INTRODUCTION TO ALGEBRAIC THEORY OF QUANDLES
Lecture 2

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All quandles (racks) form a category. We can define free objects in this category.

Definition

A free quandle on a non empty set A is a quandle $FQ(A)$ together with a map $\varphi : A \to FQ(A)$ such that for any other map $\rho : A \to Q$, where Q is a quandle, there exists a unique homomorphism $\overline{\rho} : FQ(A) \to Q$ such that $\overline{\rho} \circ \varphi = \rho$.

The free quandle is unique up to isomorphism. The definition of the free rack $FR(A)$ is the same.

On the free group $F = F(A)$ define operation $*$ and its inverse $*^{-1}$:

$$x * y = y^{-1}xy, \quad x *^{-1} y = yxy^{-1}.$$

Then $(F, *)$ is the conjugacy quandle $Conj(F)$.

Free quandles may be faithfully represented as unions of conjugacy classes in free groups.

Theorem (Joyce)

Let A be a set and $F = F(A)$ be the free group on A. Then the free quandle $FQ(A)$ on A appears as the subquandle Q of $Conj(F)$ consisting of the conjugates of the generators of F.

In some sense, the theory of quandles is ‘the theory of conjugation’.

Corollary

Any equation holding in $Conj(G)$ for all groups G holds in all quandles.

We can formulate

Question

Is it true that any quandle is a subquandle of some conjugacy quandle?

Although the equations true for conjugation in groups are also true in quandles, there are many statements true for conjugation in groups but false for quandles in general.

Example

The implication

$$p * q = p \;\Rightarrow\; q * p = q$$

holds in $Conj(G)$ for all groups G but is false for the 3-element quandle:

$$Cs(4) = \langle x, y, z \mid x{*}y = z, x{*}z = x, z{*}x = z, z{*}y = x, y{*}x = y, y{*}z = y \rangle.$$

Definition

A crossed set is a quandle $(X, *)$ which further satisfies

$$x * y = x \Rightarrow y * x = y.$$

Question

Is it true that for any crossed set X there exists a group G such that X is isomorphic to some subquandle of $Conj(G)$?

As for the free group it is possible to prove

Theorem (Ivanov-Kadantsev-Kuznetsov)

A subquandle of a free quandle is free.

The following construction of the free rack $FR(A)$ was suggested by R. Fenn and C. Rourke. Let $F(A)$ be the free group with the free generators A. On the set $A \times F(A)$ define the operation

$$(a, u) * (b, v) = (a, uv^{-1}bv)$$

for $a, b \in A$, $u, v \in F(A)$. The algebraic system $FR(A) = (A \times F(A), *)$ is the free rack on A.

For $(a, u), (b, v) \in FR(A)$ the element $(a, u) *^{-1} (b, v)$ is given by

$$(a, u) *^{-1} (b, v) = (a, uv^{-1}b^{-1}v).$$

The free quandle $FQ(A)$ on A can be constructed as the quotient of $FR(A)$ modulo the equivalence relation $(a,w) \sim (a,aw)$ for $a \in A$, $w \in F(A)$.

If $A = \{x_1, x_2, \ldots, x_n\}$ contains n elements, we shall denote $FR(A)$ by FR_n and $FQ(A)$ by FQ_n.

We say that a quandle Q is algebraically connected or simply connected if the inner automorphism group $\mathrm{Inn}(Q)$ acts transitively on Q.

In other words, Q is connected iff for each pair a, b in Q there are $a_1, a_2, \dots, a_n$ in Q and $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ in $\{1, -1\}$ such that

$$(\dots((a *^{\varepsilon_1} a_1) *^{\varepsilon_2}) \dots) *^{\varepsilon_n} a_n = b.$$

1) Dihedral quandles R_n: If n is odd, then R_n is connected. If n is even, then R_n is not connected.

2) Any fundamental knot quandle is connected.

3) For any quandle epimorphism $f : X \to Y$, if X is connected, so is Y.

Definition

Quandles for which left multiplications are bijections are called Latin quandles, i.e. Q is a Latin if for every $a \in Q$ the map $L_a(x) = a * x$ is a bijection.

In particular, if Q is finite, then every row of multiplication table contains all elements of Q.

It is evident, that any Latin quandle is connected.

The following example shows that not every connected quandle is Latin.

Example

Let $Q = \{(1,2),\ (1,3),\ (1,4),\ \ (2,3),\ (2,4),\ (3,4)\}$ be the subquandle of $Conj(\Sigma_4)$, which contains all transpositions.. Then it has multiplication table

$$\begin{pmatrix} 1 & 4 & 5 & 2 & 3 & 1 \\ 4 & 2 & 6 & 1 & 2 & 3 \\ 5 & 6 & 3 & 3 & 1 & 2 \\ 2 & 1 & 4 & 4 & 6 & 5 \\ 3 & 5 & 1 & 6 & 5 & 4 \\ 6 & 3 & 2 & 5 & 4 & 6 \end{pmatrix},$$

where we denote: $1 = (1,2),\ 2 = (2,3),\ 3 = (1,4),\ 4 = (1,3),$ $5 = (2,4),\ 6 = (3,4).$

For arbitrary quandle Q let $\bar{Q} = \{O_i\}_{i \in I}$ be the set of connected components. The operation $*$ defines the operation $*$ on the orbits.

Proposition

1) Q is the disjoint union $Q = \coprod_{i \in I} O_i$;

2) $\bar{Q}$ is the homomorphic image of Q and $(\bar{Q}, *)$ is the trivial quandle.

From this proposition follows that it is need to study connected quandles.

The commutator subgroup $(\mathrm{Inn}(Q))'$ of $\mathrm{Inn}(Q)$ is always contained in $\mathrm{Trans}(Q)$.

If Q is algebraically connected, then $(\mathrm{Inn}(Q))'$ equals $\mathrm{Trans}(Q)$.

Indeed, in that case each generator $S_x S_y^{-1}$ of $\mathrm{Trans}(Q)$ is of the form $S_x a^{-1} S_x^{-1} a$, since there is an a in $\mathrm{Inn}(Q)$ such that $y = x \cdot a$ and so

$$S_y = S_{x \cdot a} = a^{-1} S_x a.$$

We describe the approach of Joyce that constructs any quandle from group.

Definition

Let G be a group, and H be a subgroup of G. Let $\rho : G \to G$ be an automorphis such that $\rho(h) = h$ for any $h \in H$. On the left cosets $H \backslash G$ define the operation $[Hx] * [Hy] := [H\rho(xy^{-1})y]$.

It is easy to prove

Proposition

The algebraic system $(H \backslash G, *)$ is a quandle.

For example, if $\rho = \widehat{z}$ is the inner automorphism $g \mapsto z^{-1}gz$ for some $z \in G$ and if z commutes with any elements of H, then the quandle operation on $H\backslash G$ is given by

$$[Hx] * [Hy] = [Hz^{-1}xy^{-1}zy] \in H\backslash G,$$

for any $x, y \in G$. We shall denote this quandle by the triple (G, H, z).

Example

Triples (G, H, z) for the dihedral quandle R_m: In this case G is the dihedral group $\mathbb{Z}/m \rtimes \mathbb{Z}/2$, and $H = \{0\} \rtimes \mathbb{Z}/2$, $z = (0, 1) \in H$.

The following construction generalizes the quandle (G, H, z).

Definition

Given a group G, a set of its subgroups $H_i \leq G$ and elements $z_i \in G$, which lie in the centralizers of H_i, i.e. $z_i h_i = h_i z_i$, where $i \in I$ for some set of indexes I. Then, we can define a quandle, denoted by $(\{H_i\}_{i \in I} \backslash G; z_i, i \in I)$, as the disjoint union of the left cosets

$$X := \coprod_{i \in I} (H_i \backslash G)$$

with the quandle operation

$$[H_i x] * [H_j y] = [H_i z_i^{-1} x y^{-1} z_j y], \quad x, y \in G.$$

Define a right action of the group $Ad(X)$ on a quandle X by the rule

$$x \cdot e_y := x * y \text{ for } x, y \in X.$$

We see that it is the same action as action of $\mathrm{Inn}(X)$. Hence, X is connected if the action of $Ad(X)$ on X is transitive.

If $a \in X$, then the stabilizer of a in $Ad(X)$ is the subgroup

$$Stab(a) = \{h \in Ad(X) \mid a \cdot h = a\}.$$

Now we can formulate the main result on structure of quandles.

Theorem (Joyce)

1) Let X be a connected quandle and $x_0 \in X$ is some fixed element. Let H be the stabilizer of x_0. Then the map

$$Ad(X) \to X, \quad \varphi \mapsto x_0 \cdot \varphi$$

induces a quandle isomorphism $(H \backslash Ad(X), z) \cong X$, where $z = e_{x_0}$.

2) Every quandle X is representable as $(\{H_i\}_{i \in I} \backslash G; z_i, i \in I)$.

The representation of connected quandle X as triple (G, H, z), constructed in the theorem is not unique.

For example, we may replace $Ad(X)$ by the automorphism group $\mathrm{Aut}(X)$ or by the inner automorphism group $\mathrm{Inn}(X)$.

In group theory people study varieties of groups: abelian groups, nilpotent groups, solvable groups, p-groups and so on.

Let us define varieties of quandles. Suppose that

$$u_i = u_i(x_1, x_2, \ldots, x_n), \quad v_i = v_i(x_1, x_2, \ldots, x_n), \quad i \in I,$$

is a set of words which define elements of the free quandle FQ_n. The set of identities is the set of equalities

$$V = \{u_i = v_i \mid i \in I\}.$$

We say that a quandle Q satisfies the identities of V if for any n-tuple $(q_1, q_2, \dots, q_n) \in Q^n$ the following equalities

$$u_i(q_1, q_2, \dots, q_n) = v_i(q_1, q_2, \dots, q_n), \quad i \in I,$$

hold in Q.

The V-variety of quandles we call the set of all quandles which satisfy the identity of V.

1) The variety of abelian quandle is a defined by one identity:

$$(x_1 * x_2) * (x_3 * x_4) = (x_1 * x_3) * (x_2 * x_4).$$

2) The variety of n-quandles is a defined by one identity:

$$x_1 *^n x_2 = x_1.$$

For $n = 2$ we get the variety of involutory quandles.

By analogy with free Burnside groups $B(m,n)$, we introduce.

Definition

The free Burnside quandle $BQ(m,n)$, $m,n \geq 2$, is the quandle that is m-generated n-quandle, i.e.

$$BQ(m,n) = \langle x_1, x_2, \ldots, x_m \mid x *^n y = x \text{ for all elements } x, y \rangle.$$

By analogy with the famous Burnside problem in group theory we can formulate

Problem

Find the structure of the free Burnside quandles $BQ(m,n)$.

If we have some V-variety of quandles $Var_q(V)$ we can formulate

Problems

1) Let X be an arbitrary quandle. Find the maximal quotient of X which lies in $Var_q(V)$.

2) What can we say on quandles which are free in $Var_q(V)$?

Suppose that Q be a set and G be a group. We say that G acts on Q if for any $q \in Q$, $g \in G$ is defined an element $q \cdot g \in Q$ and

$$(q \cdot g_1) \cdot g_2 = q \cdot (g_1 g_2), \quad q \cdot e = q \text{ for all } q \in Q, g_1, g_2 \in G.$$

The set

$$qG = \{q \cdot g \mid g \in G\}$$

is called the orbit of q. Then Q is the disjoint union of orbits:

$$Q = \coprod_{i \in I} q_i G \text{ for some set } \{q_i\}_{i \in I} \subseteq Q.$$

The stabilizer of an element $q \in Q$ is the set

$$Stab_G(q) = \{g \in G \mid q \cdot g = q\}.$$

It is not difficult to see that $Stab_G(q)$ is a subgroup of G for any $q \in Q$.

Definition

An augmented quandle (Q, G) consists of a set Q, a group G equipped with a right action on the set Q, and a function $\varepsilon : Q \to G$ called the augmentation map which satisfy axioms

AQ1. $q \cdot \varepsilon(q) = q$ for all $q \in Q$;

AQ2. $\varepsilon(q \cdot x) = x^{-1}\varepsilon(q)x$ for all $q \in Q, x \in G$.

Given an augmented quandle (Q,G), we can define quandle operations on Q by

$$p * q = p \cdot \varepsilon(q), \quad p *^{-1} q = p \cdot \varepsilon(q)^{-1}.$$

Then Q is a quandle, the action of G on Q is by quandle automorphisms, and the augmentation map is a quandle homomorphism $\varepsilon : Q \to Conj(G)$.

Conversely, given a quandle X, we can construct some augmented quandles. More precisely, for any quandle X, we have the augmented quandle $(X, Ad(X))$, where the augmentation map ε sends x to e_x.

Definition

A morphism of augmented quandles from (Q, G) to (P, H) consists of a group homomorphism $g : G \to H$ and a function $f : Q \to P$ such that the diagram

$$\begin{array}{ccccc} Q \times G & \longrightarrow & Q & \xrightarrow{\varepsilon} & G \\ {\scriptstyle f\times g}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow \\ P \times H & \longrightarrow & P & \xrightarrow{\varepsilon} & H \end{array}$$

commutes.

It follows that f is a quandle homomorphism.

Fix a quandle Q. Two examples of augmented quandles with underlying quandle Q are $(Q, \mathrm{Aut}(Q))$ and $(Q, \mathrm{Inn}(Q))$.

The set of all augmented quandles (Q, G) with different G forms a category and it called the category of augmentations of Q .

In this category $(Q, \mathrm{Aut}(Q))$ is the terminator (final object). That is, for each augmentation (Q, G) there is a unique group homomorphism $f : G \to \mathrm{Aut}(Q)$ such that the diagram

$$\begin{array}{ccccc} Q \times G & \longrightarrow & Q & \xrightarrow{\varepsilon} & G \\ {\scriptstyle id \times f} \downarrow & & {\scriptstyle id} \downarrow & & {\scriptstyle f} \downarrow \\ Q \times \mathrm{Aut}(Q) & \longrightarrow & Q & \xrightarrow{S} & \mathrm{Aut}(Q) \end{array}$$

commutes. The map f is readily defined from the action $Q \times G \to Q$.

Let (Q, G) be an augmented quandle and N be a normal subgroup of G. Let G/N be the quotient group with elements denoted as xN for x in G.

Let us define the quotient of quandle Q/N. The elements of Q/N correspond to equivalence classes

$$qN = \{q \cdot n \in Q \mid n \in N\}$$

for q in Q. The action

$$Q/N \times G/N \to Q/N$$

is given by

$$(qN) \cdot (xN) = (q \cdot x)N,$$

and the augmentation $\varepsilon : Q/N \to G/N$ is given by $\varepsilon(qN) = \varepsilon(q)N$.

Let us construct the abelianization of a quandle, that is, its largest abelian quotient. Let (Q, G) be an augmented quandle. In order that Q be abelian, we require:

$$(p * q) * (r * s) = (p * r) * (q * s).$$

Equivalently,

$$\varepsilon(q)\varepsilon(r\varepsilon(s)) = \varepsilon(r)\varepsilon(q\varepsilon(s)).$$

That is, every element of the form

$$\varepsilon(q)\varepsilon(s)^{-1}\varepsilon(r)\varepsilon(q)^{-1}\varepsilon(s)\varepsilon(r)^{-1}$$

be equal to 1.

Let N be the normal subgroup of G generated by the elements of the form

$$\varepsilon(q)\varepsilon(s)^{-1}\varepsilon(r)\varepsilon(q)^{-1}\varepsilon(s)\varepsilon(r)^{-1}.$$

Then the quotient $(Q/N, G/N)$ of (Q, G) is abelian.

It is evident that $(Q/N, G/N)$ has the universal property that each map $(Q, G) \to (P, H)$ factors uniquely through $(Q/N, G/N)$ whenever P is an abelian quandle.

Moreover, if $\varepsilon(Q)$ generates G, then Q/N is the abelianization of Q.

Theorem (Joyce)

Let A be a set and G be the group generated by A and defined by the relations $ab^{-1}c = cb^{-1}a$ for conjugates a, b, c of the generators of G. Then the free abelian quandle on A appears as the conjugates of the generators of G as a subquandle of $Conj(G)$.

Theorem (Joyce)

Let (Q, G) be an augmented quandle such that $\varepsilon(Q)$ generates G, and let n be a positive integer. Let N be the normal subgroup of G generated by elements of the form $\varepsilon(q)^n$. Then Q/N is the largest quotient of Q which is an n-quandle.

The following corollary gives description of free Burnside quandles.

Corollary

Let A be a set and G be the group presented as

$$G = \langle A \mid a^n = 1 \text{ for all } a \in A \rangle.$$

Then the free n-quandle on A consists of the conjugates of the generators of G.

In this corollary G is the free product of cyclic groups of order n. In particular, if $|A| = m$, then $BQ(m, n)$ is the subguandle of $Conj(G)$ that is the set of conjugates of the generators from A.

As we know the free Burnside group $B(m, 2)$ is abelian and if $m = 2$, then $B(2, 2) = \mathbb{Z}_2 \times \mathbb{Z}_2$. The structure of $BQ(2, 2)$, i.e. 2-generated free involutory quandle, is more complicated.

Corollary

The two generated free involutory quandle is isomorphic to $Core(\mathbb{Z})$ with generators 0 and 1.

Theorem (Joyce)

The free abelian involutory quandie on $n+1$ generators appears as

$$P = \{(k_1, \ldots, k_n) \in \mathbb{Z}^n \mid \text{at most one } k_i \text{ is odd}\}$$

as a subquandle of $Core(\mathbb{Z}^n)$. The generators are

$$e_0 = (0, 0, \ldots, 0), e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1).$$

As we know the multiplication in R_3 has the geometric interpretation. We can assume that at any vertex sits the reflection τ_i and $a_i * a_j = a_i \tau_j$.

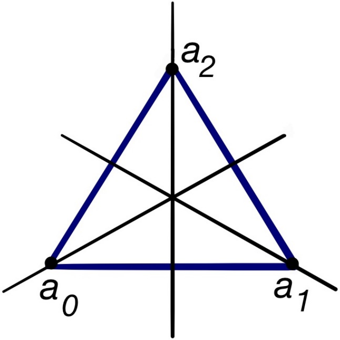

Figure: Dihedral quandle R_3

The following graph has the automorphism group $\mathbb{Z}_2 = \langle \tau \rangle$.

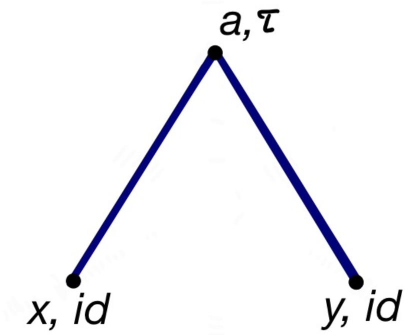

Figure: Graph with automorphisms

Define on the set of vertices $\{x, y, a\}$ the operation multiplications by the rules:

$$x * a = x\tau = y, \quad y * a = y\tau = x, \quad a * a = a\tau = a,$$

$$a * x = (a)id = a, \quad x * x = (x)id = x, \quad y * x = (y)id = y,$$

$$a * y = (a)id = a, \quad x * y = (x)id = x, \quad y * y = (y)id = y.$$

This is the second non-trivial 3-element quandle $Cz(4)$.

We can formulate

Research problem

For what finite quandles we can give the similar geometric interpretation?

To formalize this general problem we introduce some definitions and notations.

Let $\Gamma = \Gamma(V, E)$ be a simplicial graph, i.e. a graph without loops and multiple edges with the set of vertices V and the set of edges E. Any edge $e \in E$ is a pair $e = \{u, v\}$ for some $u, v \in V$.

An automorphism φ of Γ is a bijection of V onto V and E onto E which further satisfies

$$\varphi(e) = \{\varphi(u), \varphi(v)\} \text{ for any } e \in E.$$

The set of all graph automorphisms forms the automorphism group $\mathrm{Aut}(\Gamma)$.

Definition

A marked graph is a pair (Γ, S), where Γ is a graph and S is a map

$$S : V \to \mathrm{Aut}(\Gamma).$$

We denote the image of a vertex v under S by S_v.

Definition

A q-marked graph is a marked graph (Γ, S) such that $vS_v = v$ for any $v \in V$.

For simplicity we denote $S_i = S_{v_i}$.

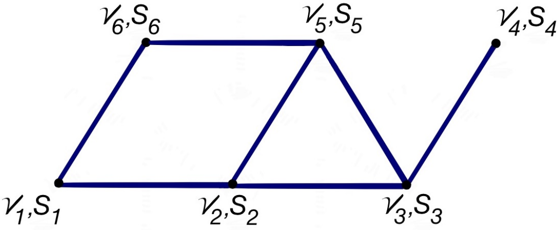

Figure: Marked graph

Suppose that (Γ, S) is a marked graph (q-marked graph). Define on the set V the binary algebraic operation $* : V \times V \to V$ by the rule

$$u * v = uS_v, \quad u, v \in V.$$

We call this algebraic system by graph groupoid (graph q-groupoid) and denote by V_S (correspondingly, V_S^q).

Research problems

1) Under what conditions V_S is a rack? (V_S^q is a quandle?)

2) Suppose that X is a rack (a quandle). Is there exist a marked graph (q-marked graph) (Γ, S) such that $X \cong V_S$ (correspondingly, $X \cong V_S^q$)?

If such graph Γ exists, does it connect with a Cayley graph of X?

If V_S is a rack we call it by graph rack. If V_S^q is a quandle we call it by graph quandle.

For construction of marked graphs for a graph Γ we have to know the automorphism group $\mathrm{Aut}(\Gamma)$.

We know these groups for trees.

You can start to study the first question for trees.

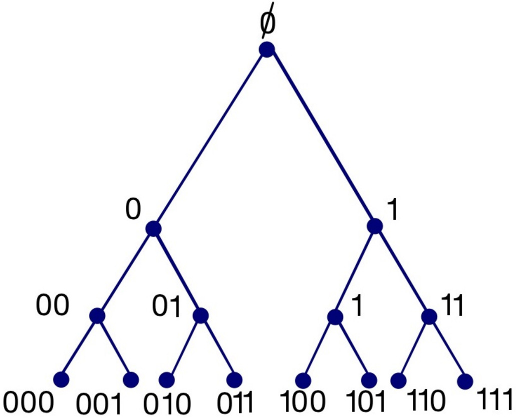

Figure: Rooted 2-tree of level 3

We know that any 3-element quandle is a graph quandle. You can start to study the second problem with the following questions.

Question

1) Is it true that any 4-element quandle is a graph quandle?

2) Is it true that any rack which contains less or equal 4 elements is a graph rack?

Let X be a non-empty set and let

$$S : X \times X \to X \times X$$

be a bijection. We say that S is a set-theoretical solution for the braid equation if

$$(S \times id)(id \times S)(S \times id) = (id \times S)(S \times id)(id \times S).$$

We shall briefly say that S is a solution or that (X, S) is a braided set.

A trivial example of a solution is the transposition

$$\tau : X \times X \to X \times X, \quad (x, y) \mapsto (y, x).$$

A map S is a solution if and only if

$$R = \tau S : X \times X \to X \times X$$

is a solution for the set-theoretic Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{ij} : X^3 \to X^3$ acts as R on the i-th and j-th factors and as identity map on the other factor.

If (X, S) is a braided set, there is an action of the braid group B_n on X^n, the standard generators σ_i acting by $S_{i,i+1}$, which means that S acts on the $i, i+1$ entries.

In particular, any braided set gives a representation of the braid group B_n by the permutations of the set X^n:

$$\rho_n : B_n \to Sym(X^n).$$

Let $X = \mathbb{Z}$ be the set of integers. Put

$$S(a, b) = (b + 1, a - 1).$$

You can check that (X, S) is a braided set. It defines the Cheng colouring from the lecture of Prof. Vesnin.

Problem

What is representation of B_n it defines? What is its kernel?

There is a close relation between racks and set-theoretical solutions for the Yang-Baxter equation, or, equivalently, for the braid equation.

Brieskorn (1988) observed that racks provide solutions for the braid equation. On the other hand, certain set-theoretical solutions for the braid equation produce racks.

Proposition (Brieskorn)

Let X be a set and let

$$* : X \times X \to X$$

be a function. Let

$$c : X \times X \to X \times X, \quad c(a, b) = (b * a, a).$$

Then c is a solution if and only if $(X, *)$ is a rack.

It is easy to check that c is a bijection if and only if

$$S_a : X \to X, \quad S_a(b) = b * a, \quad \text{is a bijection for all } a \in X.$$

c satisfies the braid equation if and only if

$$(c * b) * a = (c * a) * (b * c), \quad \text{for any } a, b, b \in X.$$

Definition

Let X, $\widetilde{X}$ be two non-empty sets and let

$$S : X \times X \to X \times X, \quad \widetilde{S} : \widetilde{X} \times \widetilde{X} \to \widetilde{X} \times \widetilde{X}$$

be two bijections. We say that (X, S) and $(\widetilde{X}, \widetilde{S})$ are equivalent if there exists a family of bijections $T^n : X^n \to \widetilde{X}^n$ such that

$$T^n S_{i,i+1} = \widetilde{S}_{i,i+1} T^n, \text{ for all } n \geq 2,\ 1 \leq i \leq n-1.$$

If (X, S) and $(\widetilde{X}, \widetilde{S})$ are equivalent and (X, S) is a solution, then $(\widetilde{X}, \widetilde{S})$ is also a solution and the T^n's intertwine the corresponding actions of the braid group B_n.

Definition

Let (X, S) be a braided set and let $f, g : X \to \mathrm{Fun}(X, X)$ be given by

$$S(a, b) = (g_a(b), f_b(a)).$$

The braided set (or the solution) is non-degenerate if the images of f and g lie inside $Sym(X)$.

The following results were found by A. Soloviev and Jiang-Hua Lu, Min Yan, Yong-Chang Zhu in 2000.

Theorem 1

Let S be a non-degenerate solution. Define $*$ by

$$b * a = f_a\left(g_{f_b^{-1}(a)}(b)\right).$$

(1) Then

– f preserves $*$, i.e. $f_a(c * b) = f_a(c) * f_a(b)$;

– $f_a f_b = f_{f_a(b)} f_{g_b(a)}$, for all $a, b \in X$.

(2) If $c : X \times X \to X \times X, \quad c(a,b) = (b * a, a)$, then c is a solution; we call it the derived solution of S. The solutions S and c are equivalent, and $(X, *)$ is a rack.

Theorem 2

Let $(X, *)$ be a rack and let $f : X \to Sym(X)$. We define $g : X \to Sym(X)$ by

$$g_a(b) = f^{-1}_{f_b(a)}(b * f_b(a)).$$

Let $S : X \times X \to X \times X$ be given by

$$S(a, b) = (g_a(b), f_b(a)).$$

Then S is a solution if and only if these conditions hold. If this happens, the solutions S and c are equivalent, and S is non-degenerate.

Let (X, S) be a non-degenerate solution and let $*$ be defined by

$$b * a = f_a\left(g_{f_b^{-1}(a)}(b)\right).$$

Then $(X, *)$ is a quandle if and only if

$$f_a^{-1}(a) = g_{f_a^{-1}(a)}(a), \quad a \in X.$$

Thank you!