Symmetries in Nuclei

Piet Van Isacker
Grand Accélérateur National d'Ions Lourds,
Caen, France

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Symmetries in Nuclei

Symmetry, mathematics and physics

Examples of symmetries in quantum mechanics

Symmetries of the nuclear shell model

Example: seniority in the nuclear shell model

What is symmetry?

Oxford Dictionary of English: "(beauty resulting from the) right correspondence of parts; quality of harmony or balance (in size, design) between parts"

Examples: disposition of a French garden; harmony of themes in a symphony.

Etymology

Ancient Greek roots:

"sun" means "with, together"

"metron" means "measure"

For the ancient Greeks symmetry was closely related to harmony, beauty and unity.

Aristotle: "The chief forms of beauty are orderly arrangement [*taxis*], proportion [*symmetria*] and definiteness [*horismenon*]."

Origin

For the ancient Greeks symmetry implied the notion of proportion.

17th century: Symmetry starts to imply also a relation of equality of elements that are opposed (*eg.* between left and right).

19th century: Definition of symmetry via the notion of invariance under transformations such as translations, rotations, reflections. Introduction of the notion of a group of transformations.

Group theory

Group theory is the mathematical theory of symmetry.

Group theory was invented (discovered?) by Evariste Galois in 1831.

Group theory became one of the pillars of mathematics (*cfr.* Klein's Erlangen programme).

Group theory has become of central importance in physics, especially in quantum physics.

The birth of group theory

Are all equations solvable algebraically?

Example of quadratic equation:

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Babylonians (from 2000 BC) knew how to solve quadratic equations in words but avoided cases with negative or no solutions.

Indian mathematicians (eg. Brahmagupta 598-670) did interpret negative solutions as `depths'.

Full solution was given in 12th century by the Spanish Jewish mathematician Abraham bar Hiyya Ha-nasi.

The birth of group theory

No solution of higher equations until dal Ferro, Tartaglia, Cardano and Ferrari solve the cubic and quartic equations in the 16th century.

$$ax^3 + bx^2 + cx + d = 0 \quad \& \quad ax^4 + bx^3 + cx^2 + dx + e = 0$$

Europe's finest mathematicians (*eg*. Euler, Lagrange, Gauss, Cauchy) attack the quintic equation but no solution is found.

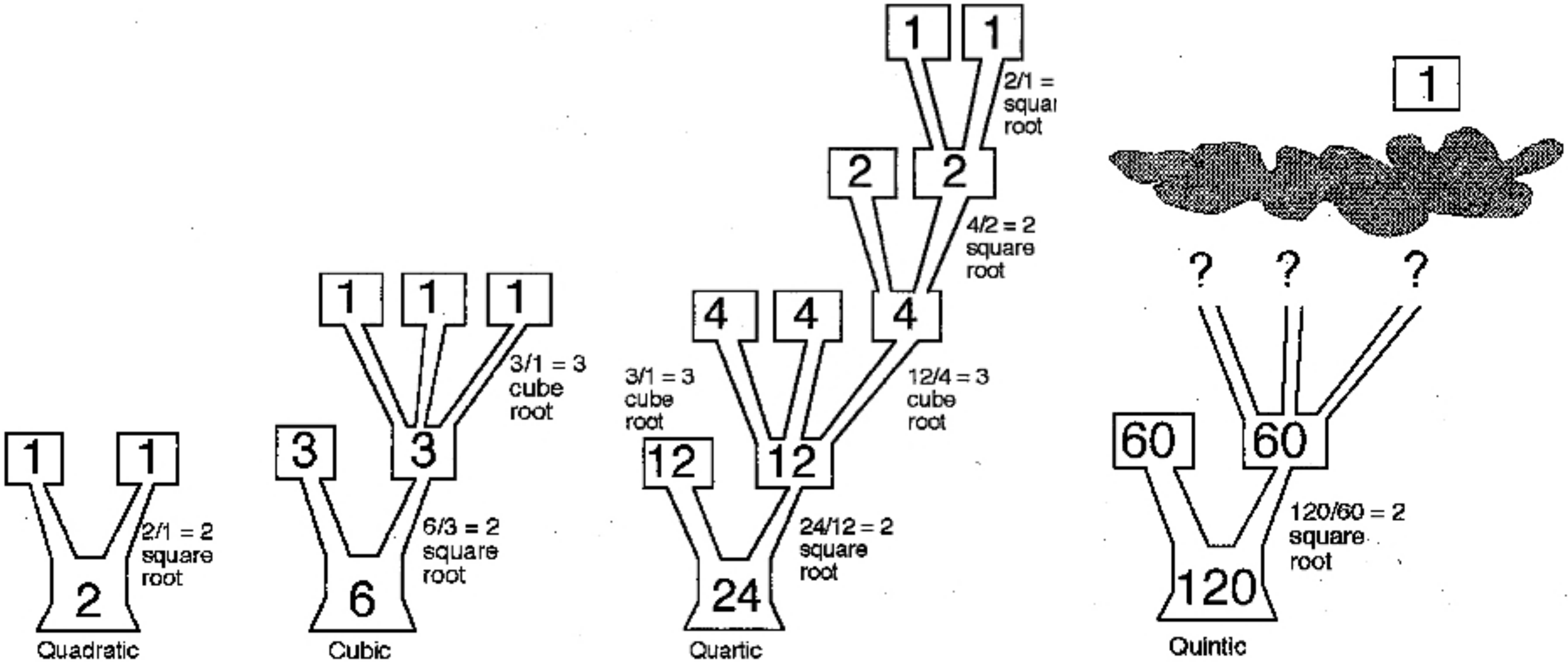
1799: proof of non-existence of an algebraic solution of the quintic equation by Ruffini?

The birth of group theory

1824: Niels Abel shows that general quintic and higher-order equations have no algebraic solution.

1831: Evariste Galois answers the solvability question: whether a given equation of degree n is algebraically solvable depends on the 'symmetry profile of its roots' which can be defined in terms of a subgroup of the group of permutations S_n.

The insolvability of the quintic



The axioms of group theory

A set *G* of elements (transformations) with an operation × which satisfies:

1. *Closure. If g_1 and g_2 belong to G, then $g_1 \times g_2$ also belongs to G.*
2. *Associativity. We always have $(g_1 \times g_2) \times g_3 = g_1 \times (g_2 \times g_3)$.*
3. *Existence of identity element. An element 1 exists such that $g \times 1 = 1 \times g = g$ for all elements g of G.*
4. *Existence of inverse element. For each element g of G, an inverse element g^{-1} exists such that $g \times g^{-1} = g^{-1} \times g = 1$.*

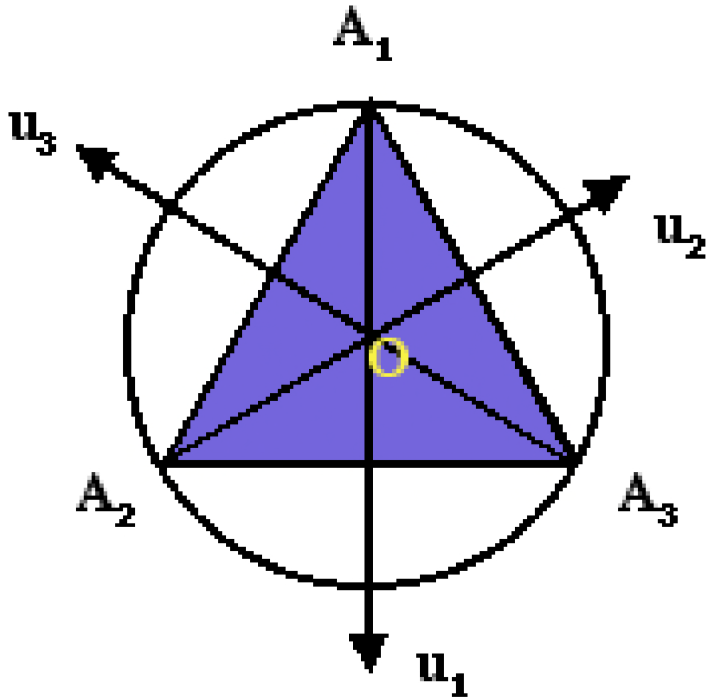
This simple set of axioms leads to an amazingly rich mathematical structure.

Example: equilateral triangle

Symmetry transformations are

- *Identity*
- *Rotation over $2\pi/3$ and $4\pi/3$ around $\mathbf{e}_z$*
- *Reflection with respect to planes $(\mathbf{u}_1,\mathbf{e}_z)$, $(\mathbf{u}_2,\mathbf{e}_z)$, $(\mathbf{u}_3,\mathbf{e}_z)$*

Symmetry group: C_{3h}.

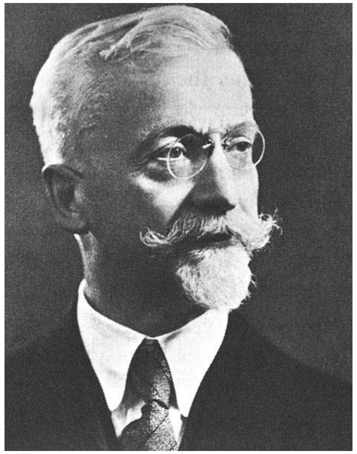


Groups and algebras

1873: Sophus Lie introduces the notion of the algebra of a continuous group with the aim of devising a theory of the solvability of *differential* equations.

1887: Wilhelm Killing classifies all Lie algebras.

1894: Elie Cartan re-derives Killing's classification and notices two exceptional Lie algebras to be equivalent.



Lie groups

A Lie group contains an *infinite* number of elements characterized by a set of *continuous* variables.

Additional conditions:

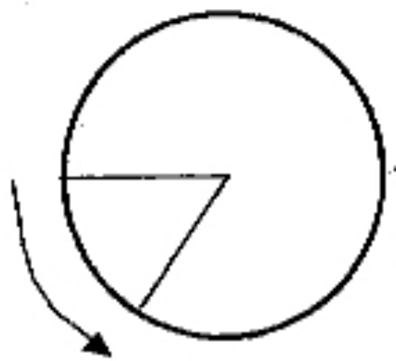
Connection to the identity element.

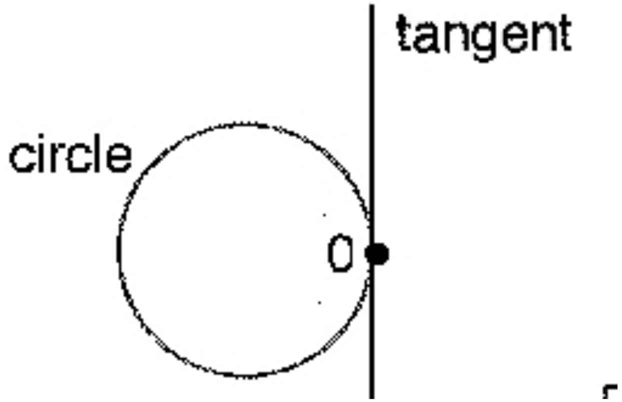
Analytic multiplication function.

Example: rotations in 2 dimensions, SO(2).

$$\hat{g}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Lie algebras





Idea: to obtain properties of the *infinite* number of elements g of a Lie *group* in terms of those of a *finite* number of elements g_i (called generators) of a Lie *algebra*.

Lie algebras

All properties of a Lie algebra follow from the commutation relations between its generators:

$$\left[\hat{g}_i,\hat{g}_j\right] \equiv \hat{g}_i \times \hat{g}_j - \hat{g}_j \times \hat{g}_i = \sum_{k=1}^{r} c_{ij}^{k}\hat{g}_k$$

Generators satisfy the Jacobi identity:

$$\left[\hat{g}_i,\left[\hat{g}_j,\hat{g}_k\right]\right]+\left[\hat{g}_j,\left[\hat{g}_k,\hat{g}_i\right]\right]+\left[\hat{g}_k,\left[\hat{g}_i,\hat{g}_j\right]\right]=0$$

Definition of the metric tensor or Killing form:

$$g_{ij} = \sum_{k,l=1}^{r} c_{ik}^{l} c_{jl}^{k}$$

Classification of Lie groups

Symmetry groups over R, C and H (quaternions) preserving a specified metric:

$$\{x \in \mathbf{R}[n]: \quad xx^* = 1, \det = 1\} \Rightarrow \mathrm{SO}(n)$$

$$\{x \in \mathbf{C}[n]: \quad xx^* = 1, \det = 1\} \Rightarrow \mathrm{SU}(n)$$

$$\{x \in \mathbf{H}[n]: \quad xx^* = 1\} \Rightarrow \mathrm{Sp}(n)$$

The five exceptional groups G_2, F_4, E_6, E_7 and E_8 are similar constructs over the normed division algebra of the octonions, O.

Rotations in 2 dimensions, SO(2)

Matrix representation of finite elements:

$$\hat{g}(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

Infinitesimal element and generator:

$$\lim_{\alpha\to 0} \hat{g}(\alpha) \approx \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \equiv \hat{e} + \alpha \hat{g}_1$$

Exponentiation leads back to finite elements:

$$\hat{g}(\alpha) = \lim_{n\to\infty} \left(\hat{e} + \frac{\alpha}{n} \hat{g}_1 \right)^n = \exp(\hat{e} + \alpha \hat{g}_1) = \exp \begin{bmatrix} 1 & \alpha \\ -\alpha & 1 \end{bmatrix} = \hat{g}(\alpha)$$

Rotations in 3 dimensions, SO(3)

Matrix representation of finite elements:

$$\begin{bmatrix} \cos\alpha_3 & -\sin\alpha_3 & 0 \\ \sin\alpha_3 & \cos\alpha_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha_2 & 0 & \sin\alpha_2 \\ 0 & 1 & 0 \\ -\sin\alpha_2 & 0 & \cos\alpha_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_1 & -\sin\alpha_1 \\ 0 & \sin\alpha_1 & \cos\alpha_1 \end{bmatrix}$$

Infinitesimal elements and associated generators:

$$\hat{g}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \hat{g}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad \hat{g}_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rotations in 3 dimensions, SO(3)

Structure constants from matrix multiplication:

$$\left[\hat{g}_k,\hat{g}_l\right]=\sum_{m=1}^{3}c_{kl}^{m}\hat{g}_m \quad \text{with} \quad c_{kl}^{m}=\varepsilon_{klm}\ (\text{Levi}-\text{Civita})$$

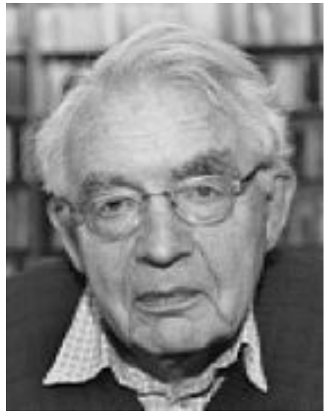
Exponentiation leads back to finite elements:

$$\hat{g}(\alpha_1,\alpha_2,\alpha_3)=\exp\left(\hat{e}+\sum_{k=1}^{3}\alpha_k\hat{g}_k\right)$$

Relation with angular momentum operators:

$$\hat{l}_k=i\hat{g}_k \Rightarrow \left[\hat{l}_k,\hat{l}_l\right]=i\sum_{m=1}^{3}\varepsilon_{klm}\hat{l}_m$$

Casimir operators



Definition: The Casimir operators $C_n[G]$ of a Lie algebra G commute with all generators of G.

The quadratic Casimir operator (n=2):

$$\hat{C}_2[G] = \sum_{i,j=1}^{r} g^{ij}\hat{g}_i\hat{g}_j \text{ with } \sum_{i,j=1}^{r} g_{ij}g^{jk} = \delta_{ik}. \quad \text{Ex}: \hat{C}_2[\mathrm{SO}(3)] = \sum_{k=1}^{3}\hat{l}_k^2$$

The number of independent Casimir operators (rank) equals the number of quantum numbers needed to characterize any (irreducible) representation of G.

Symmetry in physics

Geometrical symmetries

Example: C_{3h} symmetry (X_3 molecules, ^{12}C nucleus).

Permutational symmetries

Example: S_A symmetry of an A-body hamiltonian:

$$\hat{H} = \sum_{k=1}^{A} \frac{p_k^2}{2m_k} + \sum_{1 \le k < k'}^{A} \hat{V}_2\left(\left|\boldsymbol{r}_k - \boldsymbol{r}_{k'}\right|\right)$$

Symmetry in physics

Space-time (kinematical) symmetries

Rotational invariance, SO(3): $x_k \to x'_k = \sum_l A_{kl} x_l$

Lorentz invariance, SO(3,1): $x_\mu \to x'_\mu = \sum_\nu A_{\mu\nu} x_\nu$

Parity: $x_k \to x'_k = -x_k$

Time reversal: $t \to t' = -t$

Euclidian invariance, E(3): $x_k \to x'_k = \sum_l A_{kl} x_l + a_k$

Poincaré invariance, E(3,1): $x_\mu \to x'_\mu = \sum_\nu A_{\mu\nu} x_\nu + a_\mu$

Dilatation symmetry: $x_k \to x'_k = D_k x_k$

Symmetry in physics

Quantum-mechanical symmetries, U(n)

$$\psi_k \rightarrow \psi'_k = \sum_{l=1}^{n} B_{kl}\psi_l \quad \text{inv.} \quad \sum_l |\psi_l|^2$$

Internal symmetries. Example:

p⇔n, isospin SU(2)

u⇔d⇔s, flavour SU(3)

Gauge symmetries. Example:

Maxwell equations, U(1)

Dynamical symmetries. Example:

Coulomb problem, SO(4)

Symmetry in quantum mechanics

Assume a hamiltonian *H* which commutes with operators g_i that form a Lie algebra *G*:

$$\forall \hat{g}_i \in G : \quad \left[\hat{H}, \hat{g}_i\right] = 0$$

∴ *H* has *symmetry G* or is *invariant under G*.

Lie algebra: a set of (infinitesimal) operators that closes under commutation.

Consequences of symmetry

Degeneracy structure: If $|\gamma\rangle$ is an eigenstate of H with energy E, so is $g_i|\gamma\rangle$:

$$\hat{H}|\gamma\rangle = E|\gamma\rangle \Rightarrow \hat{H}\hat{g}_i|\gamma\rangle = \hat{g}_i\hat{H}|\gamma\rangle = E\hat{g}_i|\gamma\rangle$$

Degeneracy structure and labels of eigenstates of H are determined by algebra G:

$$\hat{H}|\Gamma\gamma\rangle = E(\Gamma)|\Gamma\gamma\rangle; \quad \hat{g}_i|\Gamma\gamma\rangle = \sum_{\gamma'} a^{\Gamma}_{\gamma'\gamma}(i)|\Gamma\gamma'\rangle$$

Casimir operators of G commute with all g_i:

$$\hat{H} = \sum_m \mu_m \hat{C}_m[G]$$

Symmetry rules

Since its introduction by Galois in 1831, group theory has become central to the field of mathematics.

Group theory remains an active field of research, (*eg.* the recent classification of all groups leading to the *Monster*.)

Symmetry has acquired a central role in all domains of physics.