepacket initial condition (*VBB, Karrasch, Moore, '19*)

$$\beta(x) = \beta - (\beta - \beta_M) e^{-(x/L)^2} \tag{8}$$

and low bulk temperature $\beta \gg 1$.

- Can probe space-time scaling by looking at **logarithmic derivatives of moments**:

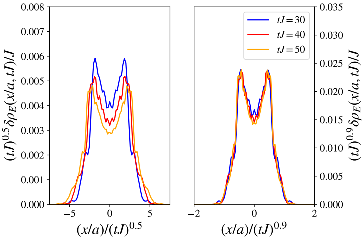
$$\frac{1}{n} \frac{d \log \langle |x|^n \rangle (t)}{d \log t} \to \alpha, \quad t \to \infty. \tag{9}$$

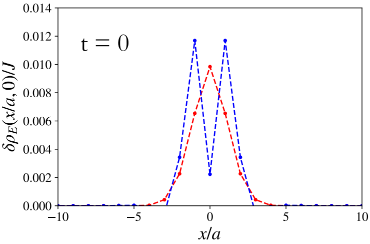
- Non-trivial prediction of our model: these should converge to the same, superdiffusive exponent, $2/3 < \alpha < 1$.
- At any non-zero bulk temperature, expect eventual crossover to diffusive behaviour on a timescale

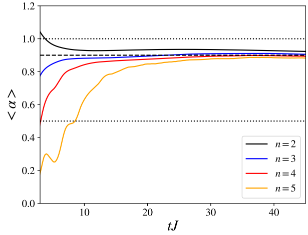
$$\tau_D(T) \sim T^{\lambda - 1}. \tag{10}$$

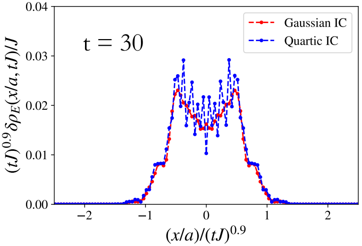
Numerical results for anomalous diffusion of heat

- For bulk temperatures $T = 1/12$, clear evidence for superdiffusive, rather than diffusive, spreading of wavepacket:









Summary of part I

- Interacting, one-dimensional metals that exhibit generic, thermalizing behaviour in all other respects (level statistics, charge transport, . . .) can exhibit superdiffusion of heat at low temperatures.
- Simple analytical model of nonlinear diffusion equation is sufficient to predict collapse of moments to a single superdiffusive exponent.
- Detailed shape of profile is a direction for future work (kinetic theory? proximate integrability?)
- Unexpected violation of Fourier's law in a well-studied class of physical systems.
- This type of superdiffusion is robust to integrability-breaking and should be observable in time-resolved experiments on quantum wires involving laser irradiation of a small region (c.f. *Hensel, Dynes, '77*).

Introduction

Superdiffusion of energy in one-dimensional metals

Kardar-Parisi-Zhang physics in isotropic magnets

Numerical observations

- A large body of work has confirmed "integrability protected" KPZ physics in one-dimensional magnets with isotropic symmetry.
- Diagnostic is the long-time behaviour of the spin autocorrelation function

$$\langle \mathbf{S}(x,t)\cdot\mathbf{S}(0,0)\rangle_\beta \sim t^{-\alpha} f(x/t^\alpha), \quad t\to\infty. \tag{11}$$

- From numerics, it is found that $\alpha = 2/3$ for integrable models and $\alpha = 1/2$ for non-integrable models (with a possible crossover from $\alpha = 2/3$).
- **Classical:** *Prosen, Žunkovič, '13, Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, Ray, Bhattacharjee, '19, Das, Damle, Dhar, Huse, Kulkarni, Mendl, Spohn, '19, Das, Kulkarni, Spohn, Dhar, '19, Krajnik, Prosen, '19*, **Quantum:** *Žnidarič, '11, Ljubotina, Žnidarič, Prosen, '17, Ljubotina, Žnidarič, Prosen, '19, de Nardis, Medenjak, Karrasch, Ilievski, '19 , Dupont, Moore, '19, Weiner, Schmitteckert, Bera, Evers, '19*

State of theory

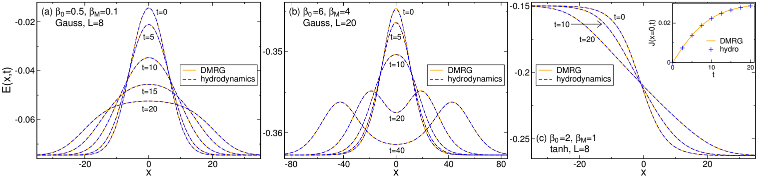
- Previously, the closest approach to a theoretical explanation for the dynamical exponent $\alpha = 2/3$ was a self-consistent derivation from generalized hydrodynamics for the spin-$1/2$ Heisenberg model (*Gopalakrishnan, Vasseur, '19*).
- However, three fundamental questions have not been addressed:
  1. Why is a collapse to universal, Kardar-Parisi-Zhang scaling functions observed numerically? (*Ljubotina, Žnidarič, Prosen, '19, Das, Kulkarni, Spohn, Dhar, '19*)
  2. Why does the same phenomenon occur for both quantum and classical systems?
  3. Why are both integrability and isotropic symmetry necessary for this phenomenon to be stable at long times? (*Dupont, Moore, '19, Krajnik, Prosen, '19*)
- These questions reflect a basic lack of understanding of the physical mechanism underlying this phenomenon.

Kardar-Parisi-Zhang universality from soft gauge modes

- The main theoretical difficulty is that according to generalized hydrodynamics (*Castro-Alvaredo, Doyon, Yoshimura '16, Bertini, Collura, De Nardis, Fagotti, '16*), the fluctuations of quasiparticle modes are purely diffusive (*Doyon, Myers, '19, de Nardis, Medenjak, Karrasch, Ilievski, '19*)
- Very recently, we pointed out that isotropic magnets support soft modes of the magnetization that carry a subextensive energy and are therefore not captured by standard hydrodynamic approaches (*VBB, '19*).
- These soft modes are nonlinear and separated in scale from short-wavelength fluctuations, yielding a channel for superdiffusive spin transport.
- This provides a physical mechanism for the emergence of KPZ physics in these models.

Why do we believe generalized hydrodynamics in spin chains?

- The theory appears to describe ballistic transport in the gapless spin-1/2 XXZ model **exactly**:



(figure from *VBB, Vasseur, Karrasch, Moore '17*)

- Another highly non-trivial test is that the spin Drude weight is captured exactly (*Zotos, '99, Ilievski, De Nardis, '17, VBB, Vasseur, Karrasch, Moore, '17, Urichuk, Oez, Klümper, Sirker, '18*).
- Recent analytical expressions for ρ_J in finite spin-1/2 XXZ and XXX chains (*Borsi, Pozsgay, Pristyák, '19, Pozsgay, '19*)

A theoretical puzzle

- If we believe GHD, then mode-coupling coefficients of linearized quasiparticle modes are known exactly. Diagonal elements of the Hessian vanish, $G^{k}_{k'k'} = 0$, implying **purely diffusive fluctuations** (*Popkov, Schadschneider, Schmidt, Schütz, '15*).
- The resolution: in isotropic integrable magnets, the generalized hydrodynamics of infinitely many conserved charges is not equivalent to the kinetic equation describing the propagation of quasiparticle modes.
- The Bethe-Boltzmann equation is a scalar equation, while isotropic magnets have a conserved vector degree of freedom.
- The kinetic theory description needs to be augmented by "gauge dynamics" - an equation of motion for the local pseudovacuum.

Example: spin-1/2 Heisenberg

- For concreteness, let us consider the spin-1/2 Heisenberg Hamiltonian

$$H = -J\sum_{i=1}^{N} \mathbf{S}_i \cdot \mathbf{S}_{i+1} - 2\mathbf{h} \cdot \sum_{i=1}^{N} \mathbf{S}_i. \tag{12}$$

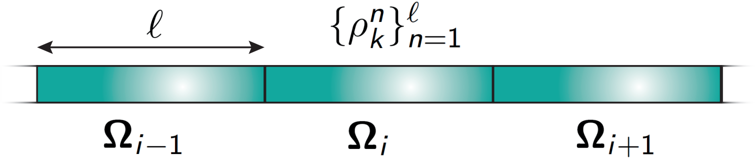
- In the absence of an applied magnetic field, $\mathbf{h} = 0$, Bethe's solution to this model exhibits an $SU(2)$ gauge symmetry - direction of the pseudovacuum, $\mathbf{\Omega} \in S^2$, is arbitrary. This extends to the thermodynamic Bethe ansatz (TBA) equations.
- An applied magnetic field $\mathbf{h} \neq 0$ breaks this symmetry; only pseudovacuum directions $\mathbf{\Omega} \parallel \mathbf{h}$ are allowed. TBA predicts

$$\langle \mathbf{S} \rangle / \ell = \mathbf{\Omega} \left[\frac{1}{2} - \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} dk \, n\rho_k^n \right]. \tag{13}$$

- In local equilibrium states, formation of a local magnetization on a fluid cell spontaneously breaks $SU(2)$ symmetry - **pseudovacuum becomes a dynamical degree of freedom**.

Landau-Lifshitz states in isotropic integrable magnets

- Consider local equilibrium states with constant quasiparticle occupancies per fluid cell and a pseudovacuum $\mathbf{\Omega}(x,t) \in S^2$ varying on a scale $\ell_{\mathbf{\Omega}} \gg \ell$:



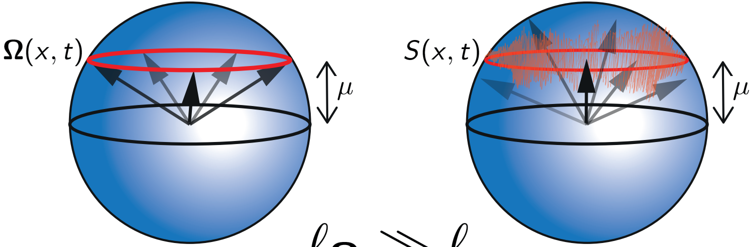
- In the limit $\ell/\ell_{\mathbf{\Omega}} \to 0$, the quasiparticle dynamics is decoupled from the gauge dynamics.
- Slow modulations of the pseudovacuum are mean-field states by definition, with Hamiltonian dynamics ($\lambda = J/2$)

$$\partial_t \mathbf{\Omega} = \lambda \mathbf{\Omega} \times \partial_x^2 \mathbf{\Omega} + \mathcal{O}(\ell_{\mathbf{\Omega}}^{-4}). \tag{14}$$

- At zero temperature, recovers previous results (*Gamayun, Miao, Ilievski, '19, Misguich, Pavloff, Pasquier, '19*). At finite temperature, Eq. (14) is coupled to a thermal bath.

The physical picture

- Certain local equilibrium states exhibit a separation of scales between nonlinear gauge dynamics and linear quasiparticle dynamics.
- When this occurs, the quasiparticle modes can act as a fluctuating bath for the nonlinear gauge dynamics.
- Origin of KPZ physics lies in qualitative difference between transverse and longitudinal hydrodynamic modes of the spin:



$$\ell_{\mathbf{\Omega}} \gg \ell$$

Coarse-grained Landau-Lifshitz dynamics I

- Effective field theory describing KPZ physics is **Landau-Lifshitz dynamics at finite temperature**. We therefore consider nonlinear fluctuating hydrodynamics of this equation. First we need the Euler hydrodynamics.
- Since integrability is broken microscopically, mean-field evolution is not integrable and there are two conserved modes. Exactly the same reasoning as discretized GPE (*Kulkarni, Huse, Spohn, '15*).
- Standard parameterizations of the sphere (e.g. spherical polar) are not gauge invariant. An elegant solution is to regard x as arc-length and $\mathbf{\Omega}$ as the tangent vector of a space curve (*Lakshmanan, Ruijgrok, Thompson, '76*).
- In terms of the curvature κ and torsion τ of this curve, the Landau-Lifshitz evolution becomes

$$\dot{\kappa} + \lambda(2\kappa'\tau + \kappa\tau') = 0, \quad \dot{\tau} + \lambda(\tau^2 - \kappa''/\kappa - \kappa^2/2)' = 0. \tag{15}$$

Coarse-grained Landau-Lifshitz dynamics II

- Discarding the dispersive term and changing variable from κ to energy density $\mathcal{E} = \kappa^2/2$ yields

$$\partial_t \mathcal{E} + \partial_x[\lambda(2\mathcal{E}\tau)] = 0, \tag{16}$$

$$\partial_t \tau + \partial_x[\lambda(\tau^2 - \mathcal{E})] = 0. \tag{17}$$

- Linearizing, we find imaginary velocities and violation of sum rules - **instability**. Suggests that two-mode hydrodynamics of Landau-Lifshitz is unphysical.
- Intuition: soft modes can't transport extensive energy. More precisely, total energy in a fluid cell of characteristic length $\ell_{\mathbf{\Omega}}$ is subextensive, $\mathcal{E} \sim 1/\ell_{\mathbf{\Omega}}$. Thus energy is not a true hydrodynamic variable and tends to zero as $\ell_{\mathbf{\Omega}} \to \infty$.
- This leaves a single Burgers equation for the torsion (i.e. magnetization density),

$$\partial_t \tau + \partial_x(\lambda\tau^2) = 0. \tag{18}$$

Coarse-grained Landau-Lifshitz dynamics III

- Mesosopic coupling to noise and dissipation yields stochastic Burgers equation

$$\partial_t \tau + \partial_x(\lambda\tau^2 - D\partial_x\tau + \sigma\zeta_\tau) = 0, \tag{19}$$

  with coefficients constrained by fluctuation dissipation relation $\langle\tau\tau\rangle_\mu = 2\sigma D$.
- Follows that "height function" η, defined by $\tau = \partial_x\eta$, satisfies the KPZ equation, and that the correlation functions of the torsional mode have superdiffusive scaling form $C(x,t) = \mathbb{E}[\tau(x,t)\tau(0,0)] = f_{KPZ}(x/(\Gamma t)^{3/2})/(\Gamma t)^{3/2}$, where $\Gamma = 2\sqrt{2}\lambda$ (*Spohn, '16*).
- No fitting parameters, and for spin-1/2 Heisenberg and Faddeev-Takhtajan, predicts the coupling strength $\lambda/J = 0.5$, well within typical NLFH error of observations, $\lambda/J \approx 0.65$ and $\lambda/J \approx 0.68$ resp.
- c.f. mode-coupling prediction $\lambda_{pred} = 1.97$ vs numerical observation $\lambda_{obs} = 13.8$ for heat peak in FPU (*Das, Dhar, Saito, Mendl, Spohn, '14*)

The story so far

- Our effective theory provides a simple model for why isotropic quantum and classical magnets exhibit KPZ physics - they support nonlinear, long-wavelength modes of the magnetization.
- These modes are “soft” and transport subextensive energy, explaining the puzzling insensitivity of the observed KPZ physics to energy conservation (*Ljubotina, Žnidarič, Prosen, ’19, Krajnik, Prosen, ’19*).
- Leaves question of **integrability protection** - why does KPZ appear to cross over to diffusion in non-integrable isotropic magnets?

A hydrodynamic Higgs mechanism

- One way to understand the emergence of nonlinear modes in these models: a separation of scales $\ell_{\mathbf{\Omega}} \gg \ell$ acts like a **Higgs mechanism for the nonlinearities** in the mode-coupling theory.
- Schematically:

$$\left.\begin{matrix} G^{\parallel} = 0 \\ G^{\perp} = 0 \end{matrix}\right\} \xrightarrow{\ell_{\mathbf{\Omega}} \gg \ell} \left\{\begin{matrix} G^{\parallel} = 0 \\ G^{\perp} \neq 0 \end{matrix}\right. \tag{20}$$

($G^{\perp}$, $G^{\parallel}$ indicate nonlinear terms in the mode-coupling theory of the transverse and longitudinal components of local magnetization, $\langle \mathbf{S} \rangle^{\perp}$ and $\langle \mathbf{S} \rangle^{\parallel}$ respectively)

- Thus the key question becomes: why is this separation of scales protected by integrability?

A hydrodynamic explanation for integrability protection

- Short answer: integrable models support distinct "gauge" and "quasiparticle" excitations of spin. No such distinction exists in non-integrable models.
- For the "Landau-Lifshitz" states that we considered earlier, the only difference in the short-wavelength hydrodynamics is the nature of the scalar bath.
- **Non-integrable models**: two scalar conserved modes $\{S, E\}$ per fluid cell. Total variance scales as $\sigma_\ell^2 \sim \ell^{-1}$. As $\ell \to \infty$, hydrodynamics of S becomes deterministic and recouples to slow dynamics of $\mathbf{\Omega}$.
- **Integrable models**: extensively many scalar modes $\{S, E, Q_3, \ldots, Q_n\}_{n \sim \ell}$. the variance scales as $\sigma_\ell^2 \sim \ell^0$. As $\ell \to \infty$, S continues to fluctuate as part of a bath.
- The idea of KPZ physics arising from long-wavelength modes coupled to a quasiparticle bath previously appeared in 1D Bose gases (*Arzamasovs, Bovo, Gangardt, '14*)

Some novel predictions for the torsional mode

- Consider "tilted" weak domain walls, which differ by an angle $0 < \varphi < \pi$ (like *Ljubotina, Žnidarič, Prosen, '17* but near infinite temperature).
- Previously, no predictions for dynamics in this set-up. Our model implies scaling collapse with φ for energy density and spin autocorrelation functions:

$$\langle \mathbf{S}_\varphi(x,t) \cdot \mathbf{S}_\varphi(0,0) \rangle / \cos\varphi = t^{-2/3} f(x/t^{2/3}), \tag{21}$$

$$\mathcal{E}_\varphi(x,t) / \sin\varphi/2 = t^{-2/3} g(x/t^{2/3}). \tag{22}$$

- For magnetization $\mu \ll 1$ small but above the noise scale, further predict ballistic spin dynamics with speed $v_{ball} \sim \mu$.
- Derived in spin-1/2 Heisenberg from GHD (*Gopalakrishnan, Vasseur, Ware, '19*). Here, we argue true for classical models also. Consistent with latest numerics (*Krajnik, Prosen, '19*).
- Further testing is work in progress with M. Dupont and J. E. Moore.

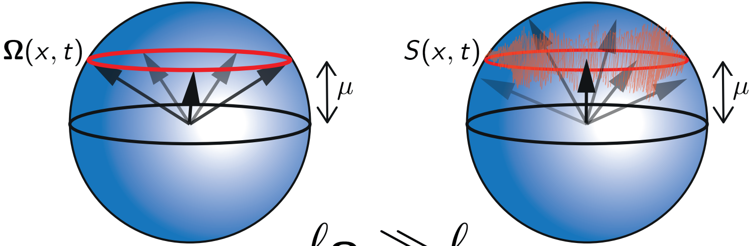
Summary of part II

- We have identified the nonlinear modes that give rise to KPZ scaling in isotropic quantum and classical magnets.
- Our analysis explains many hitherto puzzling numerical observations. It also gives rise to novel predictions for the superdiffusive mode, which could be tested in classical or quantum numerical simulations.
- Importantly, the physics giving rise to superdiffusion comes from long-wavelength spin dynamics, which is unrelated to integrability (even if "integrability protected")
- An open question: can these modes be described using Bethe ansatz? Subextensivity of energy suggests that they reside in finite-size corrections to thermodynamics.

Thank you for listening!

Relevant papers:

- **Superdiffusion of energy in one-dimensional metals**: *VBB, Christoph Karrasch, Joel E. Moore*, arXiv 1904.09287 (2019)
- **Kardar-Parisi-Zhang universality from soft gauge modes**: *VBB*, arXiv 1910.08266 (2019)



$$\ell_{\mathbf{\Omega}} \gg \ell$$

A primer on nonlinear diffusion

- The **nonlinear diffusion equation** is given by

$$\partial_t u = D\nabla^2 u^m \tag{23}$$

i.e. effective diffusion constant is nonlinear,

$$D_{eff}[u] = mDu^{m-1}.$$

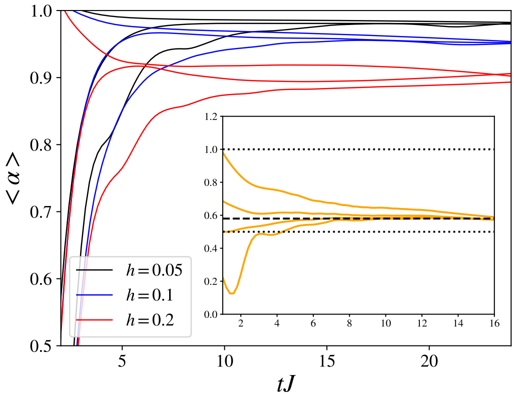
- For $m = 1$, recover normal diffusion. For $m > 1$, this is the "porous medium equation". For $m < 1$, this is the "fast diffusion equation" (for a review: *Vázquez, '06*).
- Fundamental solutions for $m \neq 1$ are non-Gaussian. Instead, nonlinearity yields **Barenblatt-Pattle** profiles, characterized by anomalous space-time scaling.
- e.g. in $d = 1$, these have the form

$$u_{B.P.}(x,t) = t^{-\alpha}\max[(C - k(x/t^{\alpha})^2)^{-\frac{1}{m-1}}, 0] \tag{24}$$

with space-time scaling exponent $\alpha = 1/(m+1)$, $k = k(m)$ constant and C fixed by initial area.

Numerical results for anomalous diffusion of heat II

- Increasing strength of integrability-breaking field lowers superdiffusive exponent (main figure):



- Sanity checks : expansion into ground state yields superdiffusion, higher temperatures begin to recover normal diffusion (inset, at $\beta = 1$, $h = 0.49$).