Symmetries in Nuclei

Symmetry, mathematics and physics

Examples of symmetries in quantum mechanics

Symmetries of the nuclear shell model

Example: seniority in the nuclear shell model

(Dynamical) symmetries in quantum mechanics

Symmetry in quantum mechanics

The hydrogen atom

The harmonic oscillator

Isospin symmetry in nuclei

Dynamical symmetry

Symmetry in quantum mechanics

Assume a hamiltonian *H* which commutes with operators g_i that form a Lie algebra *G*:

$$\forall \hat{g}_i \in G: \quad \left[\hat{H}, \hat{g}_i\right] = 0$$

$\therefore$ *H* has *symmetry G* or is *invariant under G*.

Lie algebra: a set of (infinitesimal) operators that closes under commutation.

Consequences of symmetry

Degeneracy structure: If $|\gamma\rangle$ is an eigenstate of *H* with energy *E*, so is $g_i|\gamma\rangle$:

$$\hat{H}|\gamma\rangle = E|\gamma\rangle \Rightarrow \hat{H}\hat{g}_i|\gamma\rangle = \hat{g}_i\hat{H}|\gamma\rangle = E\hat{g}_i|\gamma\rangle$$

Degeneracy structure and labels of eigenstates of *H* are determined by algebra *G*:

$$\hat{H}|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle; \quad \hat{g}_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(i)|\Gamma\gamma'\rangle$$

Casimir operators of *G* commute with all g_i:

$$\hat{H} = \sum_m \mu_m \hat{C}_m[G]$$

The hydrogen atom

The hamiltonian of the hydrogen atom is

$$\hat{H} = \frac{p^2}{2M} - \frac{\alpha}{r}$$

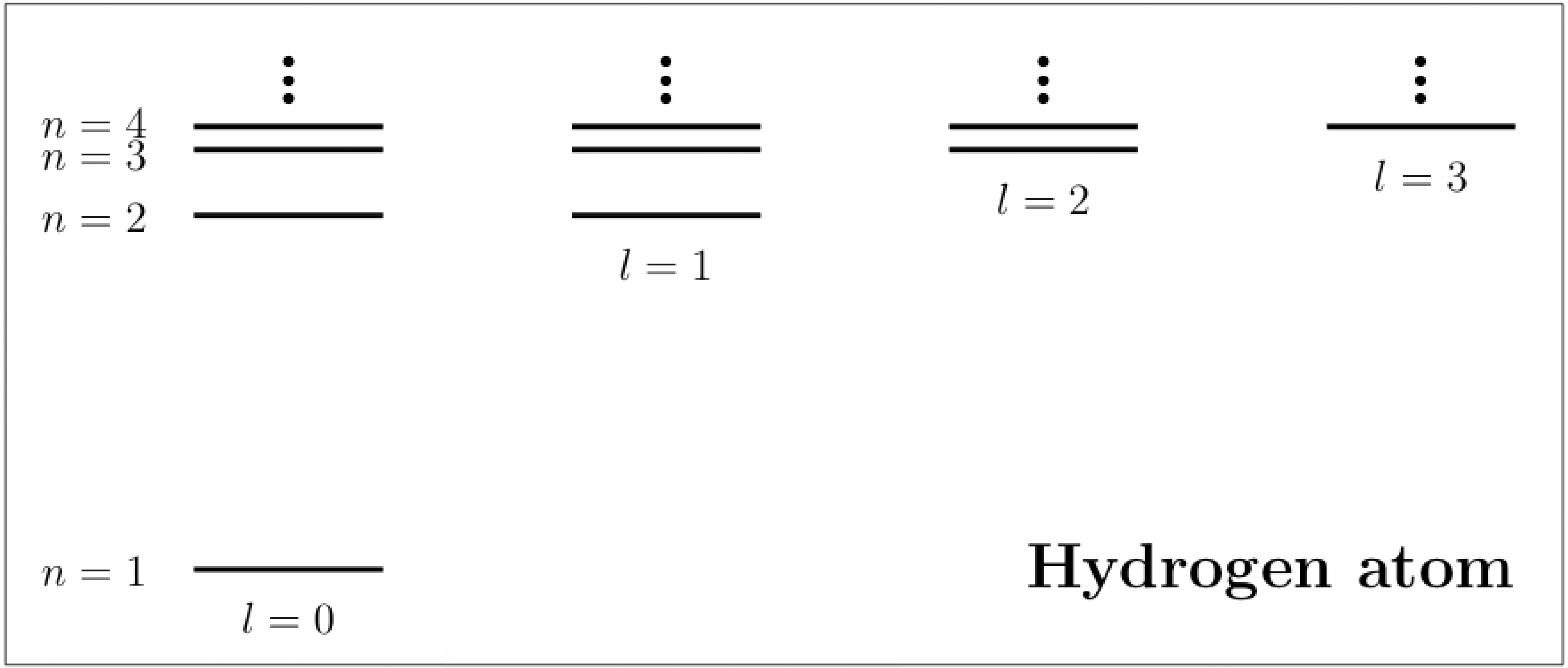
Standard wave quantum mechanics gives

$$\hat{H}\Psi_{nlm}(r,\theta,\varphi) = -\frac{M\alpha^2}{2\hbar^2 n^2}\Psi_{nlm}(r,\theta,\varphi)$$

$$\text{with } n = 1,2,\ldots;\ l = 0,1,\ldots,n-1;\ m = -l,\ldots,+l$$

Degeneracy in *m* originates from rotational symmetry. *What is the origin of l-degeneracy?*

Degeneracies of the H atom



Classical Kepler problem

Conserved quantities:

Energy (a=length semi-major axis):

$$E = -\frac{\alpha}{2a}$$

Angular momentum (ε=eccentricity):

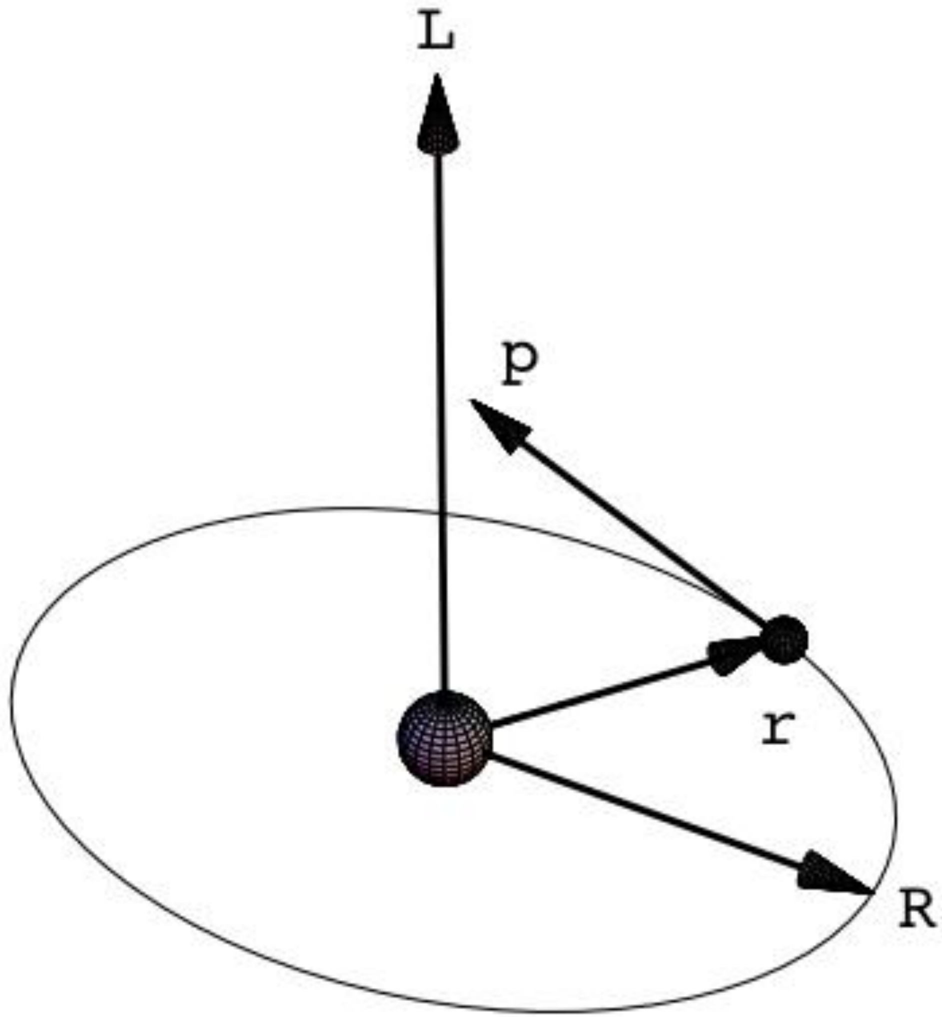
$$\boldsymbol{L} = \boldsymbol{r} \wedge \boldsymbol{p}, \quad \boldsymbol{L}^2 = M\alpha\, a\left(1-\varepsilon^2\right)$$

Runge-Lenz vector:

$$\boldsymbol{R} = \frac{\boldsymbol{p} \wedge \boldsymbol{L}}{M} - \alpha\frac{\boldsymbol{r}}{r}, \quad \boldsymbol{R}^2 = \frac{2E}{M}\boldsymbol{L}^2 + \alpha^2$$

Newtonian potential gives rise to closed orbits with constant direction of major axis.

Classical Kepler problem



Quantization of operators

From $\boldsymbol{p} \rightarrow -i\hbar\nabla$:

$$\hat{H} = -\left(\frac{\hbar^2}{2M}\nabla^2 + \frac{\alpha}{r}\right)$$

$$\hat{\boldsymbol{L}} = -i\hbar(\boldsymbol{r} \wedge \nabla)$$

$$\hat{\boldsymbol{R}} = -\frac{\hbar^2}{2M}\left[\nabla \wedge (\boldsymbol{r} \wedge \nabla) - (\boldsymbol{r} \wedge \nabla) \wedge \nabla\right] - \alpha\frac{\boldsymbol{r}}{r}$$

Some useful commutators & relations:

$$\left[\nabla, r^k\right] = k\, r^{k-2}\boldsymbol{r}, \quad \left[\nabla^2, \boldsymbol{r}\right] = 2\nabla, \quad \left[\nabla^2, r^k\right] = k\, r^{k-2}\left[(k+1) + 2\boldsymbol{r} \cdot \nabla\right]$$

$$\hat{\boldsymbol{R}}^2 = \frac{2\hat{H}}{M}\left(\hat{\boldsymbol{L}}^2 + \hbar^2\right) + \alpha^2$$

Conservation of angular momentum

The angular momentum operators ***L*** commute with the hydrogen hamiltonian:

$$\left[\hat{H},\hat{\boldsymbol{L}}\right] \propto \left[\nabla^2 + 2\kappa r^{-1}, \boldsymbol{r}\wedge\nabla\right] \qquad \left(\kappa = M\alpha/\hbar^2\right)$$

$$= \left[\nabla^2, \boldsymbol{r}\right]\wedge\nabla + 2\kappa\boldsymbol{r}\wedge\left[r^{-1},\nabla\right] = 0 + 0$$

L operators generate SO(3) algebra:

$$\left[\hat{L}_j,\hat{L}_k\right] = i\hbar\varepsilon_{jkl}\hat{L}_l, \quad j,k,l = x,y,z$$

H has SO(3) symmetry $\Rightarrow$ *m*-degeneracy.

Conserved Runge-Lenz vector

The Runge-Lenz vector ***R*** commutes with *H*:

$$\left[\hat{H},\hat{\boldsymbol{R}}\right]\propto\left[\nabla^{2}+2\kappa r^{-1},\nabla\wedge(\boldsymbol{r}\wedge\nabla)-(\boldsymbol{r}\wedge\nabla)\wedge\nabla+2\kappa r^{-1}\boldsymbol{r}\right]$$
$$=\left[\nabla^{2},2\kappa r^{-1}\boldsymbol{r}\right]+\left[2\kappa r^{-1},\nabla\wedge(\boldsymbol{r}\wedge\nabla)-(\boldsymbol{r}\wedge\nabla)\wedge\nabla\right]=0$$

R does *not* commute with the kinetic and potential parts of *H separately*:

$$-\frac{\hbar^{2}}{2M}\left[\nabla^{2},\hat{\boldsymbol{R}}\right]=-\left[-\alpha r^{-1},\hat{\boldsymbol{R}}\right]=\frac{\hbar^{2}\alpha}{M}\left[\frac{1}{r}\nabla-\frac{\boldsymbol{r}}{r^{3}}(1+\boldsymbol{r}\cdot\nabla)\right]$$

Hydrogen atom has a *dynamical symmetry*.

SO(4) symmetry

L and ***R*** (almost) close under commutation:

$$\left[\hat{L}_j, \hat{L}_k\right] = i\hbar\varepsilon_{jkl}\hat{L}_l, \quad j,k,l = x,y,z$$

$$\left[\hat{L}_j, \hat{R}_k\right] = i\hbar\varepsilon_{jkl}\hat{R}_l, \quad j,k,l = x,y,z$$

$$\left[\hat{R}_j, \hat{R}_k\right] = -i\hbar\varepsilon_{jkl}\frac{2\hat{H}}{M}\hat{L}_l, \quad j,k,l = x,y,z$$

H is time-independent and commutes with ***L*** and ***R***
$\Rightarrow$ choose a subspace with given *E*.

L and ***R'***$\equiv$*(-M/2H)*$^{1/2}$***R*** form an algebra SO(4) corresponding to rotations in four dimensions.

Energy spectrum of the H atom

Isomorphism of SO(4) and SO(3)⊕SO(3):

$$\hat{\boldsymbol{F}}^{\pm} = \frac{1}{2}\left(\hat{\boldsymbol{L}} \pm \hat{\boldsymbol{R}}'\right) \Rightarrow \left[\hat{F}_j^{\pm}, \hat{F}_k^{\pm}\right] = i\hbar\varepsilon_{jkl}\hat{F}_l^{\pm}, \quad \left[\hat{F}_j^{+}, \hat{F}_k^{-}\right] = 0$$

Since ***L*** and ***R'*** are orthogonal:

$$\left\langle \hat{\boldsymbol{F}}^{+} \cdot \hat{\boldsymbol{F}}^{+} \right\rangle = \left\langle \hat{\boldsymbol{F}}^{-} \cdot \hat{\boldsymbol{F}}^{-} \right\rangle = j(j+1)\hbar^2$$

The quadratic Casimir operator of SO(4) and *H* are related:

$$\hat{C}_2\left[\mathrm{SO}(4)\right] = \hat{\boldsymbol{F}}^{+} \cdot \hat{\boldsymbol{F}}^{+} + \hat{\boldsymbol{F}}^{-} \cdot \hat{\boldsymbol{F}}^{-} = \frac{1}{2}\left(\hat{\boldsymbol{L}}^2 - \frac{M}{2H}\hat{\boldsymbol{R}}^2\right) = -\frac{M\alpha^2}{4H} - \frac{1}{2}\hbar^2$$

$$\left\langle \hat{C}_2\left[\mathrm{SO}(4)\right] \right\rangle = 2j(j+1)\hbar^2 \Rightarrow E = -\frac{M\alpha^2}{2(2j+1)^2\hbar^2}, \; j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

W. Pauli, Z. Phys. **36** (1926) 336

The (3D) harmonic oscillator

The hamiltonian of the harmonic oscillator is

$$\hat{H} = \frac{p^2}{2M} + \frac{1}{2} M\omega^2 r^2$$

Standard wave quantum mechanics gives

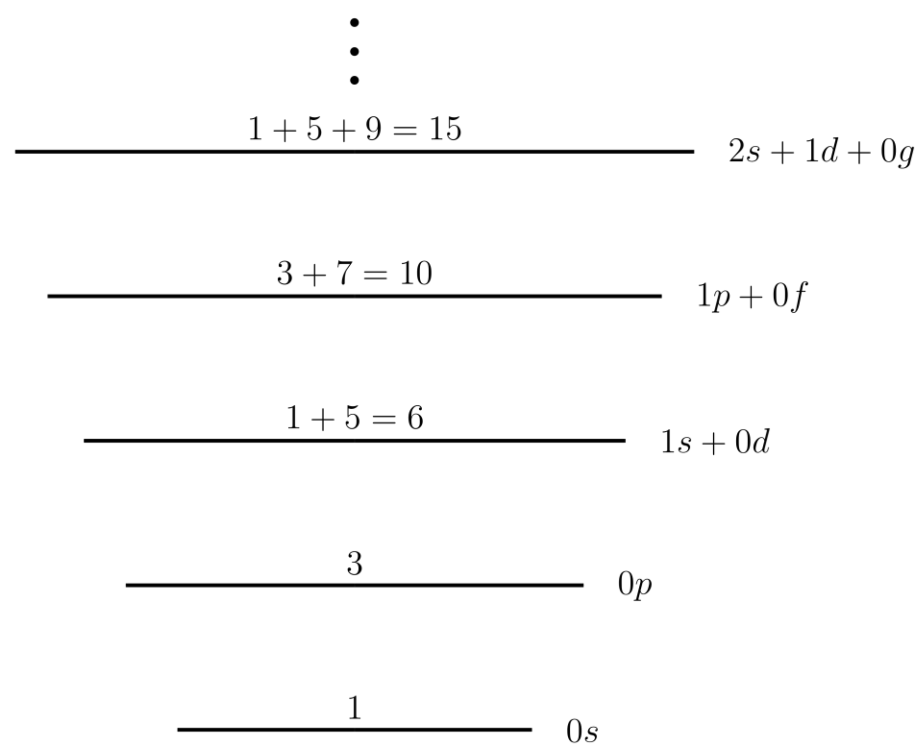
$$\hat{H}\Psi_{nlm}(r,\theta,\varphi) = \left(2n + l + \frac{3}{2}\right)\hbar\omega\Psi_{nlm}(r,\theta,\varphi)$$

$$\text{with } n = 0,1,\ldots \; ; \; l = 0,1,\ldots \; ; \; m = -l,\ldots,+l$$

Degeneracy in *m* originates from rotational symmetry. Additional degeneracy for all (*n*,*l*) combinations with *2n*+*l*=*N*.

What is the origin of this degeneracy?

Degeneracies of the 3D HO



3D harmonic oscillator

Raising and lowering operators

Introduce the raising and lowering operators

$$b_x^+ = \frac{1}{\sqrt{2}}\left(x' - \frac{\partial}{\partial x'}\right), \quad b_y^+ = \frac{1}{\sqrt{2}}\left(y' - \frac{\partial}{\partial y'}\right), \quad b_z^+ = \frac{1}{\sqrt{2}}\left(z' - \frac{\partial}{\partial z'}\right)$$

$$b_x = \frac{1}{\sqrt{2}}\left(x' + \frac{\partial}{\partial x'}\right), \quad b_y = \frac{1}{\sqrt{2}}\left(y' + \frac{\partial}{\partial y'}\right), \quad b_z = \frac{1}{\sqrt{2}}\left(z' + \frac{\partial}{\partial z'}\right)$$

$$\text{with} \quad x' = x/l, y' = y/l, z' = z/l; \quad l = \sqrt{\frac{\hbar}{M\omega}}$$

The 3D HO hamiltonian becomes

$$\hat{H} = \frac{p^2}{2M} + \frac{1}{2}M\omega^2 r^2 = \sum_{i=x,y,z}\left(b_i^+ b_i + \frac{1}{2}\right)\hbar\omega$$

M. Moshinsky, *The Harmonic Oscillator in Modern Physics*

U(3) symmetry of the 3D HO

The raising and lowering operators satisfy

$$\left[b_i, b_j\right] = 0, \quad \left[b_i^+, b_j^+\right] = 0, \quad \left[b_i, b_j^+\right] = \delta_{ij}$$

The bilinear combinations u_{ij} commute with H:

$$\hat{u}_{ij} \equiv b_i^+ b_j \Rightarrow \left[\hat{u}_{ij}, \hat{H}\right] = 0, \quad \forall i, j \in \{x, y, z\}$$

The nine operators u_{ij} generate the algebra U(3):

$$\left[\hat{u}_{ij}, \hat{u}_{kl}\right] = \hat{u}_{il}\delta_{jk} - \hat{u}_{kj}\delta_{il}$$

$\Rightarrow$ The symmetry of the harmonic oscillator in 3 dimensions is U(3).

The U(3)=U(1)⊕SU(3) algebra

The generators u_{ij} of U(3) can be written as

$$b_x^+ b_x + b_y^+ b_y + b_z^+ b_z = \frac{\hat{H}}{\hbar\omega} - \frac{3}{2}$$

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) = -i\hbar\left(b_x^+ b_y - b_y^+ b_x \right) \quad +\text{cyclic}$$

$$\hat{Q}_0 = \hbar\left(2b_z^+ b_z - b_x^+ b_x - b_y^+ b_y \right)$$

$$\hat{Q}_{\pm 1} = \hbar\sqrt{\frac{3}{2}}\left(\pm b_z^+ b_x \pm b_x^+ b_z - i b_y^+ b_z - i b_z^+ b_y \right)$$

$$\hat{Q}_{\pm 2} = \hbar\sqrt{\frac{3}{2}}\left(b_x^+ b_x - b_y^+ b_y \mp i b_x^+ b_y \mp i b_y^+ b_x \right)$$

Many particles in the 3D HO

Define operators for each particle *k*=1,2,…,*A*:

$$b_{x,k}^{+}=\frac{1}{\sqrt{2}}\left(x_k'-\frac{\partial}{\partial x_k'}\right),\quad b_{y,k}^{+}=\frac{1}{\sqrt{2}}\left(y_k'-\frac{\partial}{\partial y_k'}\right),\quad b_{z,k}^{+}=\frac{1}{\sqrt{2}}\left(z_k'-\frac{\partial}{\partial z_k'}\right)$$

$$b_{x,k}=\frac{1}{\sqrt{2}}\left(x_k'+\frac{\partial}{\partial x_k'}\right),\quad b_{y,k}=\frac{1}{\sqrt{2}}\left(y_k'+\frac{\partial}{\partial y_k'}\right),\quad b_{z,k}=\frac{1}{\sqrt{2}}\left(z_k'+\frac{\partial}{\partial z_k'}\right)$$

The *total* U(3) algebra is generated by

$$\sum_{k=1}^{A} b_{i,k}^{+} b_{j,k},\quad i,j\in\{x,y,z\}$$

Many particles in the 3D HO

Many-body hamiltonian with U(3) symmetry:

$$\hat{H} = \hbar\omega\left(\sum_{k=1}^{A} b_{x,k}^{+} b_{x,k} + b_{y,k}^{+} b_{y,k} + b_{z,k}^{+} b_{z,k}\right) + \sum_{k<l=1}^{A} \hat{V}(k,l)$$

$$\left[\hat{H}, \sum_{k=1}^{A} b_{i,k}^{+} b_{j,k}\right], \quad \forall i,j \in \{x,y,z\}$$

This property is valid if the interaction equals

$$\hat{C}_2[\mathrm{SU}(3)] = \frac{1}{2}\boldsymbol{L}\cdot\boldsymbol{L} + \frac{1}{6}\boldsymbol{Q}\cdot\boldsymbol{Q} = \sum_{k,l=1}^{A}\left(\frac{1}{2}\boldsymbol{L}(k)\cdot\boldsymbol{L}(l) + \frac{1}{6}\boldsymbol{Q}(k)\cdot\boldsymbol{Q}(l)\right)$$

J.P. Elliott, Proc. Roy. Soc. A **245** (1958) 128; 562

Dynamical symmetry

Two algebras $G_1 \supset G_2$ and a hamiltonian

$$\hat{H} = \sum_m \mu_m \hat{C}_m[G_1] + \sum_n \nu_n \hat{C}_n[G_2]$$

$\therefore$ H has symmetry G_2 but *not* G_1!

Eigenstates are independent of parameters μ_m and ν_n in H.

Dynamical symmetry breaking *"splits but does not admix eigenstates"*.

Isospin symmetry in nuclei

Empirical observations:

About equal masses of n(eutron) and p(roton).

n and p have spin 1/2.

Equal (to about 1%) nn, np, pp strong forces.

This suggests an *isospin* SU(2) symmetry of the nuclear hamiltonian:

$$\mathrm{n}: \quad t=\tfrac{1}{2}, m_t=+\tfrac{1}{2}; \qquad \mathrm{p}: \quad t=\tfrac{1}{2}, m_t=-\tfrac{1}{2}$$

$$\Rightarrow \quad \hat{t}_+\mathrm{n}=0, \quad \hat{t}_+\mathrm{p}=\mathrm{n}, \quad \hat{t}_-\mathrm{n}=\mathrm{p}, \quad \hat{t}_-\mathrm{p}=0, \quad \hat{t}_z n=\tfrac{1}{2}\mathrm{n}, \quad \hat{t}_z p=-\tfrac{1}{2}\mathrm{p}$$

W. Heisenberg, Z. Phys. **77** (1932) 1
E.P. Wigner, Phys. Rev. **51** (1937) 106

Isospin SU(2) symmetry

Isospin operators form an SU(2) algebra:

$$\left[\hat{t}_z, \hat{t}_\pm\right] = \pm\hat{t}_\pm, \quad \left[\hat{t}_+, \hat{t}_-\right] = 2\hat{t}_z$$

Assume the nuclear hamiltonian satisfies

$$\left[\hat{H}_{\mathrm{nucl}}, \hat{T}_\nu\right] = 0, \quad \hat{T}_\nu = \sum_{k=1}^{A} \hat{t}_\nu(k)$$

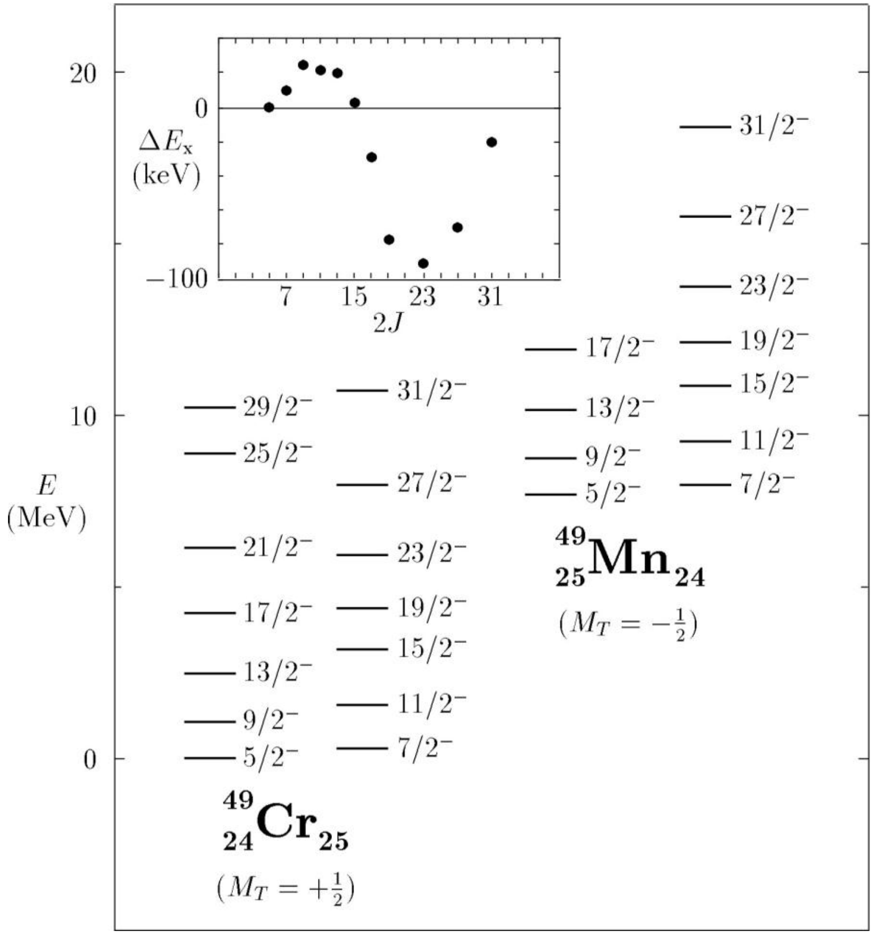
∴ H_{nucl} has SU(2) symmetry with degenerate states belonging to isobaric multiplets:

$$\left|\eta T M_T\right\rangle, \quad M_T = -T, -T+1, \ldots, +T$$

Isospin symmetry breaking: *A*=49

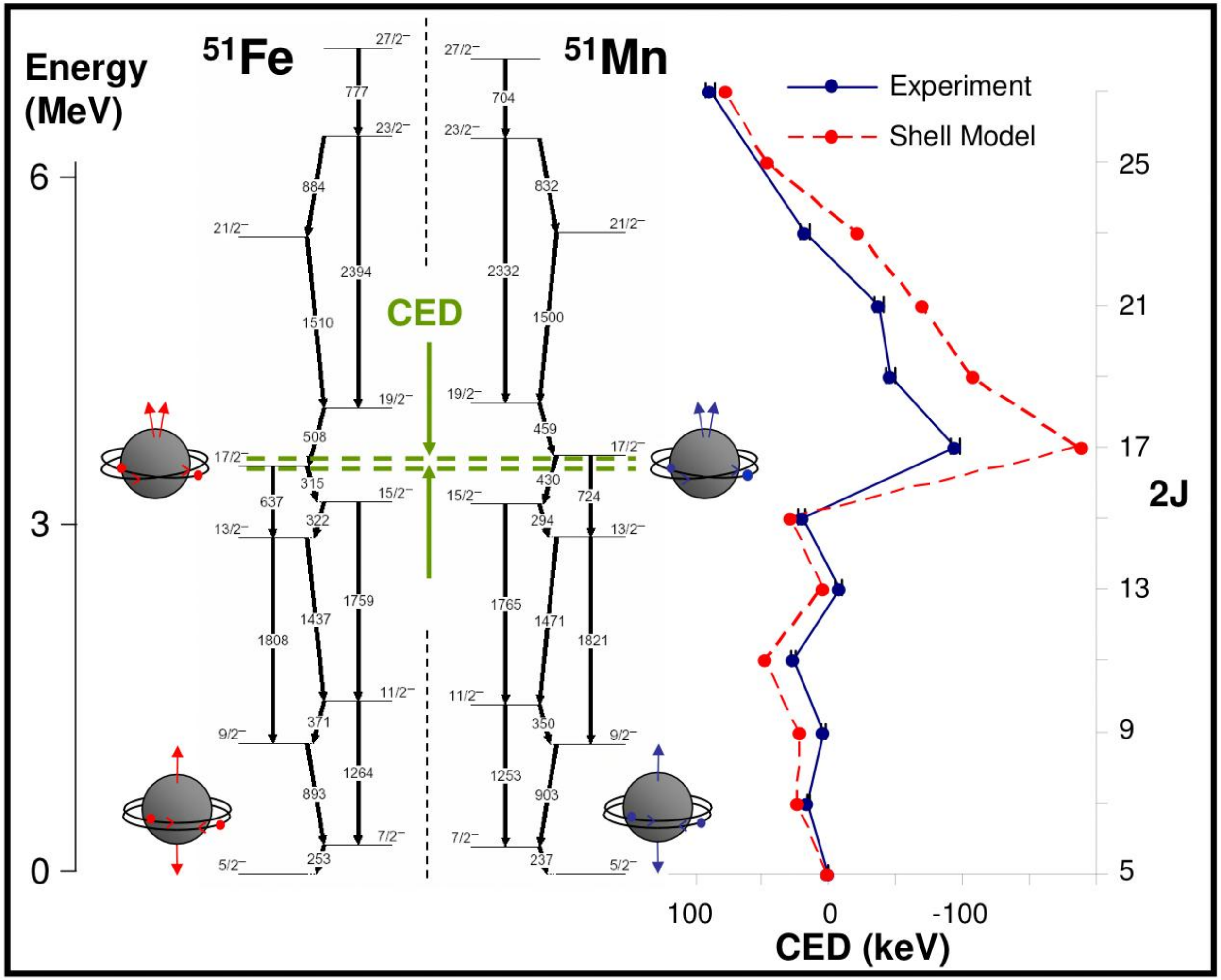
Empirical evidence from isobaric multiplets.

Example: *T*=1/2 doublet of *A*=49 nuclei.



O'Leary *et al.*, Phys. Rev. Lett. **79** (1997) 4349

Isospin symmetry breaking: *A*=51



Isospin SU(2) dynamical symmetry

Coulomb interaction can be approximated as

$$\hat{H}_{\mathrm{Coul}} \approx \kappa_0 + \kappa_1 \hat{T}_z + \kappa_2 \hat{T}_z^2 \Rightarrow \left[\hat{H}_{\mathrm{Coul}}, \hat{T}_z\right] = 0, \quad \left[\hat{H}_{\mathrm{Coul}}, \hat{T}_\pm\right] \neq 0$$

$\therefore$ H_{nucl}+H_{Coul} has SU(2) dynamical symmetry and SO(2) symmetry.

M_T-degeneracy is lifted according to

$$\hat{H}_{\mathrm{Coul}} \left|\eta T M_T\right\rangle = \left(\kappa_0 + \kappa_1 M_T + \kappa_2 M_T^2\right) \left|\eta T M_T\right\rangle$$

Summary of labelling:

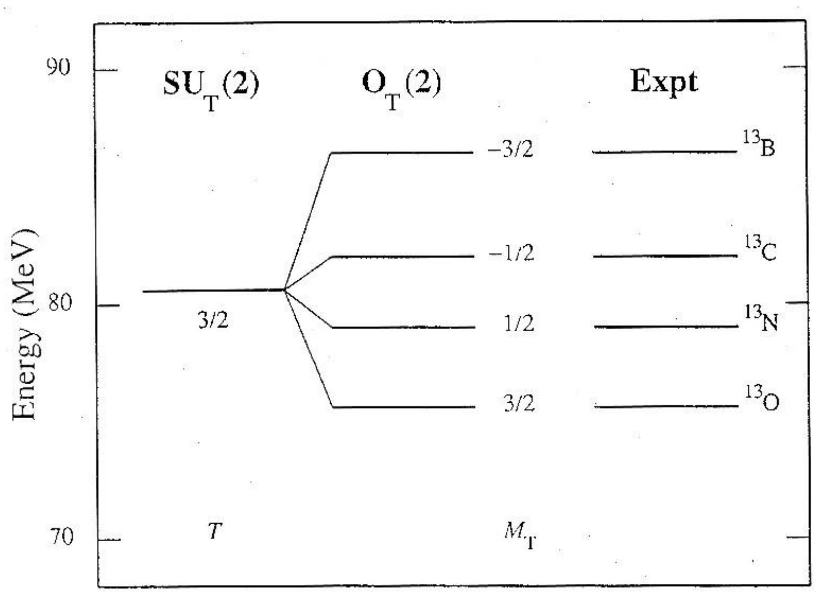
$$\begin{array}{ccc} \mathrm{SU}(2) & \supset & \mathrm{SO}(2) \\ \downarrow & & \downarrow \\ T & & M_T \end{array}$$

Isobaric multiplet mass equation

Isobaric multiplet mass equation:

$$E(\eta T M_T) = \kappa(\eta, T) + \kappa_1 M_T + \kappa_2 M_T^2$$

Example: *T*=3/2 multiplet for *A*=13 nuclei.



E.P. Wigner, Proc. Welch Found. Conf. (1958) 88

Isospin selection rules

Internal E1 transition operator is isovector:

$$\hat{T}_{\mu}^{\mathrm{E1}} = \sum_{k=1}^{A} e_k \, r_{\mu}(k) = \frac{e}{2}\left(\underbrace{\sum_{k=1}^{A} r_{\mu}(k)}_{\text{CM motion}} + \underbrace{2\sum_{k=1}^{A} \hat{t}_z(k) r_{\mu}(k)}_{\text{isovector}} \right)$$

Selection rule for *N*=*Z* (M_T=0) nuclei: *No E1 transitions are allowed between states with the same isospin.*

L.E.H. Trainor, Phys. Rev. **85** (1952) 962
L.A. Radicati, Phys. Rev. **87** (1952) 521

E1 transitions and isospin mixing

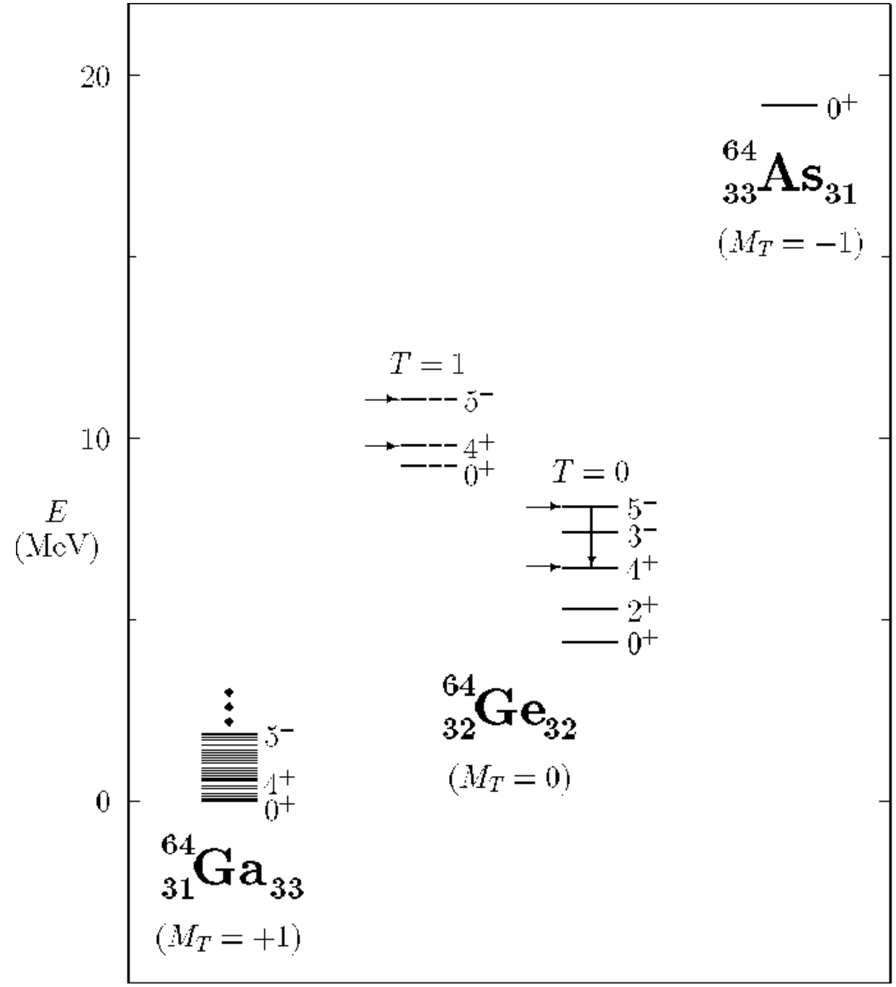
B(E1;$5^- \rightarrow 4^+$) in ^{64}Ge from:

- *lifetime of 5^- level;*
- *δ(E1/M2) mixing ratio of $5^- \rightarrow 4^+$ transition;*
- *relative intensities of transitions from 5^-.*

Estimate of minimum isospin mixing:

$$P\left(T = 1, 5^-\right) \approx P\left(T = 1, 4^+\right)$$

$$\approx 2.5\%$$



E.Farnea *et al.*, Phys. Lett. B **551** (2003) 56

Dynamical algebra

Take a generic many-body hamiltonian:

$$\hat{H} = \sum_i \varepsilon_i c_i^+ c_i + \frac{1}{4} \sum_{ijkl} \upsilon_{ijkl} c_i^+ c_j^+ c_l c_k + \cdots$$

Rewrite *H* as (bosons: *q*=0; fermions: *q*=1)

$$\hat{H} = \sum_{il} \left(\varepsilon_i \delta_{il} - (-)^q \frac{1}{4} \sum_j \upsilon_{ijlk} \right) \hat{u}_{il} + (-)^q \frac{1}{4} \sum_{ijkl} \upsilon_{ijlk} \hat{u}_{ik} \hat{u}_{jl} + \cdots$$

Operators u_{ij} generate the *dynamical algebra* U(*n*) for bosons and for fermions (*q*=0,1):

$$\hat{u}_{ij} \equiv c_i^+ c_j \Rightarrow \left[\hat{u}_{ij}, \hat{u}_{kl} \right] = \hat{u}_{il} \delta_{jk} - \hat{u}_{kj} \delta_{il}$$

Dynamical symmetry (DS)

With each chain of *nested* algebras

$$\mathrm{U}(n) = G_{\mathrm{dyn}} = G_1 \supset G_2 \supset \cdots \supset G_{\mathrm{sym}}$$

…is associated a *particular* class of many-body hamiltonian

$$\hat{H} = \sum_m \mu_m \hat{C}_m[G_1] + \sum_n \nu_n \hat{C}_n[G_2] + \cdots$$

Since *H* is a sum of commuting operators

$$\forall m,n,a,b: \quad \left[\hat{C}_m[G_a], \hat{C}_n[G_b]\right] = 0$$

…it can be solved analytically!

DS in nuclear physics

Name	G_{dyn}	G_{break}	G_{sym}	Application	Reference
Isospin	SU(2)	—	SO(2)	Isobaric multiplets, IMME	Heisenberg [4] Wigner [5]
Quasi-spin	SU(2)	—	SO(2)	Seniority spectra	Racah [6] Kerman [7]
supermultiplet SU(3) model	U(4Ω)	SU(3)	SO(3)	Wigner energy Rotational bands	Wigner [8] Elliott [9]
Interacting Boson Model	U(6)	U(5) SU(3) SO(6)	SO(3)	Vibrational nuclei Rotational nuclei γ-unstable nuclei	Arima and Iachello [10]
F-spin	SU(2)	—	SO(2)	F-spin multiplets, FMME	Brentano *et al.* [11]