Symmetries in Nuclei

Symmetry, mathematics and physics

Examples of symmetries in quantum mechanics

Symmetries of the nuclear shell model

Example: seniority in the nuclear shell model

ANUP11, Goa, November 2011

Symmetries of the nuclear shell model

The nuclear shell model

Racah's pairing model and seniority

Wigner's supermultiplet model

Elliott's SU(3) model and extensions

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The nuclear shell model

Many-body quantum mechanical problem:

$$\hat{H} = \sum_{k=1}^{A} \frac{p_k^2}{2m_k} + \sum_{k<l}^{A} \hat{V}_2(\boldsymbol{r}_k, \boldsymbol{r}_l)$$

$$= \underbrace{\sum_{k=1}^{A} \left[\frac{p_k^2}{2m_k} + \hat{V}(\boldsymbol{r}_k) \right]}_{\text{mean field}} + \underbrace{\left[\sum_{k<l}^{A} \hat{V}_2(\boldsymbol{r}_k, \boldsymbol{r}_l) - \sum_{k=1}^{A} V(\boldsymbol{r}_k) \right]}_{\text{residual interaction}}$$

Independent-particle assumption. Choose *V* and neglect residual interaction:

$$\hat{H} \approx \hat{H}_{\mathrm{IP}} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m_k} + \hat{V}(\boldsymbol{r}_k) \right]$$

Independent-particle shell model

Solution for one particle:

$$\left[\frac{p^2}{2m} + \hat{V}(\boldsymbol{r})\right]\phi_i(\boldsymbol{r}) = E_i\phi_i(\boldsymbol{r})$$

Solution for many particles:

$$\Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A) = \prod_{k=1}^{A}\phi_{i_k}(\boldsymbol{r}_k)$$

$$\hat{H}_{\mathrm{IP}}\Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A) = \left(\sum_{k=1}^{A} E_{i_k}\right)\Phi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)$$

Independent-particle shell model

Anti-symmetric solution for many particles (Slater determinant):

$$\Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A) = \frac{1}{\sqrt{A!}} \begin{vmatrix} \phi_{i_1}(\boldsymbol{r}_1) & \phi_{i_1}(\boldsymbol{r}_2) & \cdots & \phi_{i_1}(\boldsymbol{r}_A) \\ \phi_{i_2}(\boldsymbol{r}_1) & \phi_{i_2}(\boldsymbol{r}_2) & \cdots & \phi_{i_2}(\boldsymbol{r}_A) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{i_A}(\boldsymbol{r}_1) & \phi_{i_A}(\boldsymbol{r}_2) & \cdots & \phi_{i_A}(\boldsymbol{r}_A) \end{vmatrix}$$

Example for *A=2* particles:

$$\Psi_{i_1 i_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{\sqrt{2}} \left[\phi_{i_1}(\boldsymbol{r}_1)\phi_{i_2}(\boldsymbol{r}_2) - \phi_{i_1}(\boldsymbol{r}_2)\phi_{i_2}(\boldsymbol{r}_1) \right]$$

Hartree-Fock approximation

Vary ϕ_i (*i.e.* *V*) to minimize the expectation value of *H* in a Slater determinant:

$$\delta \frac{\int \Psi^*_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A)\hat{H}\Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A)\,d\boldsymbol{r}_1 d\boldsymbol{r}_2 \ldots d\boldsymbol{r}_A}{\int \Psi^*_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A)\Psi_{i_1 i_2 \ldots i_A}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_A)\,d\boldsymbol{r}_1 d\boldsymbol{r}_2 \ldots d\boldsymbol{r}_A} = 0$$

Poor man's Hartree-Fock

Choose a simple, analytically solvable V that approximates the microscopic HF potential:

$$\hat{H}_{\mathrm{IP}} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m} + \frac{m\omega^2}{2} r_k^2 - \zeta\, \boldsymbol{l}_k \cdot \boldsymbol{s}_k - \kappa\, l_k^2 \right]$$

Contains

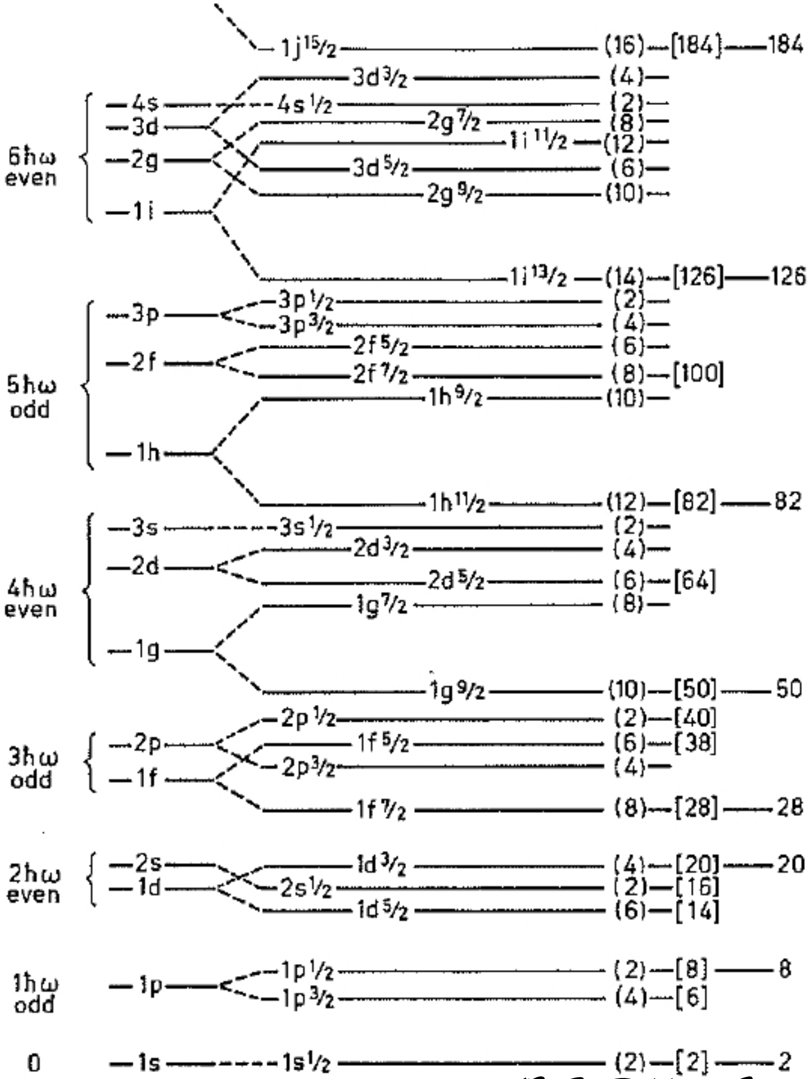
Harmonic oscillator potential with constant ω.

Spin-orbit term with strength ζ.

Orbit-orbit term with strength κ.

Adjust ω, ζ and κ to best reproduce HF.

Single-particle energy levels



Typical parameter values:

$$\hbar\omega \approx 41\, A^{-1/3}\ \text{MeV}$$

$$\zeta\,\hbar^2 \approx 20\, A^{-2/3}\ \text{MeV}$$

$$\kappa\,\hbar^2 \approx 0.1\,\text{MeV}$$

$$\therefore b \approx 1.0\, A^{1/6}\ \text{fm}$$

'Magic' numbers at 2, 8, 20, 28, 50, 82, 126, 184, …

The nuclear shell model

Hamiltonian with one-body term (mean field) and two-body (residual) interactions:

$$\hat{H}_{\mathrm{SM}} = \sum_{k=1}^{A} \hat{U}(\boldsymbol{r}_k) + \sum_{1 \leq k < l}^{A} \hat{W}_2(\boldsymbol{r}_k, \boldsymbol{r}_l)$$

Entirely equivalent form of the same hamiltonian in second quantization:

$$\hat{H}_{\mathrm{SM}} = \sum_{i} \varepsilon_i a_i^+ a_i + \frac{1}{4} \sum_{ijkl} \upsilon_{ijlk} a_i^+ a_j^+ a_k a_l$$

ε, υ: single-particle energies & interactions

ijkl: single-particle quantum numbers

Boson and fermion statistics

Fermions have half-integer spin and obey Fermi-Dirac statistics:

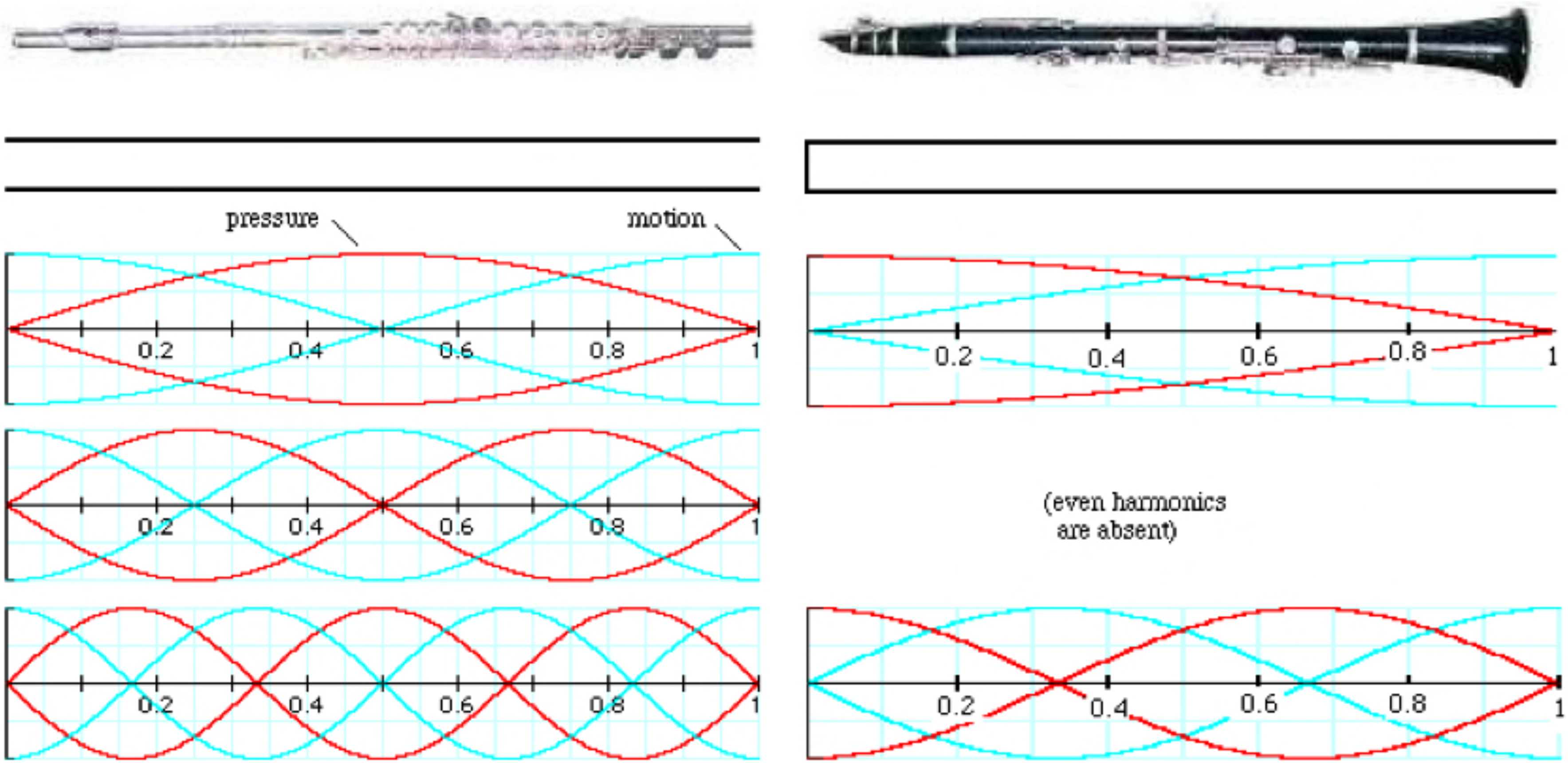
$$\left\{a_i, a_j^+\right\} \equiv a_i a_j^+ + a_j^+ a_i = \delta_{ij}, \quad \left\{a_i, a_j\right\} = \left\{a_i^+, a_j^+\right\} = 0$$

Bosons have integer spin and obey Bose-Einstein statistics:

$$\left[b_i, b_j^+\right] \equiv b_i b_j^+ - b_j^+ b_i = \delta_{ij}, \quad \left[b_i, b_j\right] = \left[b_i^+, b_j^+\right] = 0$$

Matter is carried by fermions. Interactions are carried by bosons. Composite matter particles can be fermions or bosons.

Bosons and fermions



Symmetries of the shell model

Three *bench-mark* solutions:

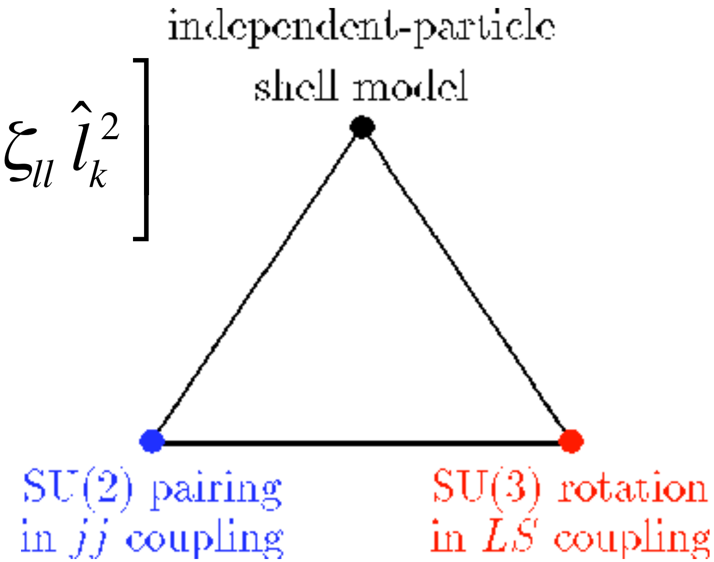
No residual interaction ⇒ IP shell model.

Pairing (in jj coupling) ⇒ Racah's SU(2).

Quadrupole (in LS coupling) ⇒ Elliott's SU(3).

Symmetry triangle:

$$\hat{H} = \sum_{k=1}^{A}\left[\frac{p_k^2}{2m} + \frac{1}{2}m\omega^2 r_k^2 - \zeta_{ls}\,\hat{\boldsymbol{l}}_k\cdot\hat{\boldsymbol{s}}_k - \zeta_{ll}\,\hat{l}_k^2\right] + \sum_{1\le k<l}^{A}\hat{W}_2(\boldsymbol{r}_k,\boldsymbol{r}_l)$$

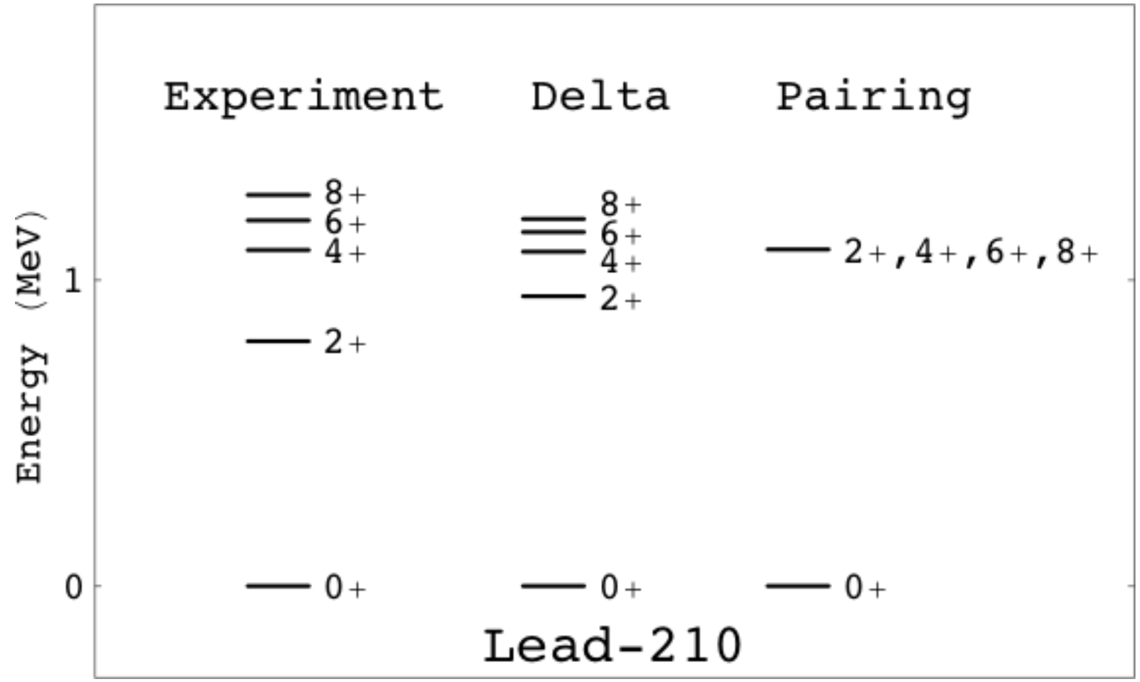


Racah's SU(2) pairing model

Assume pairing interaction in a single-j shell:

$$\left\langle j^2 J M_J \left| \hat{V}_{\text{pairing}} \right| j^2 J M_J \right\rangle = \begin{cases} -\frac{1}{2}(2j+1)g_0, & J=0 \\ 0, & J \neq 0 \end{cases}$$

Spectrum ^{210}Pb:



G. Racah, Phys. Rev. **63** (1943) 367

Pairing SU(2) dynamical symmetry

The pairing hamiltonian,

$$\hat{H} = -g_0 \hat{S}_+ \cdot \hat{S}_-, \quad \hat{S}_+ = \tfrac{1}{2}\sum_m a^+_{jm} a^+_{j\bar{m}}, \quad \hat{S}_- = \left(\hat{S}_+\right)^+$$

…has a *quasi-spin* SU(2) algebraic structure:

$$\left[\hat{S}_+, \hat{S}_-\right] = \tfrac{1}{2}\left(2\hat{n} - 2j - 1\right) \equiv -2\hat{S}_z, \quad \left[\hat{S}_z, \hat{S}_\pm\right] = \pm \hat{S}_\pm$$

H has SU(2) ⊃ SO(2) dynamical symmetry:

$$-g_0 \hat{S}_+ \cdot \hat{S}_- = -g_0\left(\hat{S}^2 - \hat{S}_z^2 + \hat{S}_z\right)$$

Eigensolutions of pairing hamiltonian:

$$-g_0 \hat{S}_+ \cdot \hat{S}_- \left|SM_S\right\rangle = -g_0\left(S(S+1) - M_S\left(M_S - 1\right)\right)\left|SM_S\right\rangle$$

A. Kerman, Ann. Phys. (NY) **12** (1961) 300
K. Helmers, Nucl. Phys. **23** (1961) 594

Interpretation of pairing solution

Quasi-spin labels S and M_S are related to nucleon number n and seniority υ:

$$S = \tfrac{1}{4}(2j - 2\upsilon + 1), \quad M_S = \tfrac{1}{4}(2n - 2j - 1)$$

Energy eigenvalues in terms of n, j and υ:

$$\left\langle j^n \upsilon J M_J \right| -g_0 \hat{S}_+ \cdot \hat{S}_- \left| j^n \upsilon J M_J \right\rangle = -g_0 \tfrac{1}{4}(n - \upsilon)(2j - n + \upsilon + 3)$$

Eigenstates have an S-pair character:

$$\left| j^n \upsilon J M_J \right\rangle \propto \left(\hat{S}_+\right)^{(n-\upsilon)/2} \left| j^\upsilon \upsilon J M_J \right\rangle$$

Seniority υ is the number of nucleons *not* in S pairs (pairs coupled to J=0).

Pairing between identical nucleons

Analytic solution of the pairing hamiltonian based on SU(2) symmetry. *E.g.* energies:

$$\left\langle j^n \upsilon J \right| \sum_{1\leq k<l}^{n} \hat{V}_{\mathrm{pairing}}(k,l) \left| j^n \upsilon J \right\rangle = -g_0 \tfrac{1}{4}(n-\upsilon)(2j-n-\upsilon+3)$$

Seniority υ (number of nucleons not in pairs coupled to *J*=0) is a good quantum number.

Correlated ground-state solution (*cf.* BCS).

Nuclear superfluidity

Ground states of pairing hamiltonian have the following *correlated* character:

Even-even nucleus (υ=0): $\left(\hat{S}_+\right)^{n/2}|\mathrm{o}\rangle, \quad \hat{S}_+ = \frac{1}{2}\sum_m a^+_{jm} a^+_{j\bar{m}}$

Odd-mass nucleus (υ=1): $a^+_{jm}\left(\hat{S}_+\right)^{n/2}|\mathrm{o}\rangle$

Nuclear superfluidity leads to

Constant energy of first 2^+ in even-even nuclei.

Odd-even staggering in masses.

Smooth variation of two-nucleon separation energies with nucleon number.

Two-particle (2n or 2p) transfer enhancement.

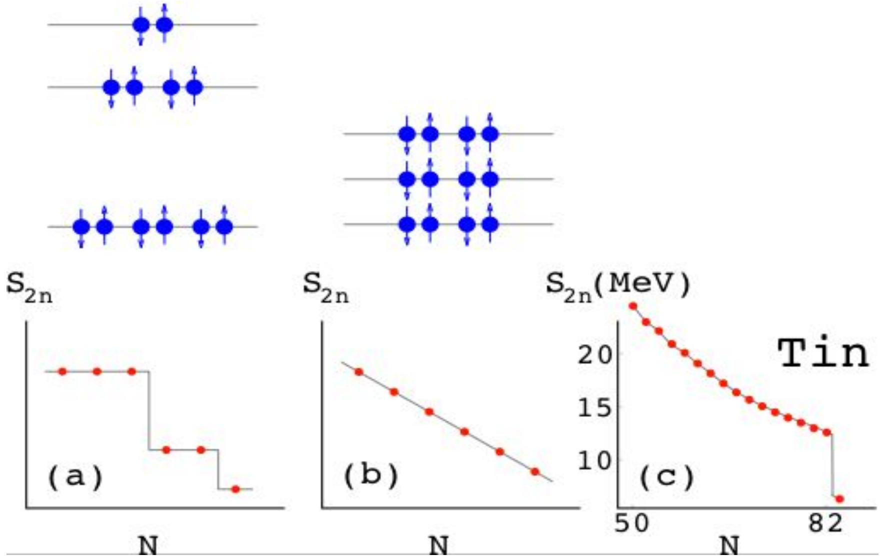
Two-nucleon separation energies

Two-nucleon separation energies S_{2n}:

(a) Shell splitting dominates over interaction.

(b) Interaction dominates over shell splitting.

(c) S_{2n} in tin isotopes.



Pairing gap in semi-magic nuclei

Even-even nuclei:

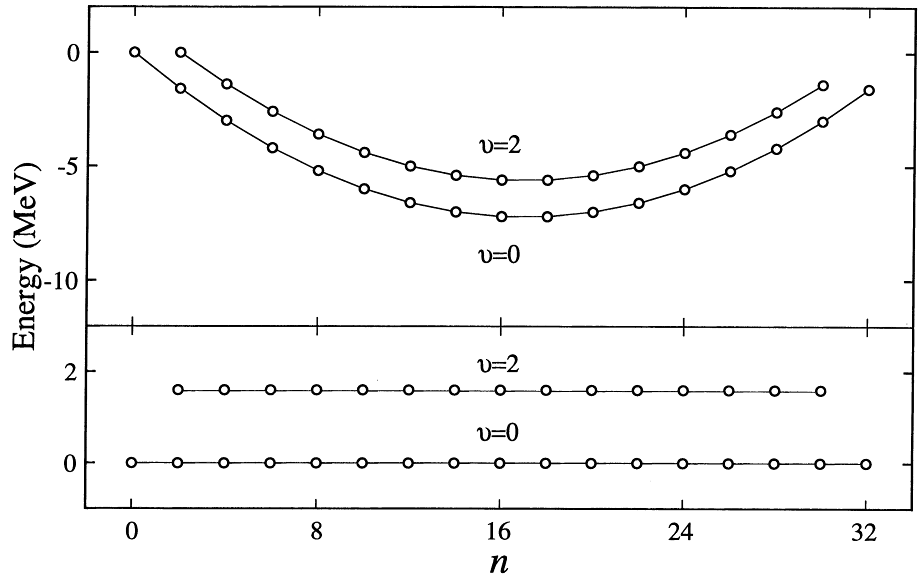
Ground state: υ=0.

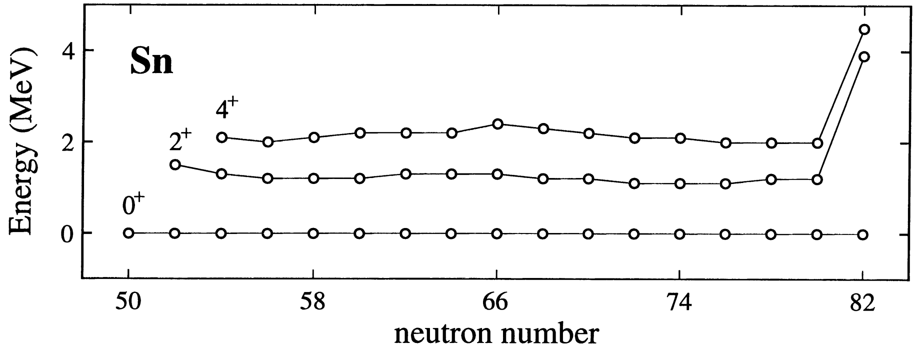
First-excited state: υ=2.

Pairing produces constant excitation energy:

$$E_x(2_1^+) = \tfrac{1}{2}(2j+1)g_0$$

Example of Sn isotopes:





Generalized seniority models

Trivial generalization from a single-*j* shell to several **degenerate** *j* shells.

Non-degenerate shells:

- *Generalized seniority (Talmi).*
- *Integrable pairing models (Richardson, Gaudin, Dukelsky).*

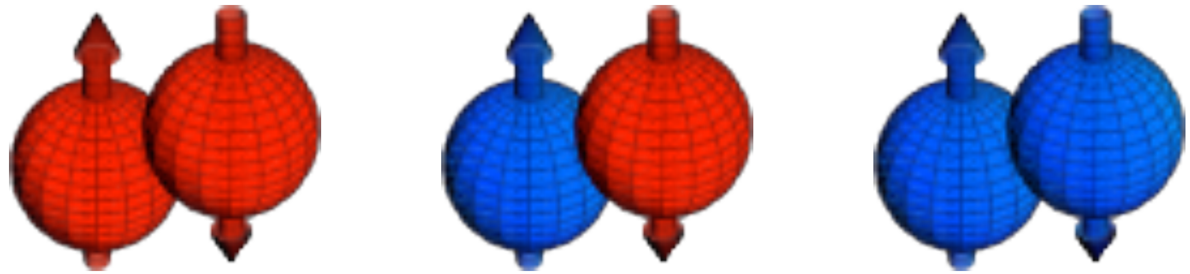
Pairing with neutrons and protons (**isospin**):

- *SO(5) T=1 pairing (Racah, Flowers, Hecht).*
- *SO(8) T=0 & T=1 pairing (Flowers and Szpikowski).*

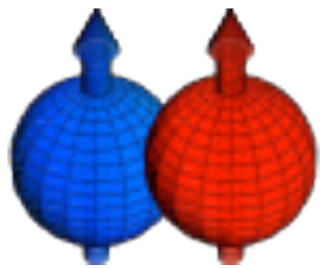
Pairing with neutrons and protons

For neutrons and protons *two* pairs and hence *two* pairing interactions are possible:

$^1S_0$ *isovector or spin singlet (S=0,T=1):* $\hat{S}_+ = \sum_{m>0} a^+_{m\downarrow} a^+_{\bar{m}\uparrow}$



$^3S_1$ *isoscalar or spin triplet (S=1,T=0):* $\hat{P}_+ = \sum_{m>0} a^+_{m\uparrow} a^+_{\bar{m}\uparrow}$



Neutron-proton pairing hamiltonian

The nuclear hamiltonian has two pairing interactions

$$\hat{V}_{\text{pairing}} = -g_0 \hat{S}_+ \cdot \hat{S}_- - g_1 \hat{P}_+ \cdot \hat{P}_-$$

SO(8) algebraic structure.

Integrable and solvable for g_0=0, g_1=0 and g_0=g_1.

B.H. Flowers & S. Szpikowski, Proc. Phys. Soc. **84** (1964) 673

Quartetting in *N=Z* nuclei

Pairing ground state of an *N=Z* nucleus:

$$\left(\cos\theta\,\hat{S}_{+}\cdot\hat{S}_{+}-\sin\theta\,\hat{P}_{+}\cdot\hat{P}_{+}\right)^{n/4}|\mathrm{o}\rangle$$

$\Rightarrow$ Condensate of "α-like" objects.

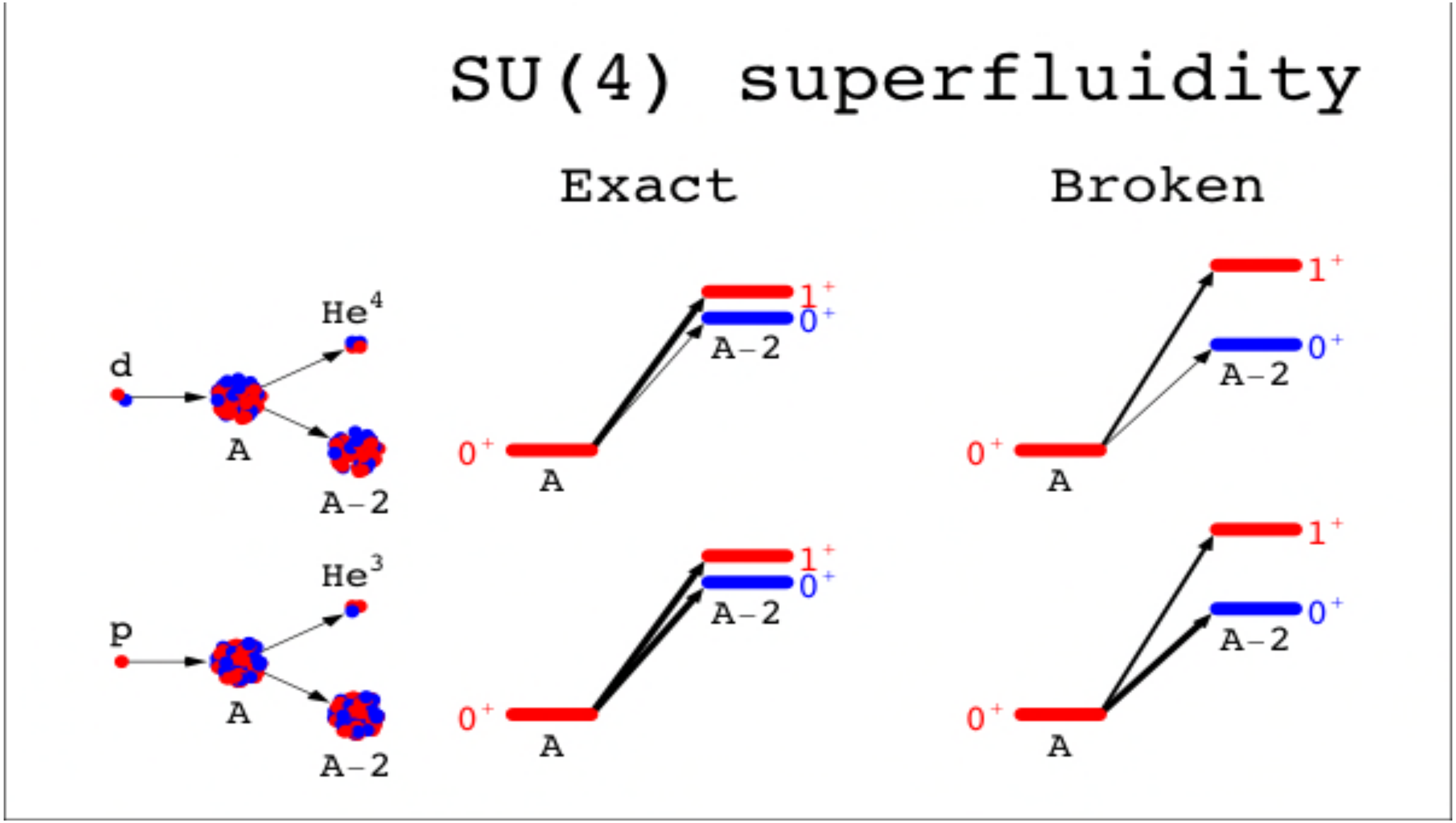
Observations:

Isoscalar component in condensate survives only in N≈Z nuclei, if anywhere at all.

Spin-orbit term reduces *isoscalar component.*

J. Dobes & S. Pittel, Phys. Rev. C **57** (1998) 6883

(d,α) and (p,^{3}He) transfer



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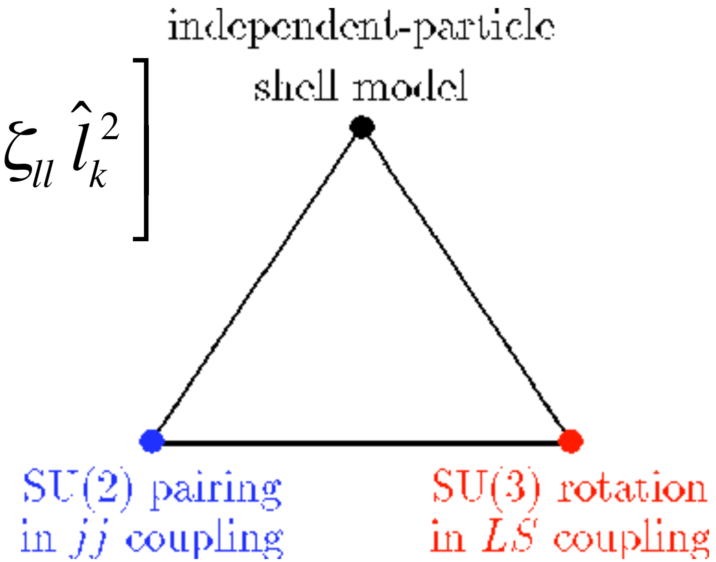


Wigner's SU(4) symmetry

Assume the nuclear hamiltonian is invariant under spin *and* isospin rotations:

$$\left[\hat{H}_{\text{nucl}},\hat{S}_{\mu}\right]=\left[\hat{H}_{\text{nucl}},\hat{T}_{\nu}\right]=\left[\hat{H}_{\text{nucl}},\hat{Y}_{\mu\nu}\right]=0$$

$$\hat{S}_{\mu}=\sum_{k=1}^{A}\hat{s}_{\mu}(k),\quad \hat{T}_{\nu}=\sum_{k=1}^{A}\hat{t}_{\nu}(k),\quad \hat{Y}_{\mu\nu}=\sum_{k=1}^{A}\hat{s}_{\mu}(k)\hat{t}_{\nu}(k)$$

Since $\{S_{\mu},T_{\nu},Y_{\mu\nu}\}$ form an SU(4) algebra:

- *H_{nucl} has SU(4) symmetry.*
- *Total spin S, total orbital angular momentum L, total isospin T and SU(4) labels (λ,μ,ν) are conserved quantum numbers.*

E.P. Wigner, Phys. Rev. **51** (1937) 106
F. Hund, Z. Phys. **105** (1937) 202

Physical origin of SU(4) symmetry

SU(4) labels specify the separate spatial and spin-isospin symmetry of the wave function.

Nuclear interaction is short-range attractive and hence *favours maximal spatial symmetry.*

particle number	spatial symmetry	L	spin–isospin symmetry	$(\lambda\mu\nu)$	(S,T)
1		$0, 2$		(100)	$(\frac{1}{2},\frac{1}{2})$
2	(S)	$0^2, 2^2, 4$	(A)	(010)	(0,1) (1,0)
	(A)	$1, 2, 3$	(S)	(200)	(0,0) (1,1)

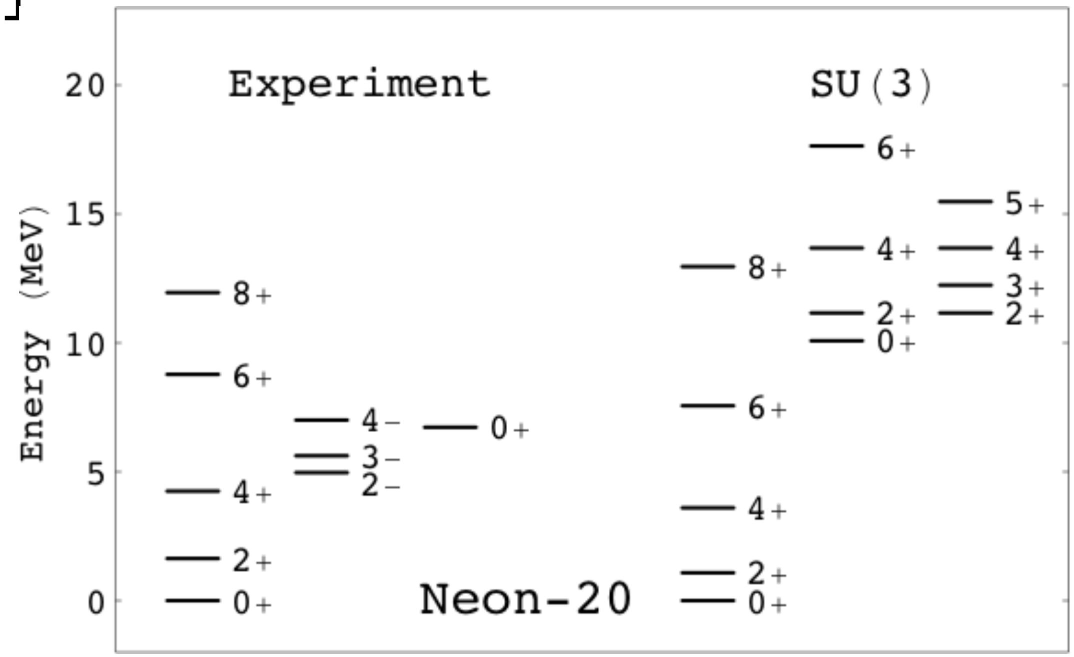
Elliott's SU(3) model of rotation

Harmonic oscillator mean field (*no* spin-orbit) with residual interaction of quadrupole type:

$$\hat{H} = \sum_{k=1}^{A} \left[\frac{p_k^2}{2m} + \frac{1}{2} m\omega^2 r_k^2 \right] - g_2 \hat{Q} \cdot \hat{Q},$$

$$\hat{Q}_\mu \propto \sum_{k=1}^{A} r_k^2 Y_{2\mu}(\hat{\boldsymbol{r}}_k)$$

$$+ \sum_{k=1}^{A} p_k^2 Y_{2\mu}(\hat{\boldsymbol{p}}_k)$$



J.P. Elliott, Proc. Roy. Soc. A **245** (1958) 128; 562

Importance & limitations of SU(3)

Historical importance:

Bridge between the spherical shell model and the liquid-drop model through mixing of orbits.

Spectrum generating algebra of Wigner's SU(4) model.

Limitations:

LS (Russell-Saunders) coupling, not jj coupling (no spin-orbit splitting) ⇒ (beginning of) sd shell.

Q is the algebraic quadrupole operator ⇒ no major-shell mixing.

Breaking of SU(4) symmetry

SU(4) symmetry breaking as a consequence of

Spin-orbit term in nuclear mean field.

Coulomb interaction.

Spin-dependence of the nuclear interaction.

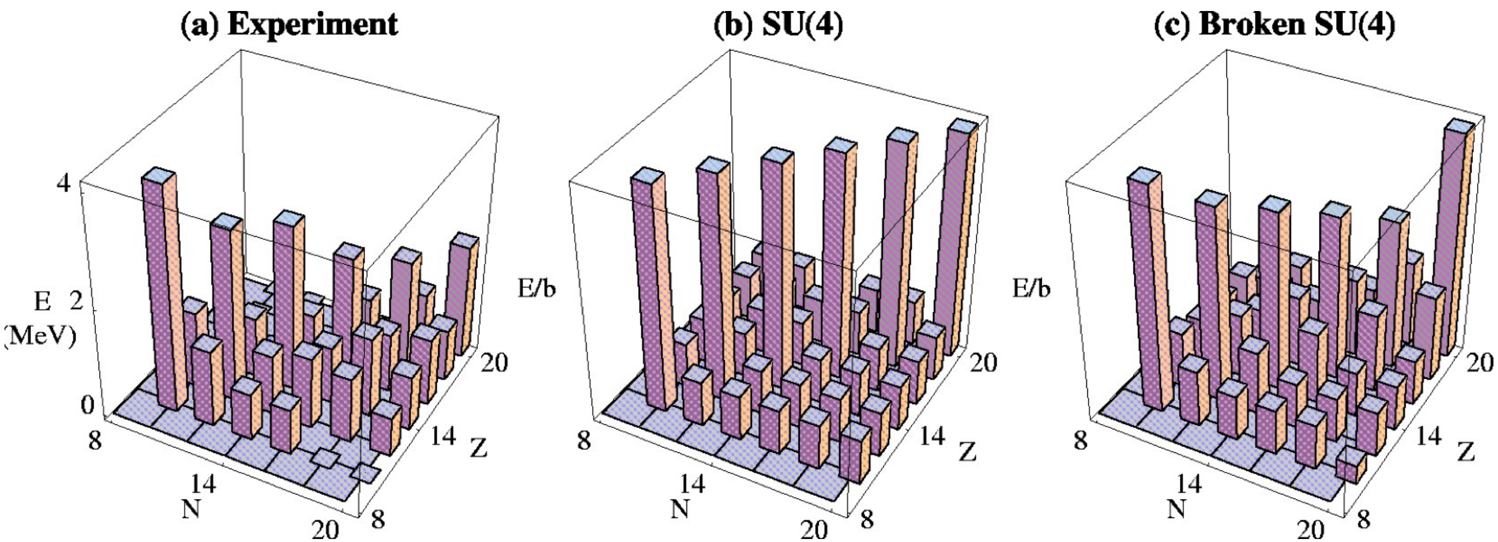
Evidence for SU(4) symmetry breaking from masses and from Gamow-Teller β decay.

SU(4) breaking from masses

Double binding energy difference δV_{np}

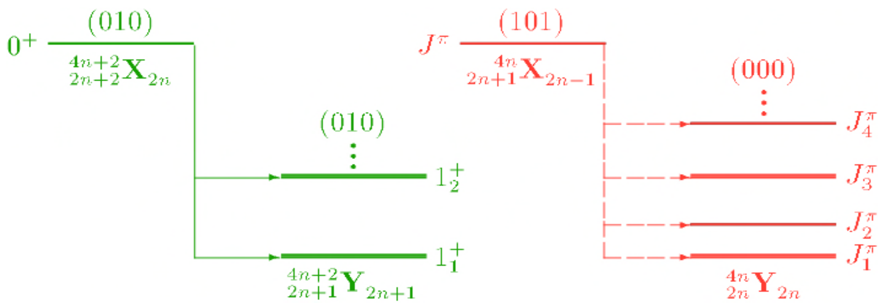
$$\delta V_{\mathrm{np}}(N,Z) = \tfrac{1}{4}\left[B(N,Z) - B(N-2,Z) - B(N,Z-2) + B(N-2,Z-2)\right]$$

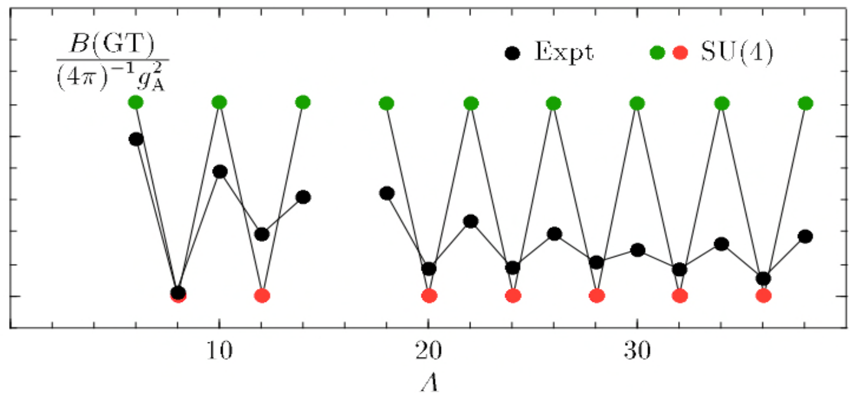
δV_{np} in *sd*-shell nuclei:



P. Van Isacker *et al.*, Phys. Rev. Lett. **74** (1995) 4607

SU(4) breaking from β decay





Gamow-Teller decay into odd-odd or even-even N=Z nuclei.

P. Halse & B.R. Barrett, Ann. Phys. (NY) **192** (1989) 204

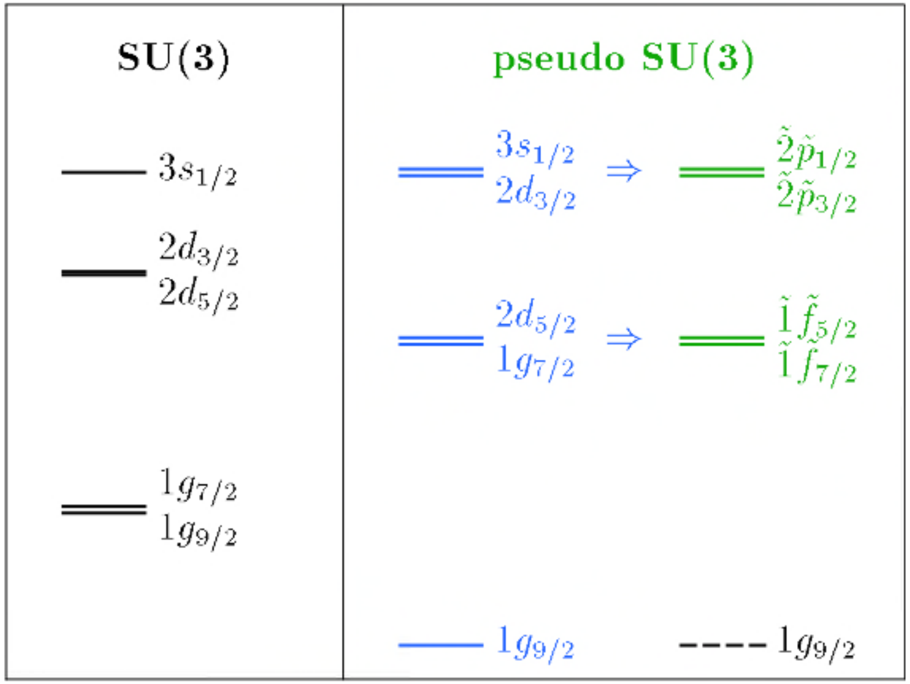
Pseudo-spin symmetry

Apply a *helicity* transformation to the spin-orbit + orbit-orbit nuclear mean field:

$$\hat{u}_k^{-1}\left(\zeta\hat{\boldsymbol{l}}_k\cdot\hat{\boldsymbol{s}}_k+\kappa\hat{\boldsymbol{l}}_k\cdot\hat{\boldsymbol{l}}_k\right)\hat{u}_k=(4\zeta-\kappa)\hat{\boldsymbol{l}}_k\cdot\hat{\boldsymbol{s}}_k+\kappa\hat{\boldsymbol{l}}_k\cdot\hat{\boldsymbol{l}}_k+\mathrm{c}^{\mathrm{te}}$$

$$\hat{u}_k=2i\frac{\hat{\boldsymbol{s}}_k\cdot\boldsymbol{p}_k}{p_k}$$

Degeneracies occur for $4\zeta=\kappa$.



K.T. Hecht & A. Adler, Nucl. Phys. A **137** (1969) 129
A. Arima *et al.*, Phys. Lett. B **30** (1969) 517
R.D. Ratna *et al.*, Nucl. Phys. A **202** (1973) 433
J.N. Ginocchio, Phys. Rev. Lett. **78** (1998) 436

Pseudo-SU(4) symmetry

Assume the nuclear hamiltonian is invariant under *pseudo*-spin and isospin rotations:

$$\left[\hat{H}_{\text{nucl}},\hat{\tilde{S}}_{\mu}\right]=\left[\hat{H}_{\text{nucl}},\hat{T}_{\nu}\right]=\left[\hat{H}_{\text{nucl}},\hat{\tilde{Y}}_{\mu\nu}\right]=0$$

$$\hat{\tilde{S}}_{\mu}=\sum_{k=1}^{A}\hat{\tilde{s}}_{\mu}(k),\quad \hat{T}_{\nu}=\sum_{k=1}^{A}\hat{t}_{\nu}(k),\quad \hat{\tilde{Y}}_{\mu\nu}=\sum_{k=1}^{A}\hat{\tilde{s}}_{\mu}(k)\hat{t}_{\nu}(k)$$

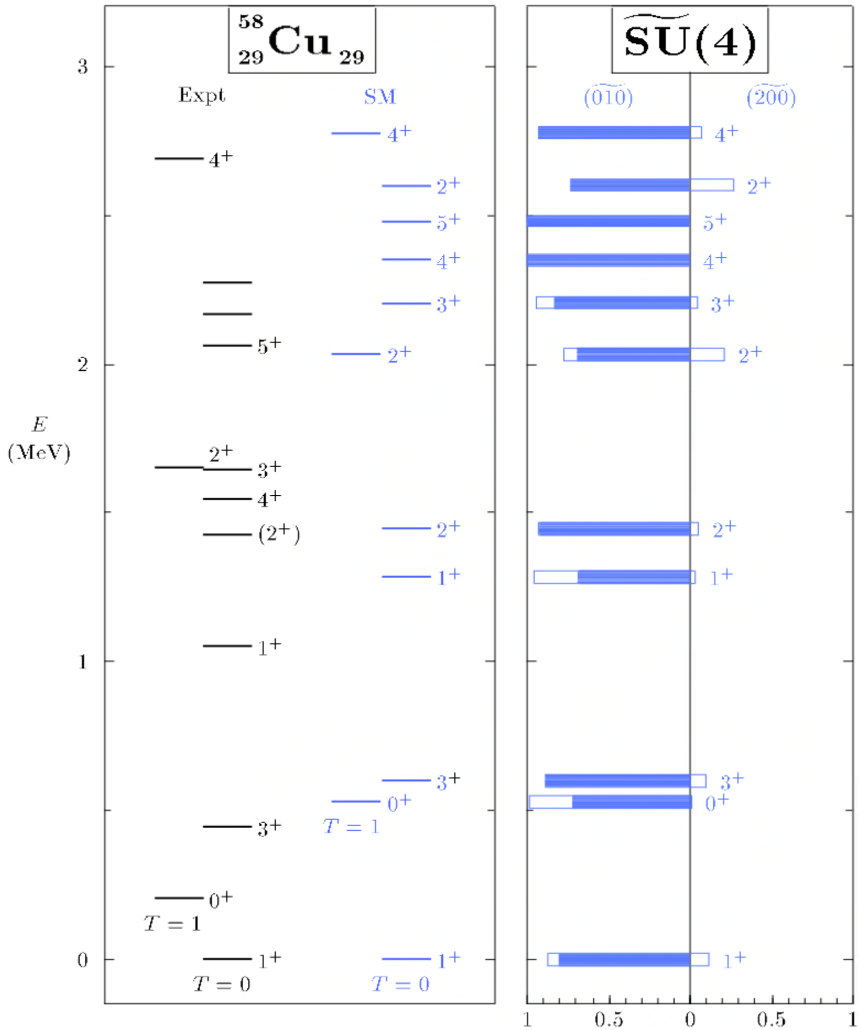
Consequences:

Hamiltonian has pseudo-SU(4) symmetry.

Total pseudo-spin, total pseudo-orbital angular momentum, total isospin and pseudo-SU(4) labels are conserved quantum numbers.

D. Strottman, Nucl. Phys. A **188** (1971) 488

Test of pseudo-SU(4) symmetry



Shell-model test of pseudo-SU(4).

Realistic interaction in $pf_{5/2}g_{9/2}$ space.

Example: ^{58}Cu.

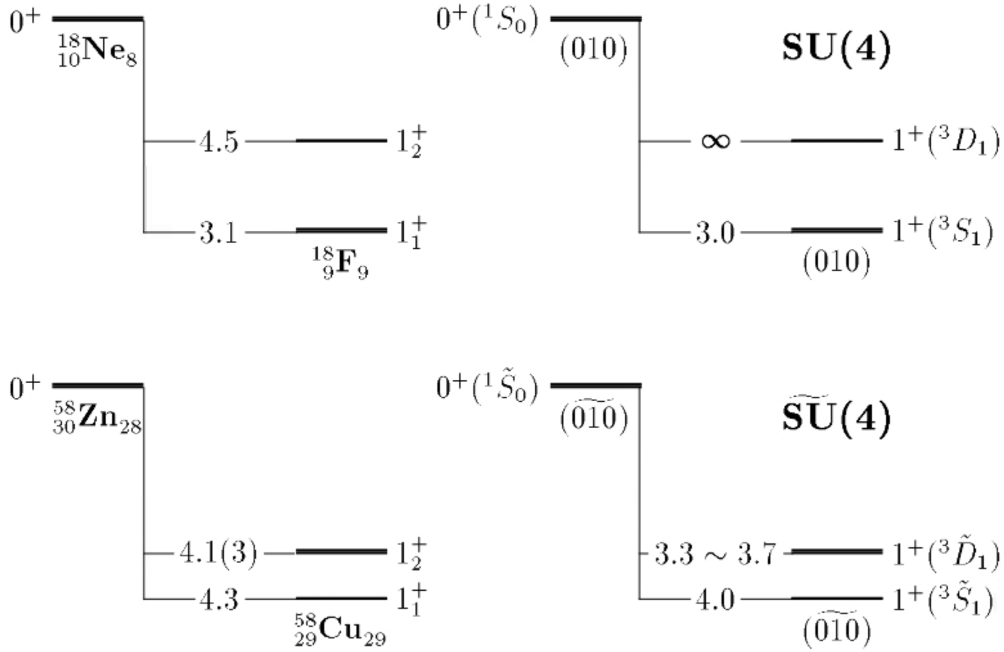
P. Van Isacker *et al.*, Phys. Rev. Lett. **82** (1999) 2060

Pseudo-SU(4) and β decay

Pseudo-spin transformed Gamow-Teller operator is *deformation dependent*:

$$\hat{\tilde{s}}_\mu \hat{t}_\nu \equiv \hat{u}^{-1}\hat{s}_\mu \hat{t}_\nu \hat{u} = -\frac{1}{3}\hat{s}_\mu \hat{t}_\nu + \sqrt{\frac{20}{3}}\,\frac{1}{r^2}\left[\left(\boldsymbol{r}\times\boldsymbol{r}\right)^{(2)}\times\hat{\boldsymbol{s}}\right]^{(1)}_\mu \hat{t}_\nu$$

Test: β decay of ^{18}Ne *vs.* ^{58}Zn.



A. Jokinen *et al.*, Eur. Phys. A. **3** (1998) 271

Three faces of the shell model