Phases to Phases

An Invitation to Topological Phases of Many Particles

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Acknowledgements

- Research funding: SERB, DST; DAE

- Key contributors:

Adhip Agarwala

Arijit Haldar

Overview: Phases to Phases

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$$e^{i\theta} \longrightarrow$$

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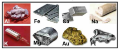
$$e^{i\theta} \longrightarrow$$

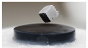
- Gapless (single fermion excitation)[1]



1. The filled Fermi sea

- Gapped states

2. Band insulator



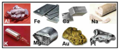
3. BCS superconductor

4. Filled Landau level

Overview: Phases to Phases

$$e^{i\theta}$$

- Gapless (single fermion excitation)[1]



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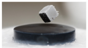

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- Background – Electronic phases: A brief excursion of quantum condensed matter physics
- Electronic Phases – A quick illustration of “topology”
- Electronic Phases – The Tenfold Way
- Advertisement: Topological phases in amorphous systems
- Brief mention: what we do not discussed, open issues

Topological Matter Matters

Photo: A. Mahmoud

David J. Thouless

Prize share: 1/2

Photo: A. Mahmoud

F. Duncan M. Haldane

Prize share: 1/4

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J. Michael Kosterlitz

Prize share: 1/4

The Nobel Prize in Physics 2016 was awarded with one half to David J. Thouless, and the other half to F. Duncan M. Haldane and J. Michael Kosterlitz *"for theoretical discoveries of topological phase transitions and topological phases of matter"*.

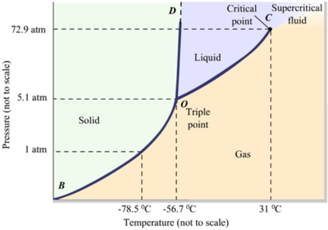
Phases of Electrons

Matter...and its Phases

- Matter broadly appears in three distinct phases (at human scales)

(Internet)

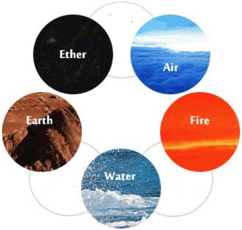
- ...gas, liquid and solid



(Phase Diagram of CO_2, source:Internet)

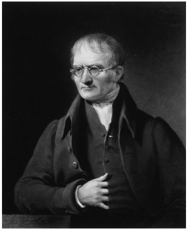
Whence Phases?

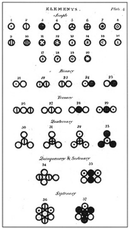
- Ancient wisdom...the panchabhootas...



(Wikipedia)

- ...to ideas of Dalton...the atomic hypothesis (early 1800s)

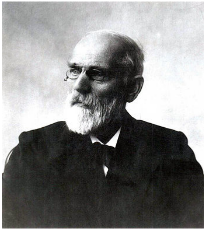
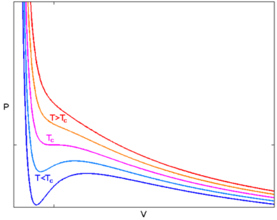




(Wikipedia)

Atomic Theory...to Theory of Phases

- van der Waals (later part of 1800s) showed how liquids and gases can arise from the same constituent atoms/molecules...

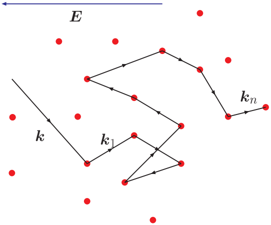
...offering a framework to understand gas, liquid and solid phases

- Interactions between constituent atoms (in large numbers) can lead to different phases
- Puzzle: Why are there insulators and conductors?

Electrons...and Electronic Phases

- Key milestone: Discovery of the electron (Thomson, 1890s)...

- ...to the first theory of electronic phases – Drudé (1900)

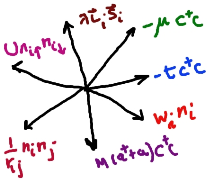


  with remarkable success...resolution of a 50 year old puzzle – the Wiedemann-Franz law ($\frac{\kappa}{\sigma T}$ = universal number)...

- **But**...with a confounding new puzzle: Drude predicts $C_V = 3R + \frac{3}{2}R = \frac{9}{2}R$ at "high" temperatures...measured $C_V = 3R$!

Quantum Condensed Matter Physics – Bird's Eye View

- Resolution of the puzzle: Quantum mechanics
- Electrons in materials "see" many things...

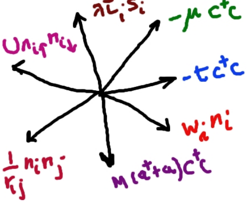


...space of all possible Hamiltonains (includes kinetic energy, spin-orbit coupling, Coulomb interaction, interaction with the lattice, disorder etc..)

- Quantum condensed matter physics aims to study and classify the phases of many electrons (many identical particles, in general)
- Many of the modern technologies from cell phones to night vision goggles owe much to this area of physics!

Taking Stock...

- Quantum theory of many electrons – offers insights into insulators and metals...



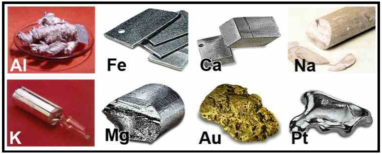
Traditional ideas of classifying many-fermion phases

- Symmetry: A magnet breaks spin-rotation symmetry (Landau)
- "Properties": Metals and insulators

Distinct phases are separated by *phase transitions*

Summary of a Grad Course!

- Gist of our understanding...*essentially* captured by **four** states
- Gapless (single fermion excitation)[1]



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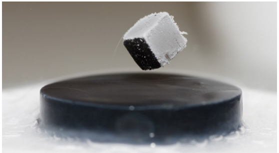

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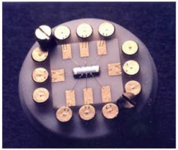

4. Filled Landau level

[1] All images are from the internet

Current Status

- Recent developments – **two** ideas (in addition to symmetry, properties etc.)
  1. **Topology:** Hinted by the quantum hall effect
  2. **Entanglement:** How “complicated” is the state?

  Focus of this talk: “Topology”
- Recent developments – Complete “topological” classification of gapped (non-interacting) many fermion systems

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(Kitaev (2009), Ryu et. al. (2010), Ludwig:1512:08882)

- The tenfold way!

Electronic Phases

...What and Why Topology?

Essential Quantum Ideas

- Quantum mechanics
  - Kinematics : Identical particles are indistinguishable
    Fermions obey Pauli's principle
    For two fermionic particles

$$\Psi(x_2, x_1) = -\Psi(x_1, x_2)$$

  - Dynamics "encoding" the Heisenberg uncertainty principle
- Immediate goal: Construct simplest electronic phases with focus on *non-interacting* systems

Warm Up: Atomic Structure – Few Electrons

- Carbon atom: Six electrons in the nuclear potential with charge +6
- Electronic structure (wave function) (NCERT – 11th Class Chemistry)

C 

$$\Psi_C(x_1,\ldots,x_6) = \begin{vmatrix} 1s_\uparrow(x_1) & 1s_\uparrow(x_2) & 1s_\uparrow(x_3) & 1s_\uparrow(x_4) & 1s_\uparrow(x_5) & 1s_\uparrow(x_6) \\ 1s_\downarrow(x_1) & 1s_\downarrow(x_2) & 1s_\uparrow(x_3) & 1s_\downarrow(x_4) & 1s_\downarrow(x_5) & 1s_\downarrow(x_6) \\ 2s_\uparrow(x_1) & 2s_\uparrow(x_2) & 2s_\uparrow(x_3) & 2s_\uparrow(x_4) & 2s_\uparrow(x_5) & 2s_\uparrow(x_6) \\ 2s_\downarrow(x_1) & 2s_\downarrow(x_2) & 2s_\uparrow(x_3) & 2s_\downarrow(x_4) & 2s_\downarrow(x_5) & 2s_\downarrow(x_6) \\ 2p_{X\uparrow}(x_1) & 2p_{X\uparrow}(x_2) & 2p_{X\uparrow}(x_3) & 2p_{X\uparrow}(x_4) & 2p_{X\uparrow}(x_5) & 2p_{X\uparrow}(x_6) \\ 2p_{Y\uparrow}(x_1) & 2p_{Y\uparrow}(x_2) & 2p_{Y\uparrow}(x_3) & 2p_{Y\uparrow}(x_4) & 2p_{Y\uparrow}(x_5) & 2p_{Y\uparrow}(x_6) \end{vmatrix}, \; (x \equiv (\boldsymbol{r}, \sigma))$$

...Slater determinant wave function

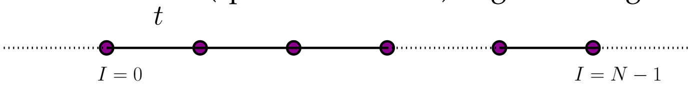
- Structure known for any atom...the periodic table

Group→ ↓Period	1	2	3		4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	1 H																		2 He
2	3 Li	4 Be												5 B	6 C	7 N	8 O	9 F	10 Ne
3	11 Na	12 Mg												13 Al	14 Si	15 P	16 S	17 Cl	18 Ar
4	19 K	20 Ca	21 Sc		22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 Cu	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
5	37 Rb	38 Sr	39 Y		40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
6	55 Cs	56 Ba	57 La	*	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
7	87 Fr	88 Ra	89 Ac	**	104 Rf	105 Db	106 Sg	107 Bh	108 Hs	109 Mt	110 Ds	111 Rg	112 Cn	113 Nh	114 Fl	115 Mc	116 Lv	117 Ts	118 Og
				*	58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dy	67 Ho	68 Er	69 Tm	70 Yb	71 Lu	
				**	90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr	

- Question: Analogous table of many (10^{23}) electron phases?

Electronic Liquid Phase – Metal

- Simple model of a metal (spinless electrons)...tight binding model



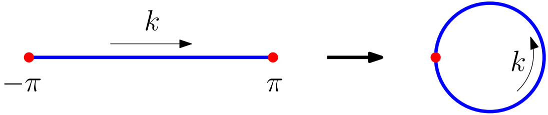
...N-site chain with periodic boundary conditions

$$H = -t \sum_I |I+1\rangle \langle I| + \text{h.c.}$$

- Diagonalized by crystal momentum k

$$|k\rangle = \frac{1}{\sqrt{N}} \sum_I e^{-ikI} |I\rangle \,, \qquad H = \sum_k \varepsilon(k) |k\rangle \langle k| \,, \quad \varepsilon(k) = -2t \cos k$$

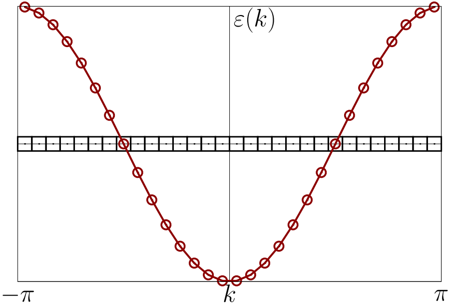
- Crystal momentum k lives in the Brillouin Zone (BZ)...$k \in [-\pi, \pi]$ with $-\pi$ and π identified...one dimensional torus



- In d-dimensions k in T^d, the d-torus

Electronic Liquid Phase – Metal

- Filling of electrons 1/2 per site, need to fill $N/2$ electrons
- In an N-site chain, there are N distinct k points
  $k = -\pi, -\pi + \frac{2\pi}{N}, -\pi + 2\frac{2\pi}{N}, \ldots, \pi - \frac{2\pi}{N}$ with $\Delta k = \frac{2\pi}{N}$



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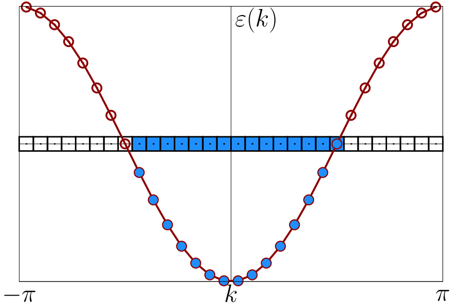

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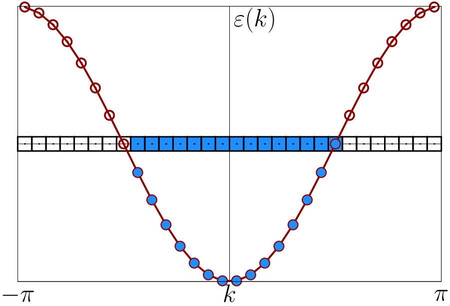
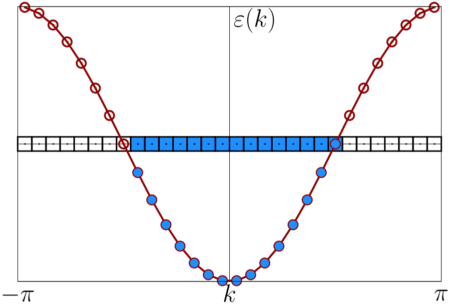


- Ground state is the filled Fermi sea (filled "pigeon holes" make up the Slater determinant wave function)

Electronic Liquid Phase – Metal

- Gapless excitations in the thermodynamic limit ($N \to \infty$)



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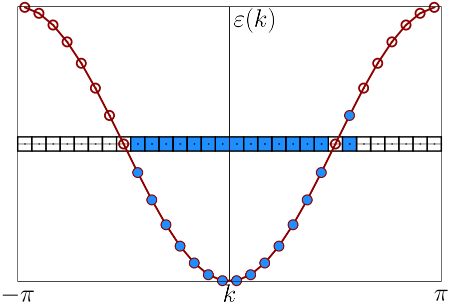

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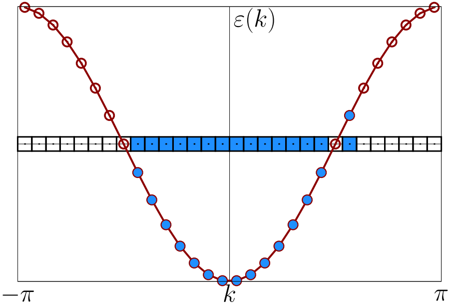

…excitation energy vanishes

$$\Delta E \sim \Delta k \sim \frac{2\pi}{N} \to 0$$

- Metal: small stimulus can produce finite responses – liquid state
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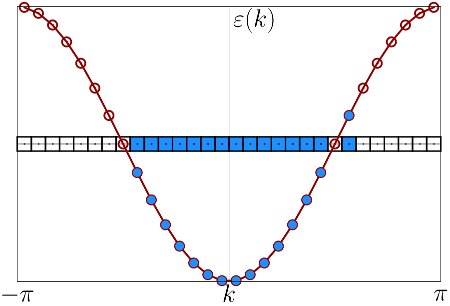

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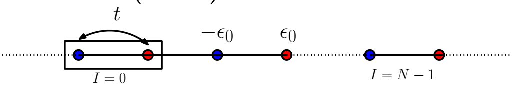
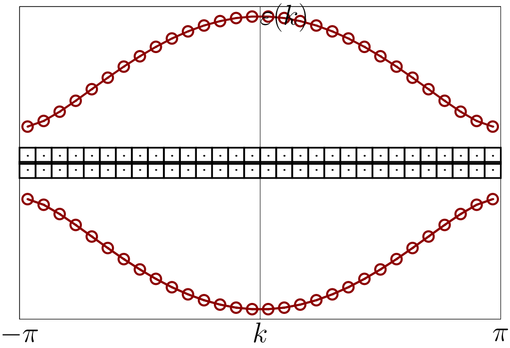
- Puzzle...Insulators?

..and Insulators

- Poor man's insulator (NaCl)



- *Two* bands (with separate pigeon holes) separated by a band gap $\sim \epsilon_0$ ...there are no electronic states in the energy gap



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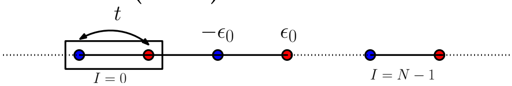

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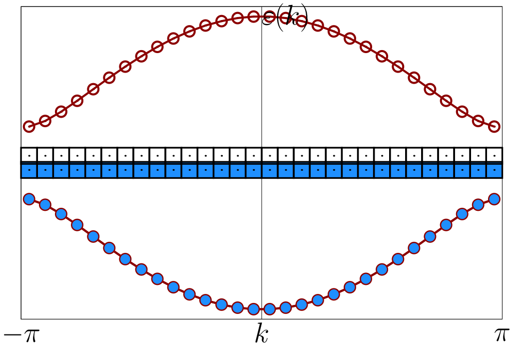

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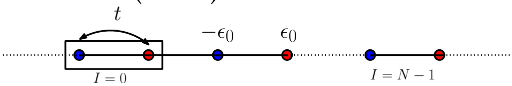

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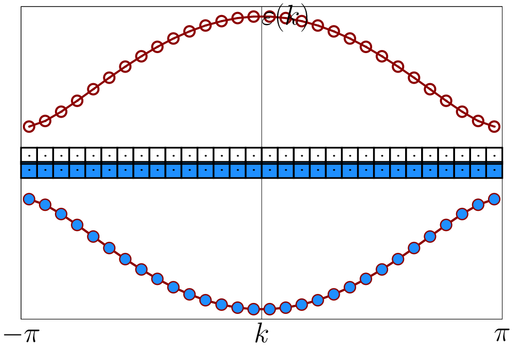
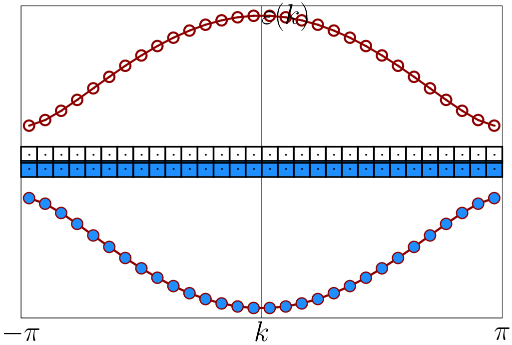


- At 1/2 electron per site, ground state is the filled "valance band" ("conduction band" is empty)

Insulators: “Electronic Solid Phase”

- Excitations are gapped



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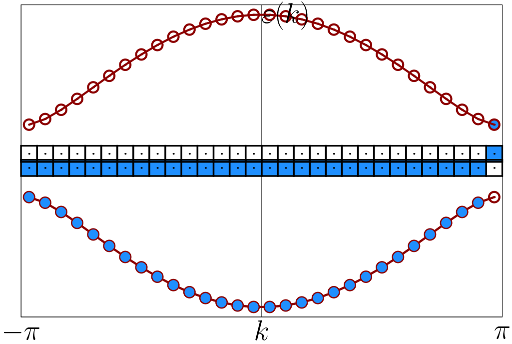

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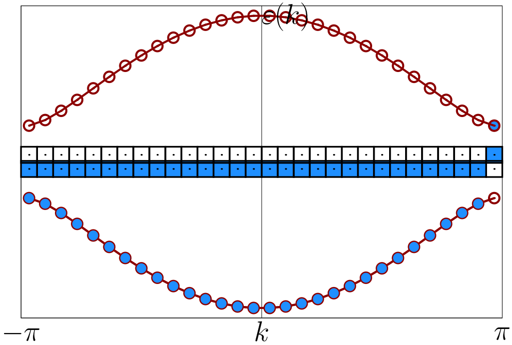

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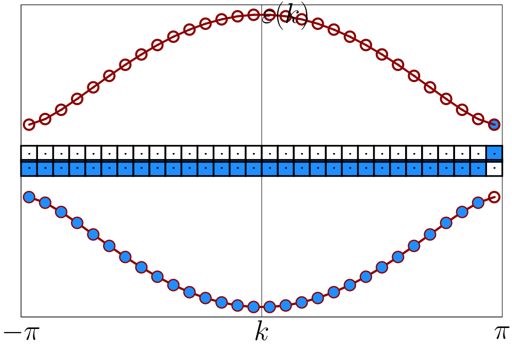

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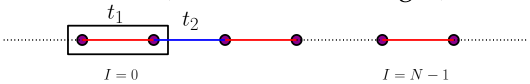
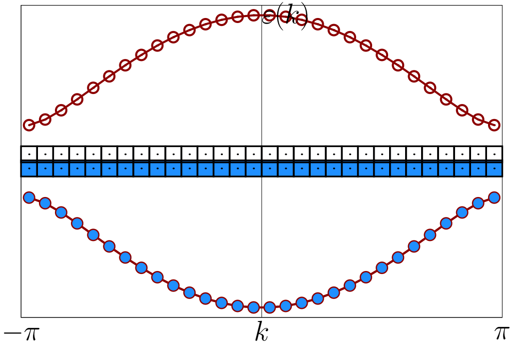
- Question: Is this the only possible insulating phase?

Topology of Electron Phases – Whetting the Appetite

- A different insulator – SSH (Su-Schrieffer-Heeger) model

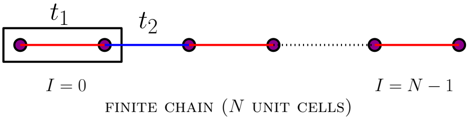


- Band structure looks qualitatively identical to the NaCl system (gap $\sim |t_2 - t_1|$)

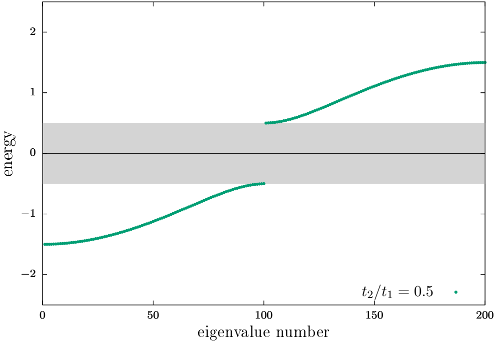


- *Cannot* distinguish between NaCl and SSH by looking at the bulk band structure

Topology...Revealed in a *Finite* Chain



- Energy eigenvalues
  - No energy levels in the gap for $t_2 < t_1$ (NaCl is similar)



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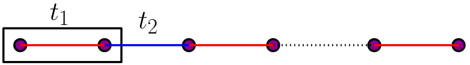
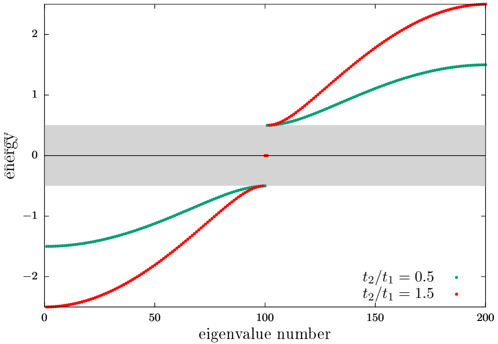


FINITE CHAIN (N UNIT CELLS)

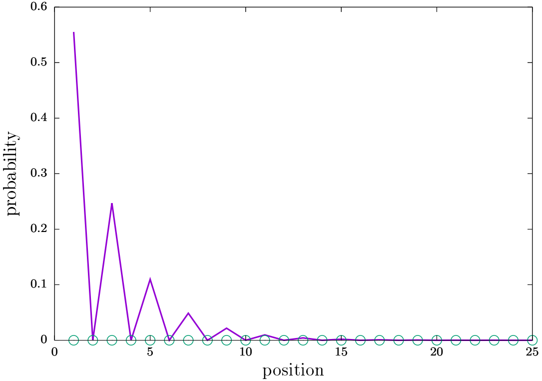
- Energy eigenvalues
  - No energy levels in the gap for $t_2 < t_1$ (NaCl is similar)



  - Two *zero energy states* when $t_2 > t_1$

Topology...on the *Edge*

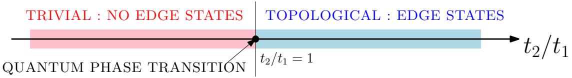
- For $t_2 > t_1$ the zero energy states are *edge states*...they are eigenstates localized near the edges ("surfaces") of the finite sample



- Insulating phase for $t_2 > t_1$ has a distinguishing character – presence of edge states

Phases Distinguished by Topology...and Phase Transitions

- Phase diagram



- Can drive a quantum phase transition by tuning t_2

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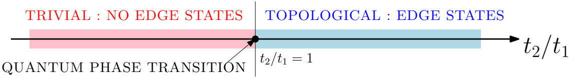

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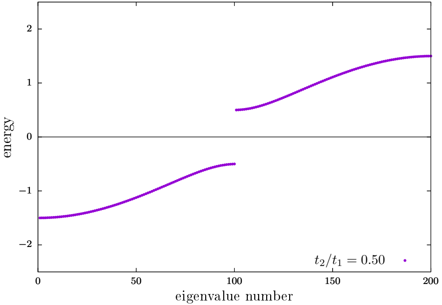

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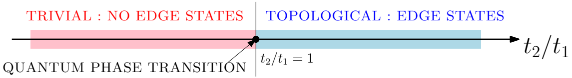

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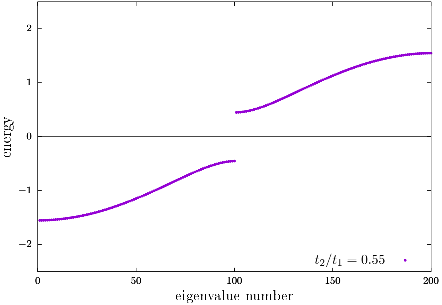

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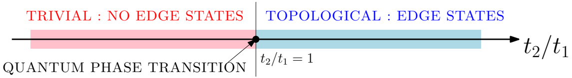

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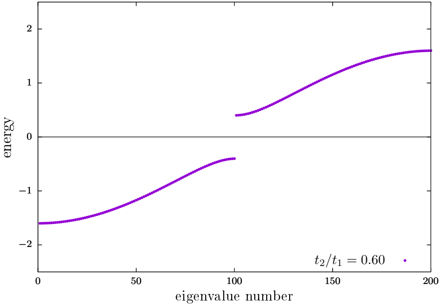

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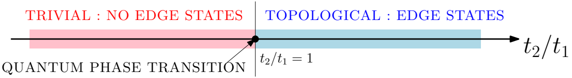

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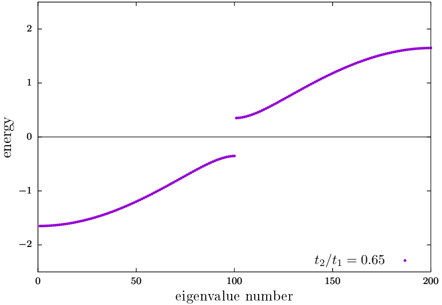

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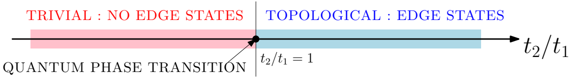

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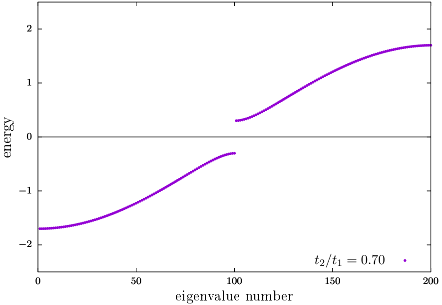

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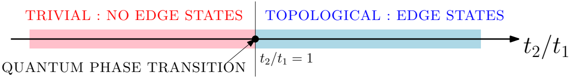

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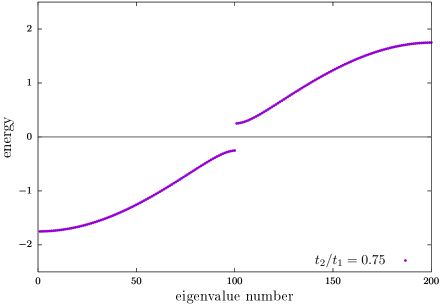

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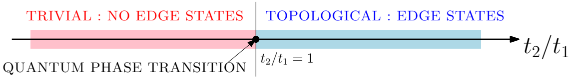

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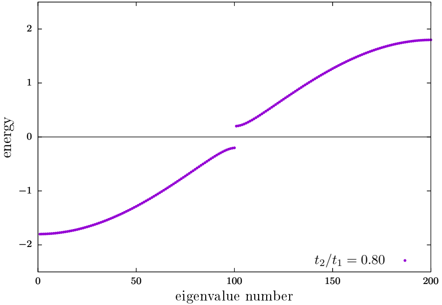

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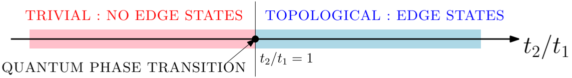

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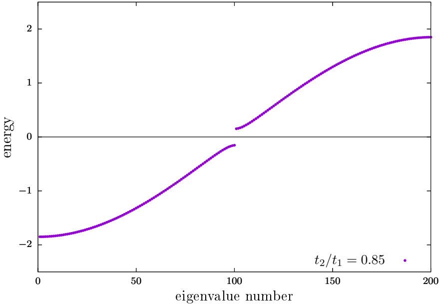

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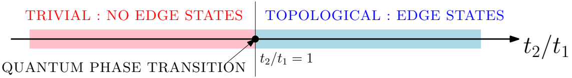

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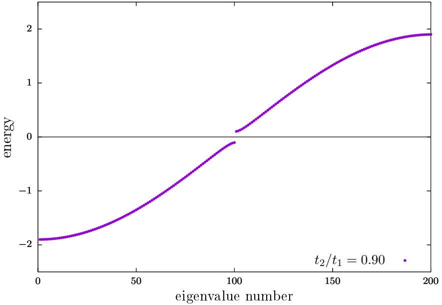

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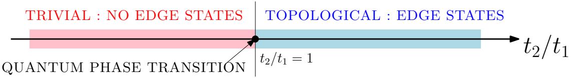

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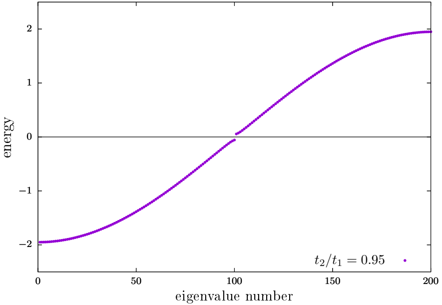

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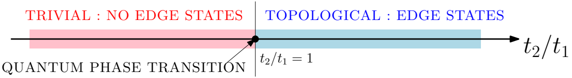

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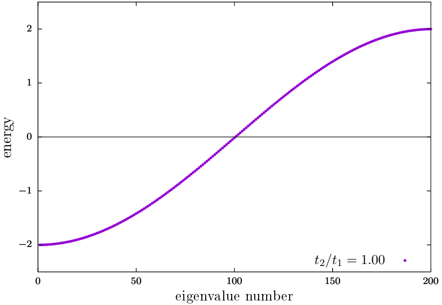

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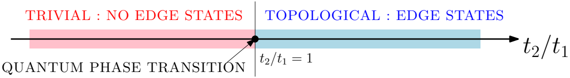

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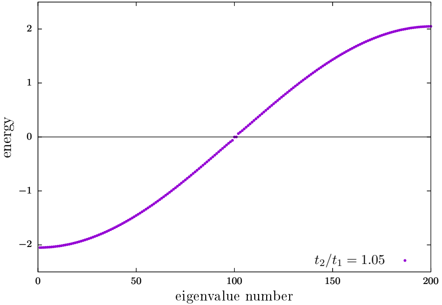

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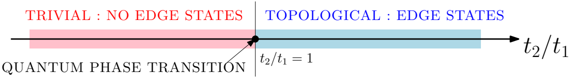

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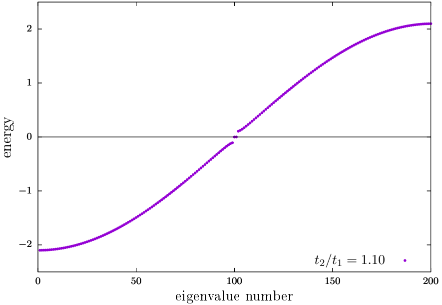

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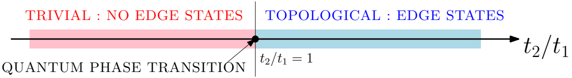

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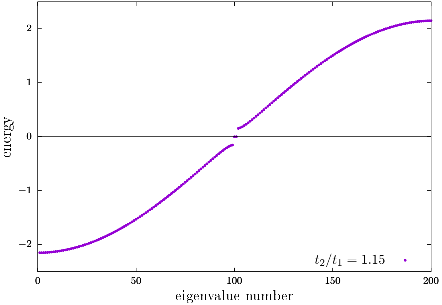

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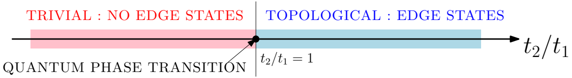

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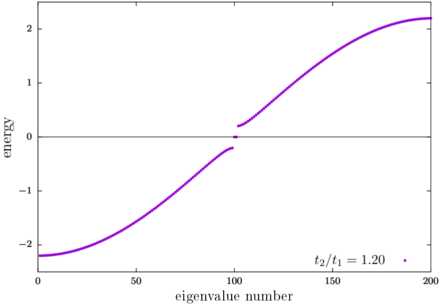

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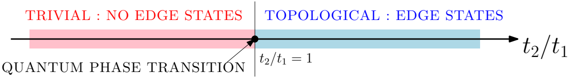

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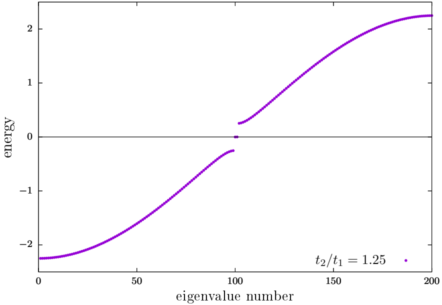

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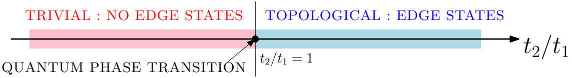

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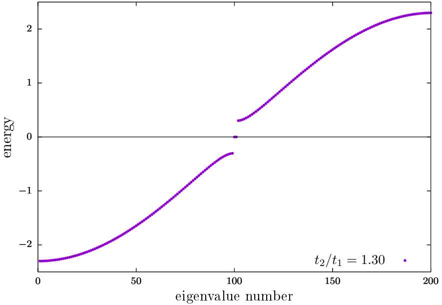

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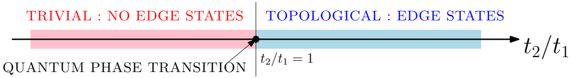

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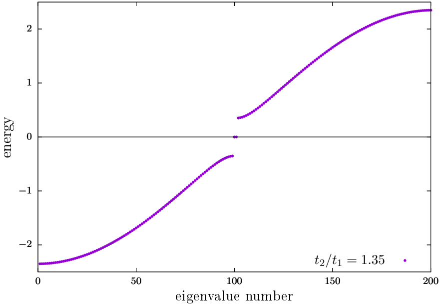

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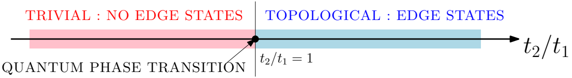

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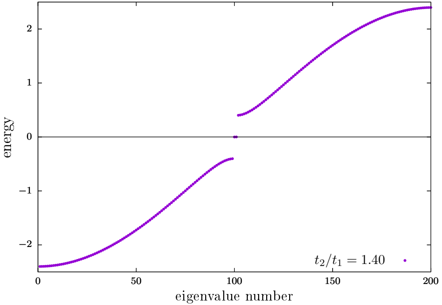

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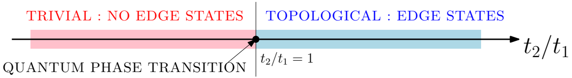

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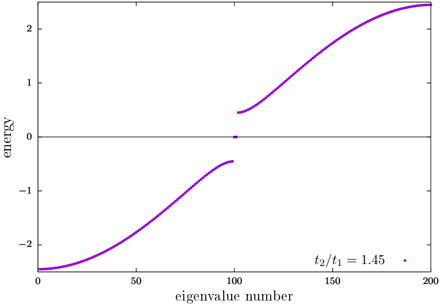

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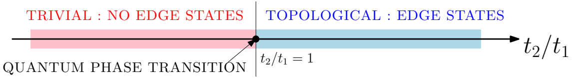

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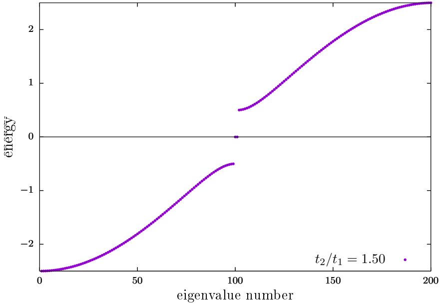

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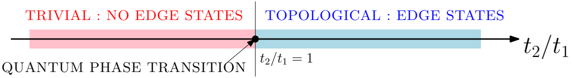

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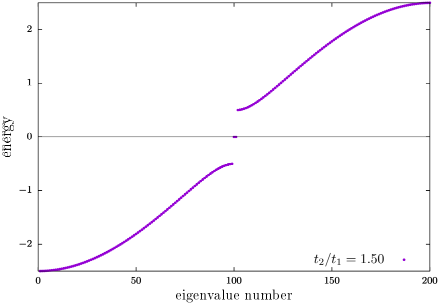

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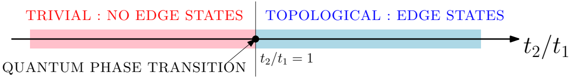

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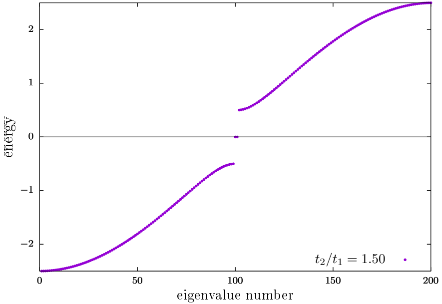

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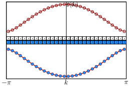
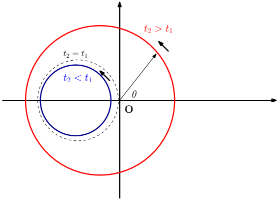
- **Question: Where *IS* the topology?**

Topology of Electron Phases

- VB wave function $\begin{pmatrix} 1 \\ e^{\mathbb{i}\theta(k)} \end{pmatrix}$ with $\theta(\pi) = \theta(-\pi) + 2\pi n$, Key: The state at k can be thought of as a two dimensional unit vector
- Ground state is filled valance band



can be viewed as an endless ribbon(Demonstration)
- For $t_2 > t_1$ the ground state ribbon is "twisted"



- The ground state of $t_1 < t_2$ cannot be deformed to that of $t_2 > t_1$ without closing the gap (tearing)
- **Topology is encoded in the *twist* of the many particle wave function**

Gapped Phases of Electrons

- What we know today about non-interacting electrons

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(Kitaev (2009), Ryu et. al. (2010), Ludwig:1512:08882)

- The tenfold way!

The Tenfold Way

- Altland and Zirnbauer, Phys. Rev. B., **55**, 1142 (1997).
- Agarwala, Haldar, VBS, arXiv:1606.05483

The Tenfold Way - Bird's-Eye View

- Summary
  - Fermionic systems can be classified according to some "intrinsic non-ordinary" symmetries
  - There are three "non-ordinary" symmetries – Time reversal, charge conjugation and sublattice
  - There are ten symmetry classes(a symmetry class is a Hilbert-Fock space along with *all* possible Hamiltonains that are allowed by symmetry)
  - There is a nice connection to geometry – the ten classes are labeled by Cartan labels of symmetric spaces
- What we address in this talk
  - What are "non-ordinary" symmetries, and why are there three of them?
  - Why are there ten classes?
  - Fleeting discussion of connection to geometry
  - ...
  - How does topology arise (an even more fleeting view)?...motivating the amorphous topological insulators

Beginnings...

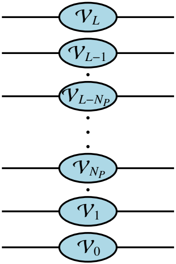
- System with L orbitals $|i\rangle$, $i = 1, \ldots, L$
- Fermions are created from the vacuum $|\mathbf{0}\rangle$ via

$$|i\rangle = \psi_i^\dagger |\mathbf{0}\rangle, \qquad \{\psi_i, \psi_j^\dagger\} = \delta_{ij}, \qquad \Psi^\dagger = \begin{bmatrix} \psi_1^\dagger & \ldots & \psi_L^\dagger \end{bmatrix}$$

- Can accommodate N_P fermions, $N_P = 0, \ldots, L$

$$\mathcal{V}_{N_P} = \mathrm{span}_{\mathbb{C}}\{\psi_{i_1}^\dagger \ldots \psi_{i_{N_P}}^\dagger |\mathbf{0}\rangle\}, \quad \dim\mathcal{V}_{N_P} = \binom{L}{N_P}$$

- Hilbert-Fock space $\mathcal{V} = \bigoplus_{N_P=0}^{L} \mathcal{V}_{N_P}$



Beginnings...II

- Noninteracting fermions – Hamiltonian

$$\mathscr{H} = \sum_{ij} H_{ij} \psi_i^\dagger \psi_j$$

($\mathbf{H}$ – matrix H_{ij})...includes BdG type superconducting hamiltonians

- The space of Hamiltonians $\mathcal{H}$ is a vector space over $\mathbb{R}$
- ...and even more, $\mathcal{H}$ is effectively a *Lie algebra*
- The Schrödinger time evolution operator (at $t = 1$)

$$e^{-\mathbb{i}\mathbf{H}}$$

runs over $U(L)$ (unitary group) as $\mathbf{H}$ runs over $\mathcal{H}$

- $\mathcal{H}$ is isomorphic to the Lie algebra $\boldsymbol{u}(L)$

$$\mathbb{i}\mathcal{H} = \boldsymbol{u}(L)$$

Symmetries...Recap

- Generic system: Hilbert space $\mathcal{V}$, Hamiltonian $\mathscr{H}$
- A *symmetry operation ("symmop")* is an invertible map $\mathscr{U} : \mathcal{V} \to \mathcal{V}$ such that, if $|\psi'\rangle = \mathscr{U}(|\psi\rangle)$, then

$$|\langle\phi'|\,\psi'\rangle| = |\langle\phi|\,\psi\rangle|, \qquad \forall\,|\phi\rangle\,,|\psi\rangle \in \mathcal{V}$$

- Wigner's theorem: Any symmop is either a linear or antilinear operator on $\mathcal{V}$
  (antilinear: $\mathscr{U}(\alpha\,|\psi\rangle + \beta\,|\phi\rangle) = \alpha^*\mathscr{U}\,|\psi\rangle + \beta^*\mathscr{U}\,|\phi\rangle$)
- The set of all symmops from a group $\mathcal{G}_O$
- A symmop $\mathscr{U}$ is a symmetry if

$$\mathscr{U}\mathscr{H}\mathscr{U}^{-1} = \mathscr{H}$$

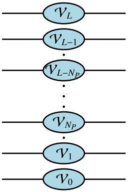
  The set of all symmetries from a group $\mathcal{G}$ (a subgroup of $\mathcal{G}_O$)
- Examples of "usual" symmetries (free particle)
  - Usual Linear: Translation
  - Usual Antilinear : Time reversal

Symmops of Fermionic Systems

- Fermionic system: $\mathcal{V} = \bigoplus_{N_P=0}^{L} \mathcal{V}_{N_P}$
- Usual symmop of fermionic systems

$$\mathscr{U}_{\mathrm{USL}}(\mathcal{V}_{N_P}) = \mathcal{V}_{N_P}, \qquad \mathscr{U}_{\mathrm{USL}} \Psi^\dagger \mathscr{U}_{\mathrm{USL}}^{-1} = \Psi^\dagger \mathbf{U}$$

described by a unitary matrix $\mathbf{U}$



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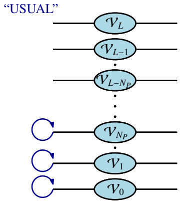

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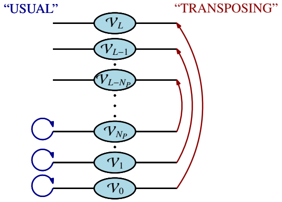


- Fermionic systems have other natural, transposing, symmops: (Altland and Zirnbauer (1997))

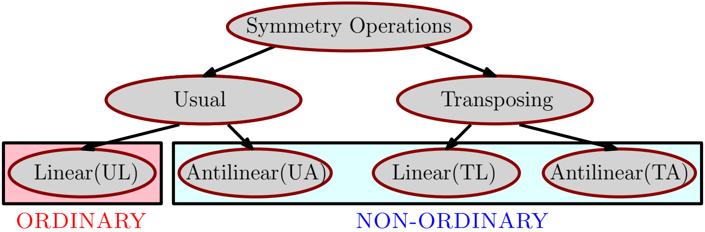
$$\mathscr{U}_{\rm TRN}(\mathcal{V}_{N_P}) = \mathcal{V}_{L-N_P}, \qquad \mathscr{U}_{\rm TRN}\Psi^\dagger \mathscr{U}_{\rm TRN}^{-1} = \Psi^T \mathbf{U}$$

...traditionally called as "particle-hole" transformations

- Transposing symmops, following Wigner, can be linear or antilinear

Symmops – Summary

- Symmops/symmetries of fermionic systems



- Central to the classification scheme are the non-ordinary symmops (UA,TL,TA)

Non-ordinary Symmops: Properties

- Product of any two non-ordinary symmop of the same type is an ordinary symmop, e. g.,

$$\mathscr{U}_{1\mathrm{TA}}\mathscr{U}_{2\mathrm{TA}} = \mathscr{U}_{\mathrm{UL}}$$

- Product of two different types non-ordinary symmops is the third type, e.g.,

$$\mathscr{U}_{\mathrm{UA}}\mathscr{U}_{\mathrm{LT}} = \mathscr{U}_{\mathrm{TA}}$$

- Multiplication table of symmops $G_O = G_O^{\mathrm{UL}} \cup G_O^{\mathrm{UA}} \cup G_O^{\mathrm{TL}} \cup G_O^{\mathrm{TA}}$

$\mathscr{U}_1 \downarrow \mid \mathscr{U}_2 \rightarrow$	G_O^{UL}	G_O^{UA}	G_O^{TL}	G_O^{TA}
G_O^{UL}	G_O^{UL}	G_O^{UA}	G_O^{TL}	G_O^{TA}
G_O^{UA}	G_O^{UA}	G_O^{UL}	G_O^{TA}	G_O^{TL}
G_O^{TL}	G_O^{TL}	G_O^{TA}	G_O^{UL}	G_O^{UA}
G_O^{TA}	G_O^{TA}	G_O^{TL}	G_O^{UA}	G_O^{UL}

"Grotesque" Fermionic Systems (GFS)

- A fermionic system is termed "grotesque" if it has no nontrivial ordinary symmetries (There is always the $U(1)$ or Z_2 symmetry associated with $\mathscr{I}_\theta = e^{\mathbb{i}\theta\mathscr{N}}$, $\mathscr{N} = \sum_i \psi_i^\dagger \psi_i$)
- Non-ordinary symmetries of a GFS are highly constrained
  1. Solitarity: There is at most one of each type of non-ordinary symmetry...they are some standard names
     - **UA** Time reversal symmetry $\mathscr{T}$
     - **TL** Charge conjugation symmetry $\mathscr{C}$
     - **TA** Sublattice symmetry $\mathscr{S}$
  2. A GFS has to make choose from one of three possibilities
     - **Type 0** No non-ordinary symmetries
     - **Type 1** A single non-ordinary symmetry (which can be any of the three)
     - **Type 3** Three non-ordinary symmetries one of each type

Further Properties of Non-Ordinary Symmetries of a GFS

- Time reversal: $\mathscr{T}^2 = (\pm 1)^{\mathscr{N}}$ denoted by $\mathrm{T} = \pm 1$
- Charge conjugation: $\mathscr{C}^2 = (\pm 1)^{\mathscr{N}}$ denoted by $\mathrm{C} = \pm 1$
- Sublattice can be chosen as : $\mathscr{S}^2 = 1$
- Why so? Illustrate this with $\mathscr{C}$: $\mathscr{C}^2 = \mathscr{I}_\theta$ for some θ; rewrite this as $\mathscr{C}^{-1}\mathscr{I}_\theta = \mathscr{I}_\theta\mathscr{C}^{-1}$; apply this relation on $\mathcal{V}_{N_P}$:

$$e^{\mathbb{i}N_P\theta} = e^{\mathbb{i}(L-N_P)\theta}, \ \ \forall \ N_P = 0, \ldots, L.$$

resulting in $e^{\mathbb{i}2\theta} = 1$, and

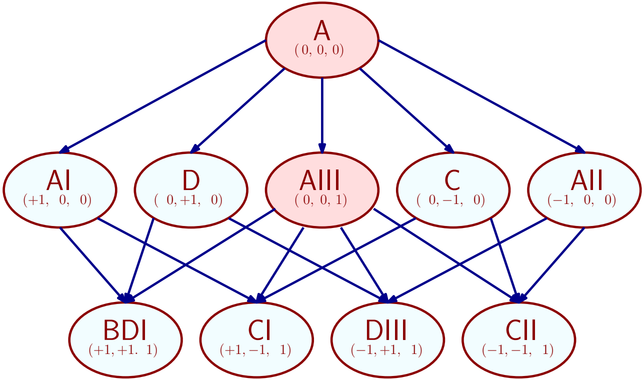
$$\mathscr{C}^2 = (\pm 1)^{\mathscr{N}} \mathscr{I}$$

Ten Classes

- Ten classes

Type 0 One class
Type 1 Five classes
Type 3 Four classes

- Each symmetry class denoted by its symmetry signature (T, C, S) (a 0 value implies absence of that symmetry)



Pink – Complex class, Cyan – Real class

Canonical Representation of Symmetries

Class	T	C	S	L	$\mathbf{U}_{\mathrm{T}}$	$\mathbf{U}_{\mathrm{C}}$	$\mathbf{U}_{\mathrm{S}}$
A	0	0	0	L	–	–	–
AI	+1	0	0	L	$\mathbf{1}$	–	–
AII	−1	0	0	$L = 2M$	$\mathbf{J}$	–	–
D	0	+1	0	L	–	$\mathbf{1}$	–
C	0	−1	0	$L = 2M$	–	$\mathbf{J}$	–
AIII	0	0	1	$L = p + q$	–	–	$\mathbf{1}_{p,q}$
BDI	+1	+1	1	$L = p + q$	$\mathbf{1}$	$\mathbf{1}_{p,q}$	$\mathbf{1}_{p,q}$
CII	−1	−1	1	$L = p + q$ $p = 2r;\ q = 2s$	$\begin{pmatrix} \mathbf{J}_{pp} & \mathbf{0}_{pq} \\ \mathbf{0}_{qp} & \mathbf{J}_{qq} \end{pmatrix}$	$\begin{pmatrix} -\mathbf{J}_{pp} & \mathbf{0}_{pq} \\ \mathbf{0}_{qp} & \mathbf{J}_{qq} \end{pmatrix}$	$\mathbf{1}_{p,q}$
CI	+1	−1	1	$L = 2M$	$\mathbf{F}$	$-\mathbf{J}$	$\mathbf{1}_{M,M}$
DIII	−1	+1	1	$L = 2M$	$\mathbf{J}$	$\mathbf{F}$	$\mathbf{1}_{M,M}$

$$\mathbf{J} = \begin{pmatrix} \mathbf{0}_{MM} & \mathbf{1}_{MM} \\ -\mathbf{1}_{MM} & \mathbf{0}_{MM} \end{pmatrix}$$

$$\mathbf{1}_{p,q} = \begin{pmatrix} \mathbf{1}_{pp} & \mathbf{0}_{pq} \\ \mathbf{0}_{qp} & -\mathbf{1}_{qq} \end{pmatrix}$$

$$\mathbf{F} = \begin{pmatrix} \mathbf{0}_{MM} & \mathbf{1}_{MM} \\ \mathbf{1}_{MM} & \mathbf{0}_{MM} \end{pmatrix}$$

Recall $\mathscr{U}_{\mathrm{USL}} \Psi^\dagger \mathscr{U}_{\mathrm{USL}}^{-1} = \Psi^\dagger \mathbf{U}$ and $\mathscr{U}_{\mathrm{TRN}} \Psi^\dagger \mathscr{U}_{\mathrm{TRN}}^{-1} = \Psi^T \mathbf{U}$

- Symmetries strongly constrain the Hilbert space, e. g., class C can be realized only when L is even!

Tenfold Way – Another View

- Abstract group of symmops: $\mathcal{K}_4$, the Klein group

$\mathcal{K}_4$	I	Θ	Ξ	Σ
I	I	Θ	Ξ	Σ
Θ	Θ	I	Σ	Ξ
Ξ	Ξ	Σ	I	Θ
Σ	Σ	Ξ	Θ	I

- Abstract symmetry group G is a subgroup of $\mathcal{K}_4$

Type 0 $\mathcal{I} = \{I\}$
Type 1 $\mathcal{Z}_2^T = \{I, \Theta\}, \mathcal{Z}_2^C = \{I, \Xi\}, \mathcal{Z}_2^S = \{I, \Sigma\}$
Type 3 $\mathcal{K}_4$

- Fermionic systems – representation spaces of G

Tenfold Way – A Group Cohomological View

- Projective representations: Each group element $g \in G$ is represented by a operator(matrix) $D(g)$ on some "fermionic" Hilbert-Fock space

$$D(g_1)D(g_2) = \omega(g_1, g_2)D(g_1 g_2)$$

where $\omega(g_1, g_2) \in U(1)$ is the "Schur multiplier" or "2-cocycle"
- Condition on Schur multipliers (associativity of the group)

$$\omega(g_1, g_2)\omega(g_1 g_2, g_3) = \omega(g_1, g_2 g_3)\omega^{g_1}(g_2, g_3)$$

where

$$\omega^g \equiv \varphi_g(\omega)$$

encodes the linearly or antilinearity of g

$$\varphi_I(\omega) = \omega, \quad \varphi_\Theta(\omega) = \omega^*$$
$$\varphi_\Xi(\omega) = \omega^*, \quad \varphi_\Sigma(\omega) = \omega$$

- Key question: How many distinct multipliers are there for a given G and a "twisting function" φ?

Tenfold Way – A Group Cohomological View

- The multipliers themselves from a group, the second cohomology group $H^2_\varphi(U(1), G)$ (after making the idea of "distinct" precise)
- The number of elements of $H^2_\varphi(U(1), G)$ determine the number of symmetry classes associated with G

Key result (1606.05483)

There are ten distinct multiplier systems for $\mathcal{I}, \mathcal{Z}_2^T, \mathcal{Z}_2^C, \mathcal{Z}_2^S, \mathcal{K}_4$...and thus ten symmetry classes

G	$H^2_\varphi(U(1), G)$	$\lvert H^2_\varphi(U(1), G)\rvert$
$\mathcal{I}$	I	1
$\mathcal{Z}_2^T$	Z_2	2
$\mathcal{Z}_2^C$	Z_2	2
$\mathcal{Z}_2^S$	I	1
$\mathcal{K}_4$	K_4	4

$$H^2_\varphi(U(1), \mathcal{K}_4) = K_4$$

$\omega(g_1 \downarrow, g_2 \rightarrow)$	I	Θ	Ξ	Σ
I	1	1	1	1
Θ	1	T	1	T
Ξ	1	TC	C	T
Σ	1	C	C	1

$$\mathrm{T} = \pm 1, \mathrm{C} = \pm 1$$

- Every GFS is a reducible projective representation associated with a multiplier system of $\mathcal{I}, \mathcal{Z}_2^T, \mathcal{Z}_2^C, \mathcal{Z}_2^S, \mathcal{K}_4$

Tenfold Way: Noninteracting Systems

Class	L	$\mathbf{H}^{(1)}$	$\dim i\mathcal{H}^{(1)}$	$i\mathcal{H}^{(1)}$	$\mathbb{U}_{\text{Schröd}}(t)$
A(0, 0, 0)	L	$\mathbf{H}^{(1)} = [\mathbf{H}^{(1)}]^\dagger$	L^2	$\mathfrak{u}(L)$	$U(L)$
AI(+1, 0, 0)	L	$\mathbf{H}^{(1)} = [\mathbf{H}^{(1)}]^*$	$L(L+1)/2$	$\mathfrak{u}(L) \setminus \mathfrak{o}(L)$	$U(L)/O(L)$
AII(−1, 0, 0)	$L = 2M$	$\begin{pmatrix} \mathbf{h}_{aa} & \mathbf{h}_{ab} \\ -\mathbf{h}^*_{ab} & \mathbf{h}^*_{aa} \end{pmatrix}$	$M(2M-1)$	$\mathfrak{u}(2M) \setminus \mathfrak{usp}(2M)$	$U(2M)/USp(2M)$
D(0, +1, 0)	L	$\mathbf{H}^{(1)} = -[\mathbf{H}^{(1)}]^*$	$L(L-1)/2$	$\mathfrak{o}(L)$	$O(L)$
C(0, −1, 0)	$L = 2M$	$\begin{pmatrix} \mathbf{h}_{aa} & \mathbf{h}_{ab} \\ \mathbf{h}^*_{ab} & -\mathbf{h}^*_{aa} \end{pmatrix}$	$M(2M+1)$	$\mathfrak{usp}(2M)$	$USp(2M)$
AIII(0, 0, 1)	$L = p + q$	$\begin{pmatrix} \mathbf{0}_{pp} & \mathbf{h}_{pq} \\ \mathbf{h}^\dagger_{pq} & \mathbf{0}_{qq} \end{pmatrix}$	$2pq$	$\mathfrak{u}(p+q) \setminus (\mathfrak{u}(p) \oplus \mathfrak{u}(q))$	$U(p+q)/(U(p) \times U(q))$
BDI(+1, +1, 1)	$L = p + q$	$\begin{pmatrix} \mathbf{0}_{pp} & \mathbf{h}_{pq} \\ \mathbf{h}^T_{pq} & \mathbf{0}_{qq} \end{pmatrix}, \mathbf{h}^*_{pq} = \mathbf{h}_{pq}$	pq	$\mathfrak{o}(p+q) \setminus (\mathfrak{o}(p) \oplus \mathfrak{o}(q))$	$O(p+q)/(O(p) \times O(q))$
CII(−1, −1, 1)	$L = p + q$, $p = 2r, q = 2s$	$\left(\begin{array}{c\|cc} \mathbf{0}_{pp} & \mathbf{h}_{rr} & \mathbf{h}_{rs} \\ & -\mathbf{h}^*_{rs} & \mathbf{h}^*_{rr} \\ \hline \text{h.c.} & \multicolumn{2}{c}{\mathbf{0}_{qq}} \end{array}\right)$	$4rs$	$\mathfrak{usp}(p+q) \setminus (\mathfrak{usp}(p) \oplus \mathfrak{usp}(q))$	$USp(2(r+s))/(USp(2r) \times USp(2s))$
CI(+1, -1, 1)	$L = 2M$	$\begin{pmatrix} \mathbf{0}_{MM} & \mathbf{h}_{MM} \\ \mathbf{h}^*_{MM} & \mathbf{0}_{MM} \end{pmatrix}, \mathbf{h}^T_{MM} = \mathbf{h}_{MM}$	$M(M+1)$	$\mathfrak{usp}(2M) \setminus \mathfrak{u}(M)$	$USp(2M)/U(M)$
DIII(-1, +1, 1)	$L = 2M$	$\begin{pmatrix} \mathbf{0}_{MM} & \mathbf{h}_{MM} \\ -\mathbf{h}^*_{MM} & \mathbf{0}_{MM} \end{pmatrix}, \mathbf{h}^T_{MM} = -\mathbf{h}_{MM}$	$M(M-1)$	$\mathfrak{o}(2M) \setminus \mathfrak{u}(M)$	$O(2M)/U(M)$

- Structure of Hamiltonian in each class
- Time evolution operator runs over a coset space
- All ten families of Cartan's symmetric spaces are realized

Interacting Systems

- Systems with *up to* N-body interactions

$$\mathscr{H} = \sum_{K=0}^{N} (\Psi^\dagger)^K \mathbf{H}^{(K)} (\Psi)^K$$

  $\mathbf{H}^{(K)}$ is a $\binom{L}{K} \times \binom{L}{K}$ matrix,... Hamiltonian specified by an $N+1$-tuple

$$\mathbf{H} = (\mathbf{H}^{(0)}, \mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(N)})$$

  Goal is to find the space of all $\mathbf{H}$ allowed by symmetry in each of the ten classes
- Main point: $\mathbf{H}^{(K)}$ will depend on $\mathbf{H}^{(R)}$ for all $K < R < N$ when the class has a transposing symmetry (recall μ in the Hubbard model at half filling depends on U)....iterative determination of $\mathbf{H}^{(K)}$ starting from $\mathbf{H}^{(N)}$
- Complete solution presented in 1606.05483

Tenfold Way: N-Body Hamiltonians (N even)

Class	L	P	Q	$\mathbf{H}^{(N)}$	$\dim i\mathcal{H}^{(N)}$	$i\mathcal{H}^{N}_{+}$
A (0,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^{\dagger}$	P^2	$\mathfrak{u}(P)$
AI (+1,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^{*}$	$P(P+1)/2$	$\mathfrak{u}(P) \setminus \mathfrak{o}(P)$
AII (-1,0,0)	$L = 2M$	$\frac{1}{2}\left\{\binom{L}{N} + \binom{M}{N/2}\right\}$	$\frac{1}{2}\left\{\binom{L}{N} - \binom{M}{N/2}\right\}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{h}^{(N)}_{PQ} \\ \left[\mathbf{h}^{(N)}_{PQ}\right]^{\dagger} & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^{*}$, $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^{*}$, $\mathbf{h}^{(N)}_{PQ} = -\left[\mathbf{h}^{(N)}_{PQ}\right]^{*}$	$\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$	$\mathfrak{u}(P+Q)$ $\setminus \mathfrak{o}(P+Q)$
D (0,+1,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = \left[\mathbf{H}^{(N)}\right]^{*}$	$P(P+1)/2$	$\mathfrak{u}(P) \setminus \mathfrak{o}(P)$
C (0,-1,0)	$L = 2M$	$\frac{1}{2}\left\{\binom{L}{N} + \binom{M}{N/2}\right\}$	$\frac{1}{2}\left\{\binom{L}{N} - \binom{M}{N/2}\right\}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{h}^{(N)}_{PQ} \\ \left[\mathbf{h}^{(N)}_{PQ}\right]^{\dagger} & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^{*}$, $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^{*}$, $\mathbf{h}^{(N)}_{PQ} = -\left[\mathbf{h}^{(N)}_{PQ}\right]^{*}$	$\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2} + PQ$	$\mathfrak{u}(P+Q)$ $\setminus \mathfrak{o}(P+Q)$
AIII (0,0,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{0}_{PQ} \\ \mathbf{0}_{QP} & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^{\dagger}$, $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^{\dagger}$	$P^2 + Q^2$	$\mathfrak{u}(P) \oplus \mathfrak{u}(Q)$
BDI (+1,+1,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{h}^{(N)}_{PP} & \mathbf{0}_{PQ} \\ \mathbf{0}_{QP} & \mathbf{h}^{(N)}_{QQ} \end{pmatrix}$ $\mathbf{h}^{(N)}_{PP} = \left[\mathbf{h}^{(N)}_{PP}\right]^{*}$, $\mathbf{h}^{(N)}_{QQ} = \left[\mathbf{h}^{(N)}_{QQ}\right]^{*}$	$\frac{P(P+1)}{2} + \frac{Q(Q+1)}{2}$	$(\mathfrak{u}(P) \setminus \mathfrak{o}(P))$ $\oplus$ $(\mathfrak{u}(Q) \setminus \mathfrak{o}(Q))$
CII (-1,-1,1)	$L = p + q$ $p = 2r \quad q = 2s$	$P = \sum_{a=1,3,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$ $A(B) = P/2$	$Q = \sum_{a=0,2,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$ $C(D) = \frac{Q}{2} \pm \sum_{a=0,2,\ldots}^{m} \frac{1}{2}\binom{r}{a/2}\binom{s}{(m-a)/2}$	$\begin{pmatrix} \begin{matrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} \\ \left[\mathbf{h}^{(N)}_{AB}\right]^{\dagger} & \mathbf{h}^{(N)}_{BB} \end{matrix} & \mathbf{0}_{PQ} \\ \mathbf{0}_{QP} & \begin{matrix} \mathbf{h}^{(N)}_{CC} & \mathbf{h}^{(N)}_{CD} \\ \left[\mathbf{h}^{(N)}_{CD}\right]^{\dagger} & \mathbf{h}^{(N)}_{DD} \end{matrix} \end{pmatrix}$ $\mathbf{h}^{(N)}_{AA(CC)} = \left[\mathbf{h}^{(N)}_{AA(CC)}\right]^{*}$, $\mathbf{h}^{(N)}_{BB(DD)} = \left[\mathbf{h}^{(N)}_{BB(DD)}\right]^{*}$, $\mathbf{h}^{(N)}_{AB(CD)} = -\left[\mathbf{h}^{(N)}_{AB(CD)}\right]^{*}$	$\frac{A(A+1)}{2} + \frac{B(B+1)}{2} + AB$ $+ \frac{C(C+1)}{2} + \frac{D(D+1)}{2} + CD$	$(\mathfrak{u}(A+B)$ $\setminus \mathfrak{o}(A+B))$ $\oplus$ $(\mathfrak{u}(C+D)$ $\setminus \mathfrak{o}(C+D))$
CI (+1,-1,1)	$L = 2M$	$P = \sum_{a=1,3,\ldots}^{N-1} \binom{M}{a}\binom{M}{N-a}$ $A(B) = \begin{cases} P/2 & ; N/2 \text{ even} \\ \frac{P}{2} \mp \binom{M}{N/2} & ; N/2 \text{ odd} \end{cases}$	$Q = \sum_{a=0,2,\ldots}^{N} \binom{M}{a}\binom{M}{N-a}$ $C(D) = \begin{cases} \frac{Q}{2} \pm \binom{M}{N/2} & ; N/2 \text{ even} \\ Q/2 & ; N/2 \text{ odd} \end{cases}$	$\begin{pmatrix} \begin{matrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} \\ \left[\mathbf{h}^{(N)}_{AB}\right]^{\dagger} & \mathbf{h}^{(N)}_{BB} \end{matrix} & \mathbf{0}_{PQ} \\ \mathbf{0}_{QP} & \begin{matrix} \mathbf{h}^{(N)}_{CC} & \mathbf{h}^{(N)}_{CD} \\ \left[\mathbf{h}^{(N)}_{CD}\right]^{\dagger} & \mathbf{h}^{(N)}_{DD} \end{matrix} \end{pmatrix}$ $\mathbf{h}^{(N)}_{AA(CC)} = \left[\mathbf{h}^{(N)}_{AA(CC)}\right]^{*}$, $\mathbf{h}^{(N)}_{BB(DD)} = \left[\mathbf{h}^{(N)}_{BB(DD)}\right]^{*}$, $\mathbf{h}^{(N)}_{AB(CD)} = -\left[\mathbf{h}^{(N)}_{AB(CD)}\right]^{*}$	$\frac{A(A+1)}{2} + \frac{B(B+1)}{2} + AB$ $+ \frac{C(C+1)}{2} + \frac{D(D+1)}{2} + CD$	-do-
DIII (-1,+1,1)	$L = 2M$	$P = \sum_{a=1,3,\ldots}^{N-1} \binom{M}{a}\binom{M}{N-a}$ $A(B) = \begin{cases} P/2 & ; N/2 \text{ even} \\ \frac{P}{2} \pm \binom{M}{N/2} & ; N/2 \text{ odd} \end{cases}$	$Q = \sum_{a=0,2,\ldots}^{N} \binom{M}{a}\binom{M}{N-a}$ $C(D) = \begin{cases} \frac{Q}{2} \pm \binom{M}{N/2} & ; N/2 \text{ even} \\ Q/2 & ; N/2 \text{ odd} \end{cases}$	$\begin{pmatrix} \begin{matrix} \mathbf{h}^{(N)}_{AA} & \mathbf{h}^{(N)}_{AB} \\ \left[\mathbf{h}^{(N)}_{AB}\right]^{\dagger} & \mathbf{h}^{(N)}_{BB} \end{matrix} & \mathbf{0}_{PQ} \\ \mathbf{0}_{QP} & \begin{matrix} \mathbf{h}^{(N)}_{CC} & \mathbf{h}^{(N)}_{CD} \\ \left[\mathbf{h}^{(N)}_{CD}\right]^{\dagger} & \mathbf{h}^{(N)}_{DD} \end{matrix} \end{pmatrix}$ $\mathbf{h}^{(N)}_{AA(CC)} = \left[\mathbf{h}^{(N)}_{AA(CC)}\right]^{*}$, $\mathbf{h}^{(N)}_{BB(DD)} = \left[\mathbf{h}^{(N)}_{BB(DD)}\right]^{*}$, $\mathbf{h}^{(N)}_{AB(CD)} = -\left[\mathbf{h}^{(N)}_{AB(CD)}\right]^{*}$	$\frac{A(A+1)}{2} + \frac{B(B+1)}{2} + AB$ $+ \frac{C(C+1)}{2} + \frac{D(D+1)}{2} + CD$	-do-

(1606.05483)

Tenfold Way: N-Body Hamiltonians (N odd)

Class	L	P	Q	$\mathbf{H}^{(N)}$	$\dim i\mathcal{H}^{(N)}$	$i\mathcal{H}_+^{(N)}$
A (0,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = [\mathbf{H}^{(N)}]^\dagger$	P^2	$\mathfrak{u}(P)$
AI (+1,0,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = [\mathbf{H}^{(N)}]^*$	$P(P+1)/2$	$\mathfrak{u}(P) \setminus \mathfrak{o}(P)$
AII (-1,0,0)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{h}_{PP}^{(N)} & \mathbf{h}_{PQ}^{(N)} \\ -[\mathbf{h}_{PQ}^{(N)}]^* & [\mathbf{h}_{PP}^{(N)}]^* \end{pmatrix}$	$P^2 + 2 \times \frac{P(P-1)}{2}$	$\mathfrak{u}(2P) \setminus \mathfrak{usp}(2P)$
D (0,+1,0)	L	$\binom{L}{N}$	–	$\mathbf{H}^{(N)} = -[\mathbf{H}^{(N)}]^*$	$P(P-1)/2$	$\mathfrak{o}(P)$
C (0,-1,0)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{h}_{PP}^{(N)} & \mathbf{h}_{PQ}^{(N)} \\ [\mathbf{h}_{PQ}^{(N)}]^* & -[\mathbf{h}_{PP}^{(N)}]^* \end{pmatrix}$	$P^2 + 2 \times \frac{P(P+1)}{2}$	$\mathfrak{usp}(2P)$
AIII (0,0,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ [\mathbf{h}_{PQ}^{(N)}]^\dagger & \mathbf{0}_{QQ} \end{pmatrix}$	$2PQ$	$\mathfrak{u}(P+Q) \setminus (\mathfrak{u}(P) \oplus \mathfrak{u}(Q))$
BDI (+1,+1,1)	$L = p + q$	$\sum_{a=1,3,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$	$\sum_{a=0,2,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ [\mathbf{h}_{PQ}^{(N)}]^T & \mathbf{0}_{QQ} \end{pmatrix}$	PQ	$\mathfrak{o}(P+Q) \setminus (\mathfrak{o}(P) \oplus \mathfrak{o}(Q))$
CII (-1,-1,1)	$L = p + q$ $p = 2r \quad q = 2s$	$P = \sum_{a=1,3,\ldots}^{N} \binom{p}{a}\binom{q}{N-a}$ $A(B) = P/2$	$Q = \sum_{a=0,2,\ldots}^{N-1} \binom{p}{a}\binom{q}{N-a}$ $C(D) = \frac{Q}{2}$	$\left(\begin{array}{c\|cc} \mathbf{0}_{PP} & \mathbf{h}_{AC}^{(N)} & \mathbf{h}_{AD}^{(N)} \\ & -[\mathbf{h}_{AD}^{(N)}]^* & [\mathbf{h}_{AC}^{(N)}]^* \\ \hline \text{h.c.} & \multicolumn{2}{c}{\mathbf{0}_{QQ}} \end{array}\right)$	PQ	$\mathfrak{usp}(P+Q) \setminus (\mathfrak{usp}(P) \oplus \mathfrak{usp}(Q))$
CI (+1,-1,1)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ [\mathbf{h}_{PQ}^{(N)}]^* & \mathbf{0}_{QQ} \end{pmatrix}$	$P(P+1)$	$\mathfrak{usp}(2P) \setminus \mathfrak{u}(P)$
DIII (-1,+1,1)	$L = 2M$	$\frac{1}{2}\binom{2M}{N}$	$\frac{1}{2}\binom{2M}{N}$	$\begin{pmatrix} \mathbf{0}_{PP} & \mathbf{h}_{PQ}^{(N)} \\ -[\mathbf{h}_{PQ}^{(N)}]^* & \mathbf{0}_{QQ} \end{pmatrix}$	$P(P-1)$	$\mathfrak{o}(2P) \setminus \mathfrak{u}(P)$

(1606.05483)

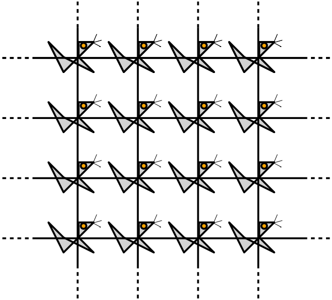
Useful in many contemporary problems: SYK model, many body localization

...on to Topology

- Kitaev, AIP Conference Proceedings, 1134, 22 (2009).
- Ryu, Schnyder, Furusaki, and Ludwig, New Journal of Physics, **12**, 065010 (2010).
- Chiu, Teo, Schnyder, Ryu, arXiv:1505.03535
- Ludwig, arXiv:1512.08882

From GFS to Lattice

- Make a lattice out of GFSs in d-dimensions



- $\Psi_I^\dagger$ – fermion operators at site I
- Non-ordinary symmetries implemented "locally" (simplest case)

$$\mathscr{U}\,\Psi_I^\dagger \mathscr{U}^{-1} = \Psi_I^\dagger \mathbf{U} \quad \text{or} \quad \Psi_I^T \mathbf{U}$$

- Hamiltonian

$$\mathscr{H} = \sum_{IJ} \Psi_I^\dagger \mathbf{H}(I,J) \Psi_J$$

Bands etc.

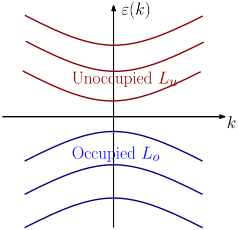
- Bloch picture

$$\mathscr{H} = \sum_{\boldsymbol{k}} \Psi_{\boldsymbol{k}}^{\dagger} \mathbf{H}(\boldsymbol{k}) \Psi_{\boldsymbol{k}}$$

  $\boldsymbol{k}$ is in the 1st Brillouin zone – T_d, the d-torus

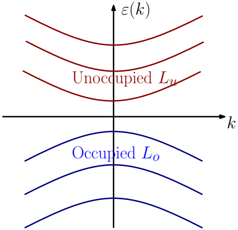
- Symmetries constrain $\mathbf{H}(\boldsymbol{k})$, e.g., time reversal implies $\mathbf{H}(-\boldsymbol{k}) = \mathbf{H}^*(\boldsymbol{k})$...i.e., symmetries determine the "character" of the Bloch states

- Focus on gapped systems...ground state obtained by "filling" bands below the chemical potential $\mu(= 0)$



Ground State...and Topology

- Ground state $|GS\rangle$ – filled bands below μ



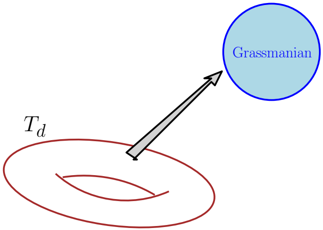
- Two systems $\mathscr{H}_1$ and $\mathscr{H}_2$ in the same symmetry class are topologically equivalent if the there is a continuous deformation of the Hamiltonian from $\mathscr{H}_1$ to $\mathscr{H}_2$ that takes $|GS_1\rangle$ to $|GS_2\rangle$ without closing the gap in the deformation process
- Key question: Given a symmetry class, how many topologically equivalent subclasses are there in d-dimensions?

Topology of Ground States

- Focus on class A: Ground state at any $\boldsymbol{k}$ is a Slater determinant of the occupied Bloch states
- "Gauge freedom" in describing this Slater determinant has to be removed – ground state at $\boldsymbol{k}$ is an object that looks like

$$\frac{U(L)}{U(L_o) \times U(L_u)}$$

a point on a Grassmanian manifold (symmetric space!)
- The ground state can be viewed as a map from T_d to the Grassmanian



- Question: how many topologically distinct ground states are there?
- Look at the homotopy group (Kitaev)

$$\pi_{T_d}(\text{Grassmanian}),$$

in general, $\pi_{T_d}(\text{Symmetric Space})$

Topology of Ground States

- Calculation of homotopy groups is hard! Remarkable simplification occurs when L is "large"

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	Z	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^3	0	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3	Z_{15}	Z_2	Z_2^2	$Z_{12}\times Z_2$	$Z_{84}\times Z_2^2$	Z_2^2
S^4	0	0	0	Z	Z_2	Z_2	$Z\times Z_{12}$	Z_2^2	Z_2^2	$Z_{24}\times Z_3$	Z_{15}	Z_2	Z_2^3	$Z_{120}\times Z_{12}\times Z_2$	$Z_{84}\times Z_2^5$
S^5	0	0	0	0	Z	Z_2	Z_2	Z_{24}	Z_2	Z_2	Z_2	Z_{30}	Z_2	Z_2^3	$Z_{72}\times Z_2$
S^6	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	Z	Z_2	Z_{60}	$Z_{24}\times Z_2$	Z_2^3
S^7	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	Z_{120}	Z_2^3
S^8	0	0	0	0	0	0	0	Z	Z_2	Z_2	Z_{24}	0	0	Z_2	$Z\times Z_{120}$

(Wikipedia)

- Homotopy groups for large L are familiar Abelian groups $(\mathbb{Z}, \mathbb{Z}_2)$

...leads to

Periodic Table

Cartan\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

(Kitaev (2009), Ryu et. al. (2010), Ludwig:1512:08882)

Key features

- In any d there are 5 classes that host topologically distinct states
- Bott periodicity: The table has a periodicity of 2 for the "complex" classes, and a periodicity of 8 for "real" classes
- The "nontrivial" classes in $d+1$ dimension are related to those in d
- Nontrivial topology will reflect in properties, gapless surface states etc...

Periodic Table...for ⟨human beings⟩

Simple illustration of the idea in $d = 0$ with $L = 2$ with a single fermion

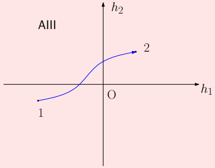
Cartan\d	0
Complex case:	
A	$\mathbb{Z}$
AIII	0
Real case:	
AI	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$
DIII	0
AII	$2\mathbb{Z}$
CII	0
C	0
CI	0

Class AIII

- Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & h_1 + \mathbb{i}h_2 \\ h_1 - \mathbb{i}h_2 & 0 \end{pmatrix}$$

- Eigenvalues $\pm\sqrt{h_1^2 + h_2^2}$, negative state occupied



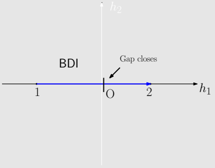
- Can deform any typical system 1 to 2 *without* closing the gap...topologically trivial

Class BDI

- Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & h_1 \\ h_1 & 0 \end{pmatrix}$$

- Eigenvalues $\pm|h_1|$, negative state occupied



- Cannot deform system 1 to 2 closing the gap...two distinct "topologies" described by a "parity" $\mathbb{Z}_2$!

Periodic Table...for ⟨human beings⟩

Simple illustration of the idea in $d = 0$ with $L = 2$ with a single fermion

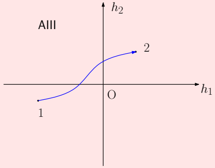
Cartan\\d	0
Complex case:	
A	$\mathbb{Z}$
AIII	0
Real case:	
AI	$\mathbb{Z}$
BDI	$\mathbb{Z}_2$
D	$\mathbb{Z}_2$
DIII	0
AII	$2\mathbb{Z}$
CII	0
C	0
CI	0

Class AIII

- Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & h_1 + \mathbb{i}h_2 \\ h_1 - \mathbb{i}h_2 & 0 \end{pmatrix}$$

- Eigenvalues $\pm\sqrt{h_1^2 + h_2^2}$, negative state occupied



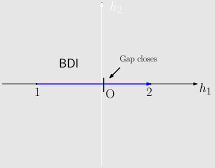
- Can deform any typical system 1 to 2 *without* closing the gap...topologically trivial

Class BDI

- Hamiltonian

$$\mathbf{H} = \begin{pmatrix} 0 & h_1 \\ h_1 & 0 \end{pmatrix}$$

- Eigenvalues $\pm|h_1|$, negative state occupied

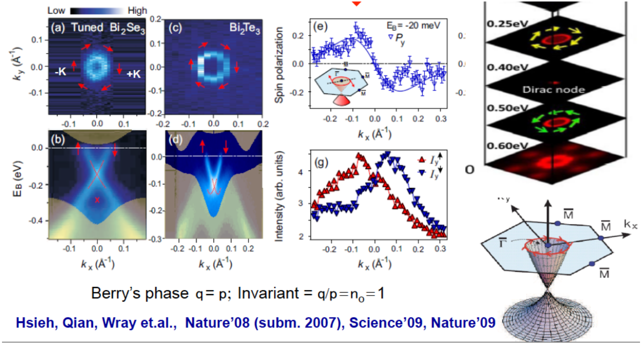


- Cannot deform system 1 to 2 closing the gap...two distinct "topologies" described by a "parity" $\mathbb{Z}_2$!

Key idea: *symmetry protected topology* - SPT phases

Topological Insulators

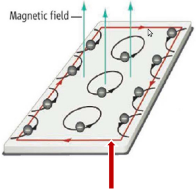
- Such physics can be realized in higher dimensions
- Topological insulators:Insulators in bulk, metals on the surface
- Realized material systems with strong *spin orbit coupling*

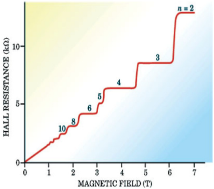


(Pinceton group, BiSb system)

Topological Insulators for Technology

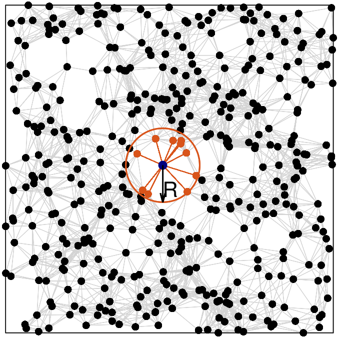
- Insulating bulk and conducting surfaces offer *many* possibilities
- Surface state transport is dissipation-less and quantized...more energy efficient devices
- Useful in metrology (standards), resistance quantized to better than one part in a billion (von Kiltzing (2012))





- Topological insulators combined with other systems such as magnets and superconductors can lead to even more interesting physics (..most recent: discovery of Majorana modes using topological phases (Science, July 21, 2017))...useful in qauntum computing
- Challenge: Finding "good" topological insulators, need topology at room temperature

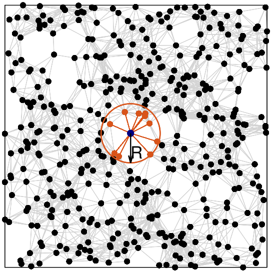
Topological Insulators in Amorphous Systems



Motivating Clues...

- Another route to periodic table: Description of topological phases in terms of non-linear sigma models (NLσM)
- Haldane's work on spin chains: If a topological term is added to NLσM, then the system will be gapless
- NLσM of disordered systems (Wegner et al.) applied to topological phases – A system in d-dimensions will be topological only if its surface, i. e., a $d-1$-dimensional system resists localization, i. e., is gapless.
- Question of if a phase is topological or not can be answered by hunting down if the $d-1$ dimensional NLσM admits a topological term
- ...in this discussion, nowhere is a crystalline lattice assumed...implying topological insulators must be present in amorphous systems

Random Lattices...



- Random set of sites characterized by a density ρ (N sites in an area/volume V; $\rho = N/V$)
- Each site I hosts L single particle states $|I\alpha\rangle, \alpha = 1, \ldots, L$
- Hopping hamiltonian

$$\mathcal{H} = \sum_{I\alpha} \sum_{J\beta} t_{\alpha\beta}(\boldsymbol{r}_{IJ}) c^{\dagger}_{I,\alpha} c_{J,\beta}$$

- Structure of hopping matrix elements

$$t_{\alpha\beta}(\boldsymbol{r} = \boldsymbol{0}) = \varepsilon_{\alpha\beta}, \quad t_{\alpha\beta}(\boldsymbol{r} \neq \boldsymbol{0}) = t(|\boldsymbol{r}|) T_{\alpha,\beta}(\hat{\boldsymbol{r}})$$

- Hopping has a finite range R

$$t(r) = C\Theta(R - r) e^{-r/a}$$

a is an “atomic” length scale

...Hamiltonians Defined

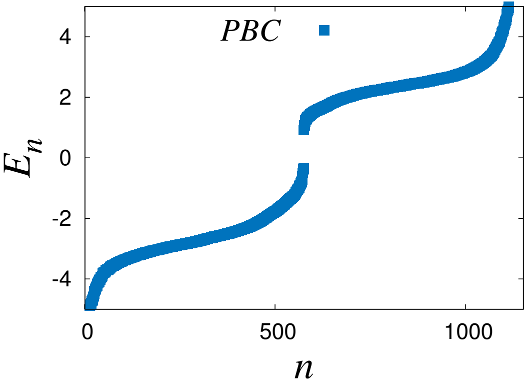
- Symmetry classes obtained by specifying $\varepsilon_{\alpha\beta}$ and $T_{\alpha,\beta}(\hat{\boldsymbol{r}})$
- In 2D ($\hat{\boldsymbol{r}} \equiv (\cos\theta, \sin\theta)$)

Class (par)	$\epsilon_{\alpha\beta}$	$T_{\alpha\beta}(\hat{\boldsymbol{r}})$
A (λ, M, t_2)	$\begin{pmatrix} 2+M & (1-i)\lambda \\ (1+i)\lambda & -(2+M) \end{pmatrix}$	$\begin{pmatrix} \frac{-1+t_2}{2} & \frac{-ie^{-i\theta}+\lambda(\sin^2\theta(1+i)-1)}{2} \\ \frac{-ie^{i\theta}+\lambda(\sin^2\theta(1-i)-1)}{2} & \frac{1+t_2}{2} \end{pmatrix}$
AII (λ, M, t_2, g)	$\begin{pmatrix} 2+M+2t_2 & -i2\lambda & 0 & 0 \\ i2\lambda & -(2+M)+2t_2 & 0 & 0 \\ 0 & 0 & 2+M+2t_2 & i2\lambda \\ 0 & 0 & -i2\lambda & -(2+M)+2t_2 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}-\frac{t_2}{2} & -\frac{i}{2}e^{-i\theta}+\frac{i\lambda}{2} & 0 & -\frac{ig}{2}e^{-i\theta} \\ -\frac{i}{2}e^{i\theta}-\frac{i\lambda}{2} & \frac{1}{2}-\frac{t_2}{2} & -\frac{ig}{2}e^{-i\theta} & 0 \\ 0 & -\frac{ig}{2}e^{i\theta} & -\frac{1}{2}-\frac{t_2}{2} & \frac{i}{2}e^{i\theta}-\frac{i\lambda}{2} \\ -\frac{ig}{2}e^{i\theta} & 0 & \frac{i}{2}e^{-i\theta}+\frac{i\lambda}{2} & \frac{1}{2}-\frac{t_2}{2} \end{pmatrix}$
D (μ, Δ)	$\begin{pmatrix} 2-\mu & 0 \\ 0 & -(2-\mu) \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & \Delta e^{i\theta} \\ -\Delta e^{-i\theta} & \frac{1}{2} \end{pmatrix}$
DIII (M, g)	$\mathrm{AII}(\lambda=0, t_2=0)$	$\mathrm{AII}(\lambda=0, t_2=0)$
C (M)	$\begin{pmatrix} 2+M & 0 \\ 0 & -(2+M) \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2}e^{-i2\theta} \\ -\frac{1}{2}e^{i2\theta} & \frac{1}{2} \end{pmatrix}$

- Focus on class A ($t_2 = 0.25t$, $\lambda = 0.5t$); for a given realization of random sites, hamiltonian tuned by changing M

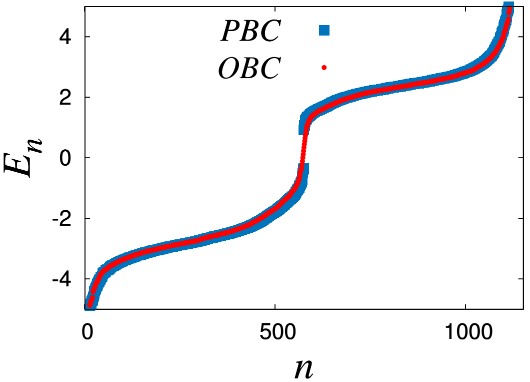
Energy Spectrum

- Study the system with periodic (PBC)...



Energy Spectrum

- Study the system with periodic (PBC)...

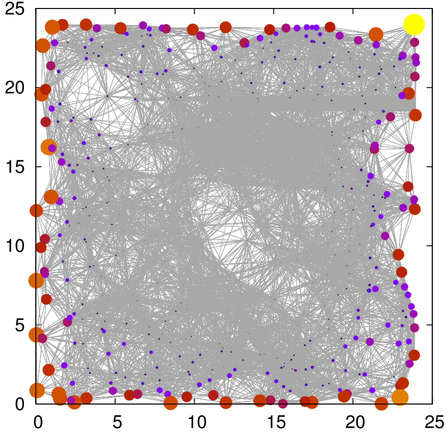


...and open (OBC) boundary conditions (filling is 1 fermion per site)

- In a range of M, PBC shows an energy gap, OBC shows midgap states!

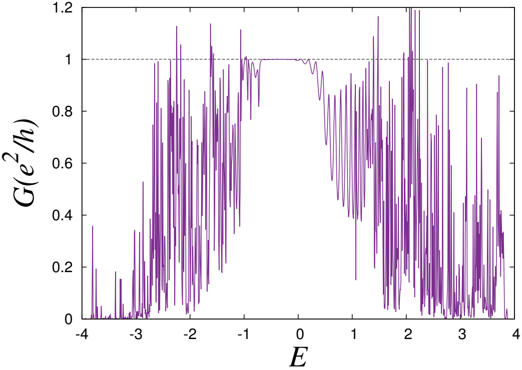
Edge States...

- Midgap states *are* edge states!



... with Quantized Transport

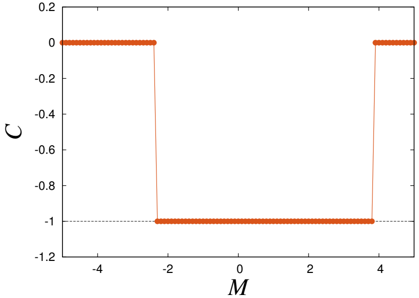
- Quantized conductance in the energy gap



...strongly indicative of a topological state

Topological Invariant

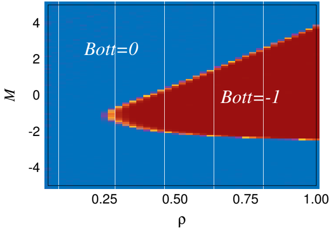
- Characterize the state using the Bott index (Adapted from Hastings and Loring, *EPL* (2010))
- Bott index changes from 0 to -1 upon changing M



- Clear demonstration of a topological state

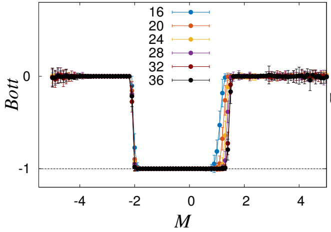
Phase Diagram

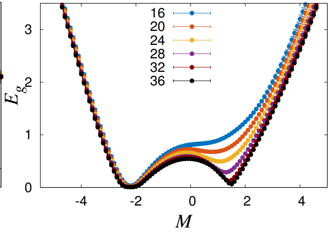
- Configuration averaged phase diagram



Need a critical density sites ρ to obtain a topological phase

- Interesting physics in the transitions between phases by tuning M (for $\rho = 0.6$)

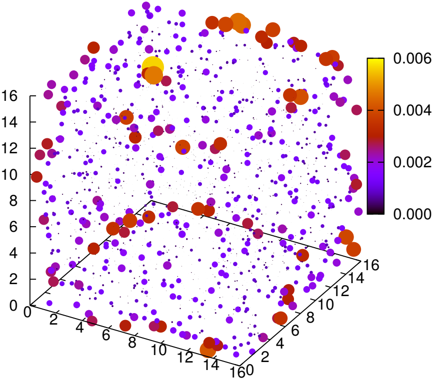




- Similar physics is found in other symmetry classes

Also in 3D

- Time reversal invariant Z_2 topological insulator in a 3D random lattice



Perspective

- New direction in the search for topological materials – **amorphous materials**

Featured in Physics   Editors' Suggestion

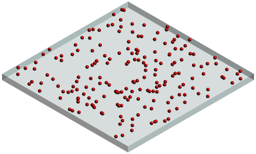
Topological Insulators in Amorphous Systems

Adhip Agarwala and VBS
Phys. Rev. Lett. **118**, 236402

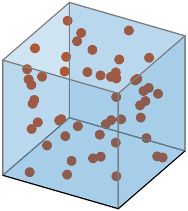
Physics See Synopsis: Glass Materials Could Be Topological Insulators

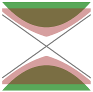
- Engineered systems

  - Randomly deposited 2D motifs on an insulating surface



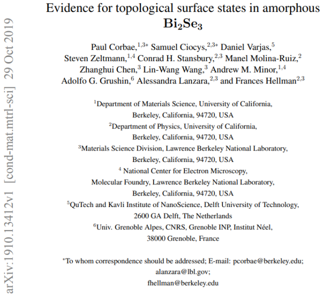
  - Impurity bands in a wide gap insulator?





- Need detailed materials science inputs/considerations

Experimental Realization!



arXiv:1910.13412v1 [cond-mat.mtrl-sci] 29 Oct 2019

Evidence for topological surface states in amorphous Bi_2Se_3

Paul Corbae,[1,3*] Samuel Ciocys,[2,3*] Daniel Varjas,[5]
Steven Zeltmann,[1,4] Conrad H. Stansbury,[2,3] Manel Molina-Ruiz,[2]
Zhanghui Chen,[3] Lin-Wang Wang,[3] Andrew M. Minor,[1,4]
Adolfo G. Grushin,[6] Alessandra Lanzara,[2,3] and Frances Hellman[2,3]

[1]Department of Materials Science, University of California,
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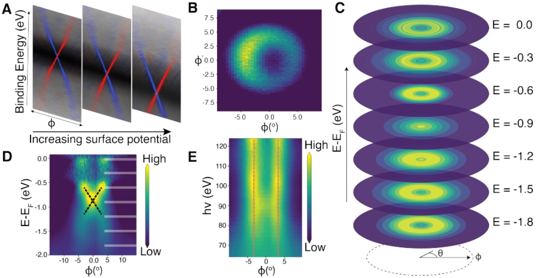
Berkeley, California, 94720, USA
[3]Materials Science Division, Lawrence Berkeley National Laboratory,
Berkeley, California, 94720, USA
[4] National Center for Electron Microscopy,
Molecular Foundry, Lawrence Berkeley National Laboratory,
Berkeley, California, 94720, USA
[5]QuTech and Kavli Institute of NanoScience, Delft University of Technology,
2600 GA Delft, The Netherlands
[6]Univ. Grenoble Alpes, CNRS, Grenoble INP, Institut Néel,
38000 Grenoble, France

*To whom correspondence should be addressed; E-mail: pcorbae@berkeley.edu;
alanzara@lbl.gov;
fhellman@berkeley.edu



- Hunting ground for amorphous topological systems: Amorphous systems with strong spin orbit coupling

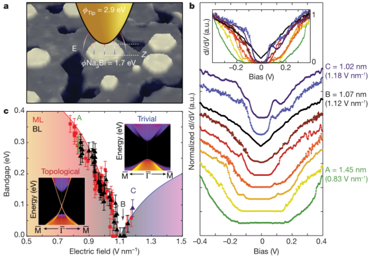
Tunable Topology?

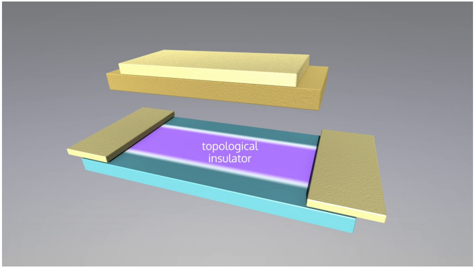
LETTER

https://doi.org/10.1038/s41586-018-0788-5

Electric-field-tuned topological phase transition in ultrathin Na_3Bi

James L. Collins[1,2,3], Anton Tadich[3,4], Weikang Wu[5], Lidia C. Gomes[6,7], Joao N. B. Rodrigues[6,8], Chang Liu[1,2,3], Jack Hellerstedt[1,2,9], Hyejin Ryu[10,11], Shujie Tang[10], Sung-Kwan Mo[10], Shaffique Adam[6,12], Shengyuan A. Yang[5,13], Michael S. Fuhrer[1,2,3] & Mark T. Edmonds[1,2,3]*





(Google images)

- Towards “ultra low energy” electronics!

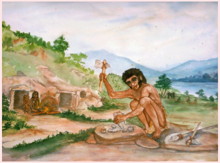
Things Not Discussed...and Open Issues

- Our discussion restricted short range entangled phases of non-interacting systems
- Not discussed: Higher order topological insulators
- Understanding certain gapless phases – Dirac materials, Weyl semi-metals (Next talk by Sumathi)
- Experimental scenario not discussed...e. g. recent discovery of Majorana modes using topological phases
- Realizations in "non-fermionic" systems – Optical/Mechanical/Magnetic systems
- Classification changes when interactions are present (Kitaev et al.: in 1D BDI which is $\mathbb{Z}$ collapses to $\mathbb{Z}_8$)...a lot is understood in 1D
- Higher dimensions is an open problem...many ideas including group cohomology (Wen et. al.) etc...are in the air
- ...and phases with long range entanglement (Wen, 1610.03911) – symmetry enriched topological phases

Key Takeaway...

Every epoch of humanity's progress was controlled by a phase of electrons:

Insulators



Stone age

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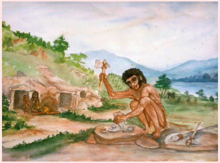

Stone age

Metals

Bronze/Iron age

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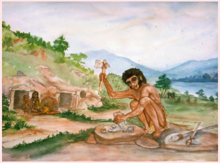

Stone age

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Semiconductors

"Mobile" age

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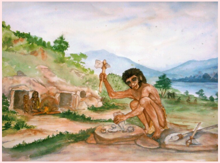


Stone age

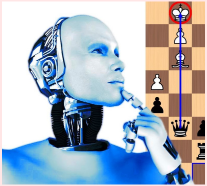
Metals

Bronze/Iron age

Semiconductors

"Mobile" age

SC/Twisted Phases?



"AI" age

Next epoch: **topological "twisted" phases?**

Summary

This talk (Ask me in person for references/review articles)

- Key message: Ideas of topology and entanglement are crucial in understanding/classifying phases of many fermions
- Tenfold way of classifying fermionic systems

Cartan\\d	0	1	2	3	4	5	6	7	8	
Complex case:										
A	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	$\cdots$
AIII	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$
Real case:										
AI	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\cdots$
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\cdots$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\cdots$
DIII	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\cdots$
AII	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	$\cdots$
CII	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\cdots$
C	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	$\cdots$
CI	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	$\cdots$

Open issues

- Topological classification in presence of interactions – **exciting times in condensed matter physics**
- Finding and using topological phases – more efficient electronics to quantum computers – **key new challenge for materials science**