Symmetric Exclusion Process with Stochastic Resetting

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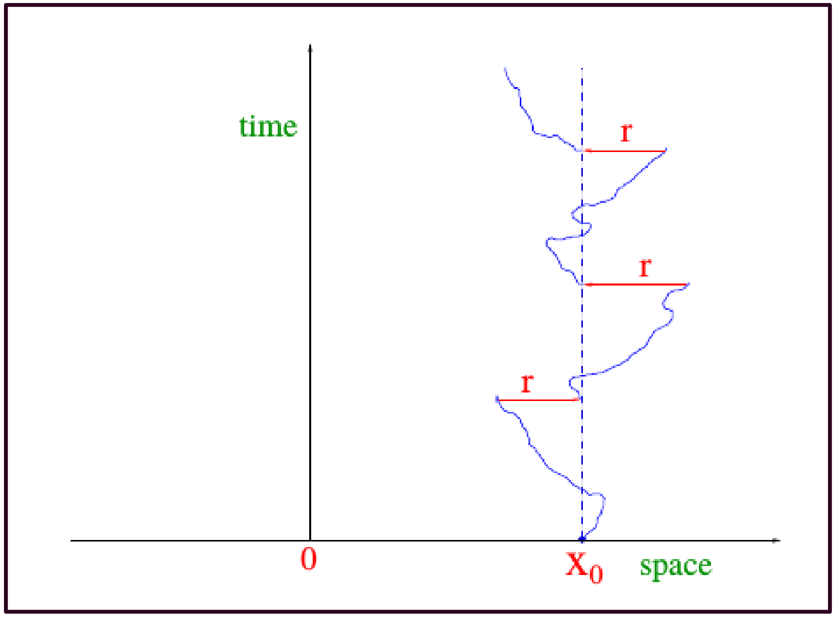
Stochastic Resetting

- Interrupt and restart a dynamical process with some rate
- Paradigmatic example: Brownian particle reset to the initial position with fixed rate r



- Leads to
  - Nontrivial stationary state
  - Non-monotonic mean first passage time
  - Anomalous persistence properties
  - …

[Evans and Majumdar, PRL 2011]

Many variations and generalizations studied …

Interacting systems

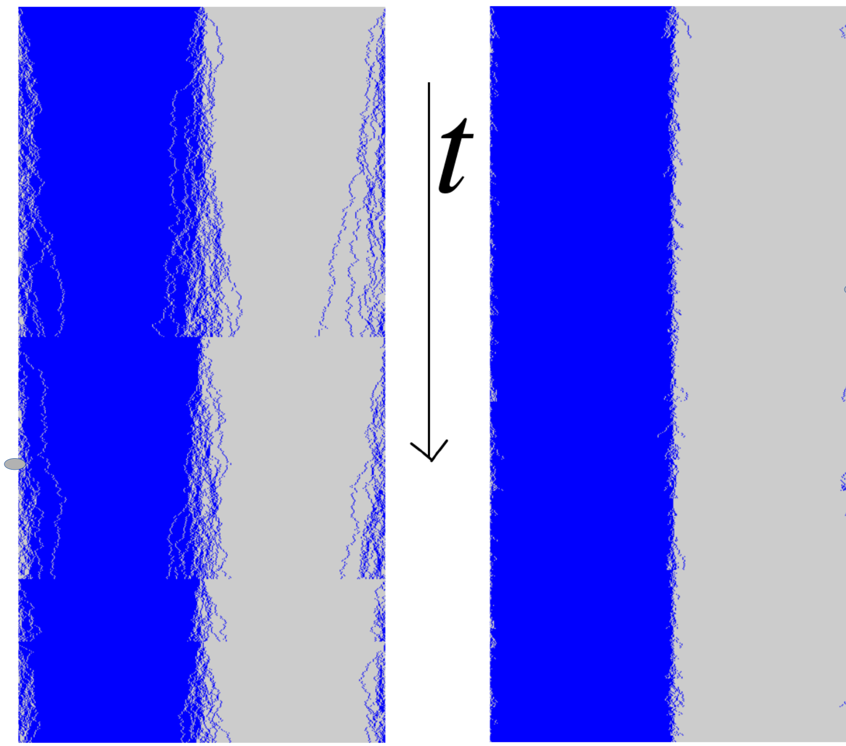
- What happens for interacting many-particle systems?
  - Stationary state?
  - Effect on particle current?
- Simple example – Symmetric exclusion process

Symmetric Exclusion Process under resetting

- Particles on 1-dimensional ring of size L
- Exclusion: each site has at most one particle
- Configuration C is $\{s_i = 0,1;\ i = 0,1,\ldots, L-1\}$
- Half-filling condition: L/2 particles in total
- Dynamics:
  - Hopping to vacant nearest neighbouring site with unit rate (ordinary SEP dynamics)
  - Resetting to a step-like configuration $C_0 = \{\ldots 111111000000\ldots\}$ with rate r .

Time evolution of configurations

Evolves as ordinary SEP between consecutive resetting events



Small
r <<1

large
r >>1

- Renewal equation for configuration weights

$$\mathcal{P}(\mathcal{C},t) \;=\; e^{-rt}\mathcal{P}_0(\mathcal{C},t) + r\int_0^t \mathrm{d}s\; e^{-rs}\mathcal{P}_0(\mathcal{C},s)$$

No resetting

One or more resetting

Weight in absence of resetting

Observables

- Density Profile: average particle density at sites

Stationary profile ?

- Particle current: net number of particles crossing the central bond during time [0,t]
  - Diffusive current (due to hopping)
  - Resetting current (due to resetting)

Statistical properties of these currents?

[UB, Kundu, Pal,
Phys. Rev. E 100, 032136 (2019)]

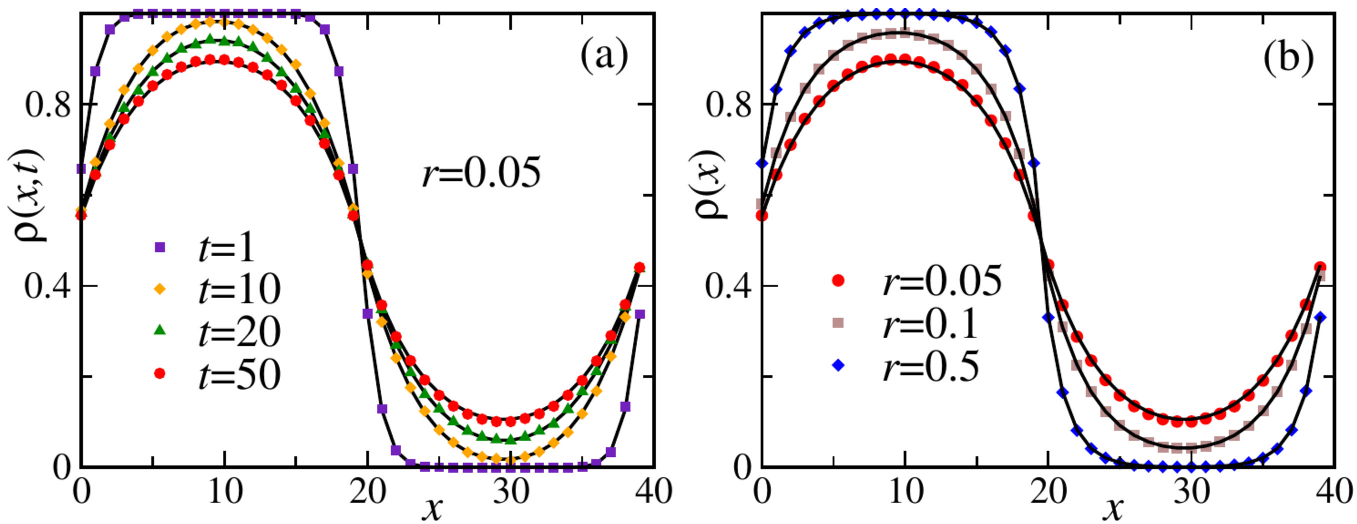
Recap: ordinary SEP

- Density profile becomes flat in the long-time limit
- Diffusive current increases with time
- For a thermodynamically large system, in the long-time regime,
  - Average current $\langle J_0(t) \rangle \simeq \sqrt{\frac{t}{\pi}}$
  - Variance $\langle J_0^2(t) \rangle - \langle J_0(t) \rangle^2 \simeq \sqrt{\frac{t}{\pi}} \left(1 - \frac{1}{\sqrt{2}} \right)$
  - Typical fluctuations of J_0: Gaussian

[B. Derrida, A. Gerschenfeld, J. Stat. Phys. (2009)]

Density Profile

- Can be calculated exactly using ordinary SEP results



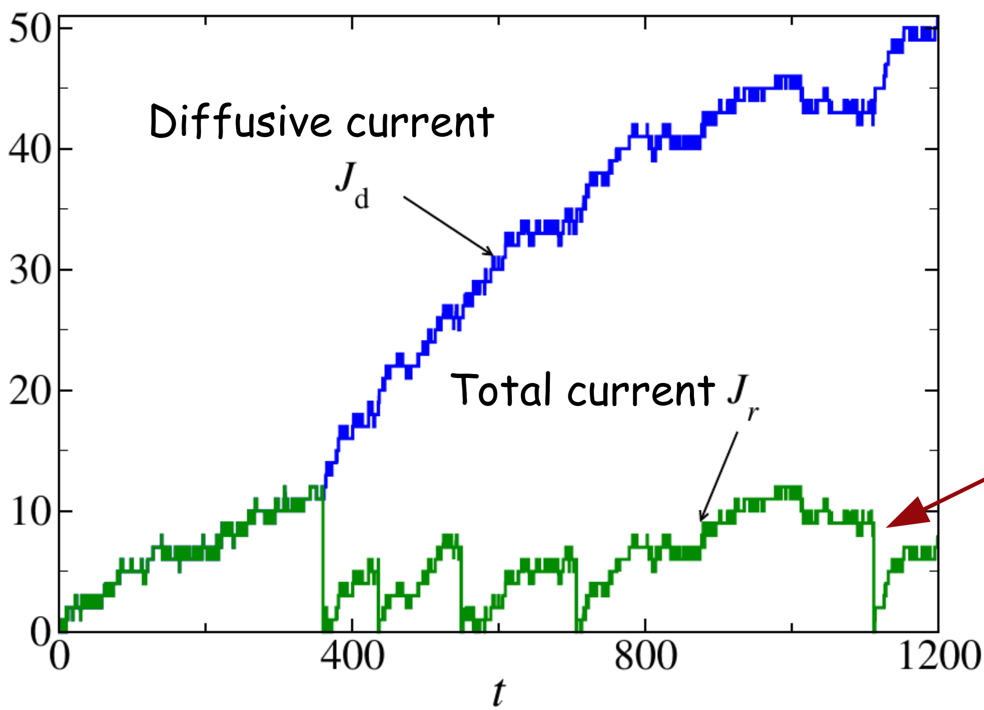
Inhomogeneous Stationary profile:

$$\rho(x) = \frac{1}{2} + \frac{r}{L} \sum_{n=1,3}^{L-1} e^{-i\frac{2\pi n x}{L}} \frac{\left(1 + i\ \cot\frac{\pi n}{L}\right)}{r + \lambda_n}$$

$$\lambda_n = 2\left(1 - \cos\frac{2\pi n}{L}\right)$$

Current

- Net number of particles crossing the central bond = #particles on the right-half of the lattice
- Total current J_r: diffusive + resetting currents



$$J_r(t) = J_{\mathrm{d}}(t) + J_{\mathrm{reset}}(t)$$

Resetting events

Total current vanishes after each resetting event …

Diffusive current J_d

- Net particles crossing the central bond due to hopping only
- Sum of diffusive currents between consecutive resettings
- For a trajectory with n resetting events,

$$J_{\mathrm{d}} = \sum_{i=1}^{n+1} J_0(t_i)$$

Hopping current for r=0

- Moments can be calculated using Laplace transform of the moment generating function

$$Q(s,\lambda) = \mathcal{L}_{t\to s}[\langle e^{\lambda J_{\mathrm{d}}}\rangle] = \int_0^\infty \mathrm{d}t\ e^{-st}\langle e^{\lambda J_{\mathrm{d}}}\rangle$$

Closed form expressions for small r

Average of J_d

- Exactly computed
- For small r

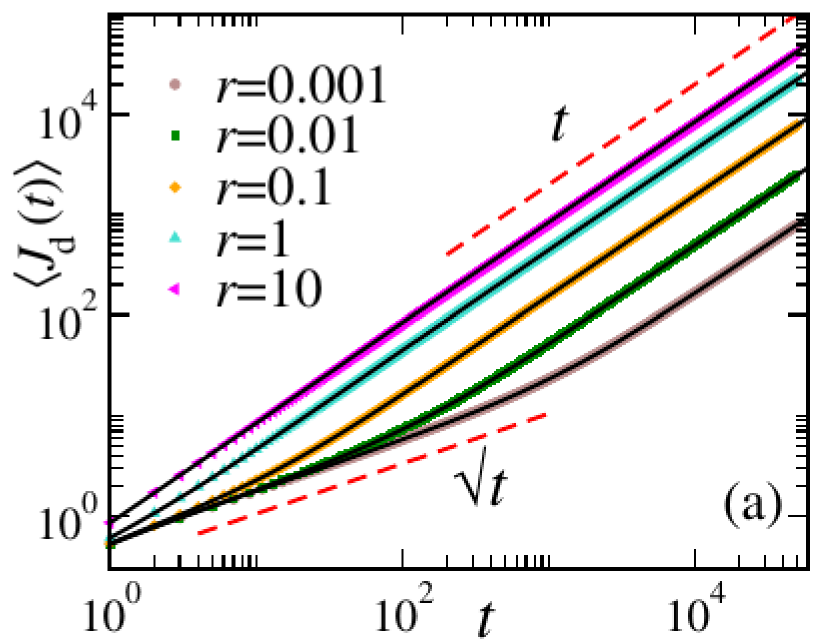
$$\langle J_{\rm d}(t)\rangle = \frac{1}{2\sqrt{r}}\left[\left(rt+\frac{1}{2}\right)\mathrm{erf}(\sqrt{rt})+\sqrt{\frac{rt}{\pi}}e^{-rt}\right]$$

- Short-time ($t \ll r^{-1}$):
  similar to SEP, grows $\sim\sqrt{t}$

- Long-time ($t \gg r^{-1}$):

Exact for all r

$$\langle J_{\rm d}(t)\rangle \simeq \sqrt{\frac{r}{r+4}}\,t.$$



★ Linear growth – much faster than the $\sqrt{t}$ behaviour for r=0 (ordinary SEP)

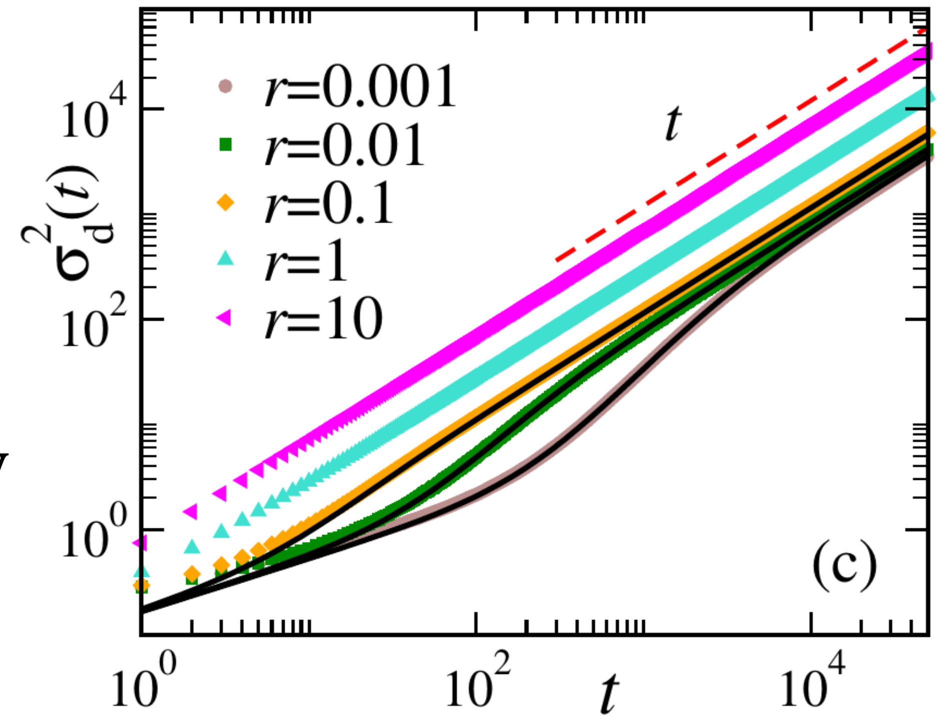
Variance of J_d

- Closed form expression for small r

$$\langle J_{\rm d}^2(t)\rangle = \frac{1}{4\pi}\left[t(\pi r t+4)+2b\sqrt{\pi t}e^{-rt} + \frac{b\pi}{\sqrt{r}}(1+2rt)\mathrm{erf}(\sqrt{rt})\right]$$

- Variance grows linearly with t at late times

$$\sigma_{\rm d}^2(t) \simeq t\left[\frac{4-\pi}{4\pi}+\frac{\sqrt{r}}{2}\left(1-\frac{1}{\sqrt{2}}\right)\right]$$



Reminder: For ordinary SEP, variance also grows ~ $\sqrt{t}$

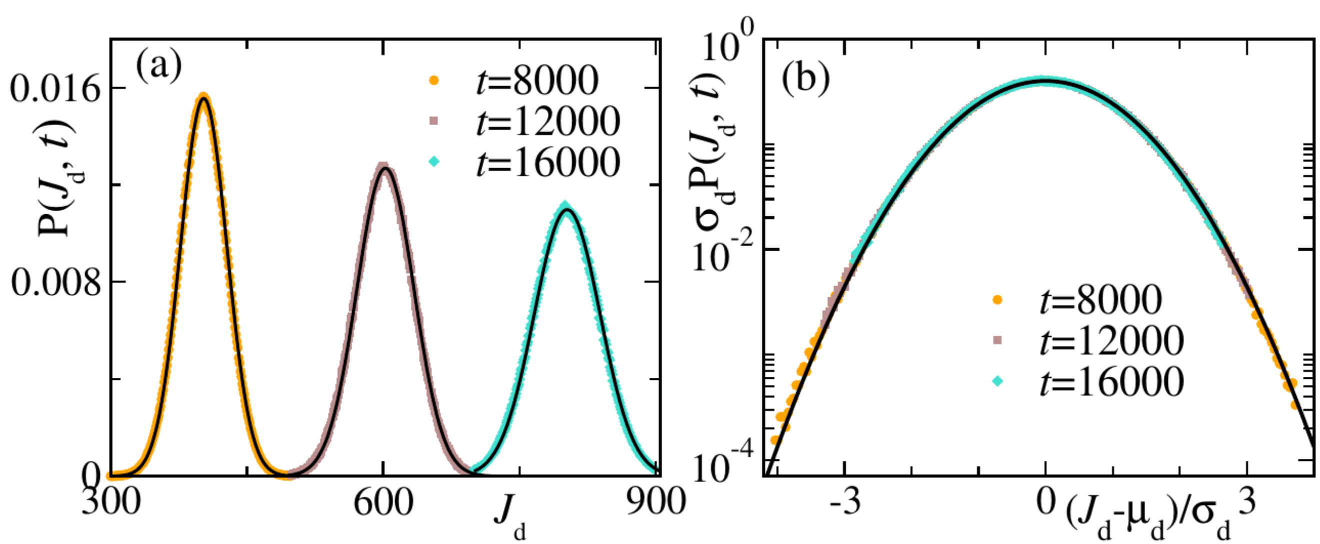
Distribution

- Sum of (n+1) independent, identically distributed variables

$$J_{\mathrm{d}} = \sum_{i=1}^{n+1} J_0(t_i)$$

- Gaussian distribution for large n, ie, at long time (rt >> 1)

$$P(J_{\mathrm{d}}, t) \simeq \frac{1}{\sqrt{2\pi\sigma_d^2(t)}} \exp\left(-\frac{[J_d - \mu_{\mathrm{d}}(t)]^2}{2\sigma_d^2(t)} \right)$$

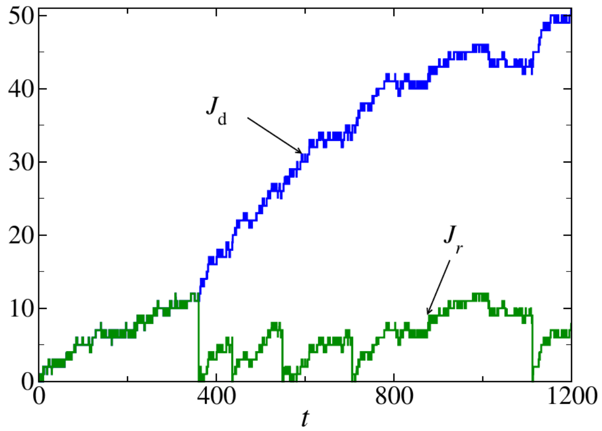


Typical fluctuations around the peak

r=0.01

Total current

- Undergoes a resetting itself – $J_r=0$ after each resetting event



- Renewal equation

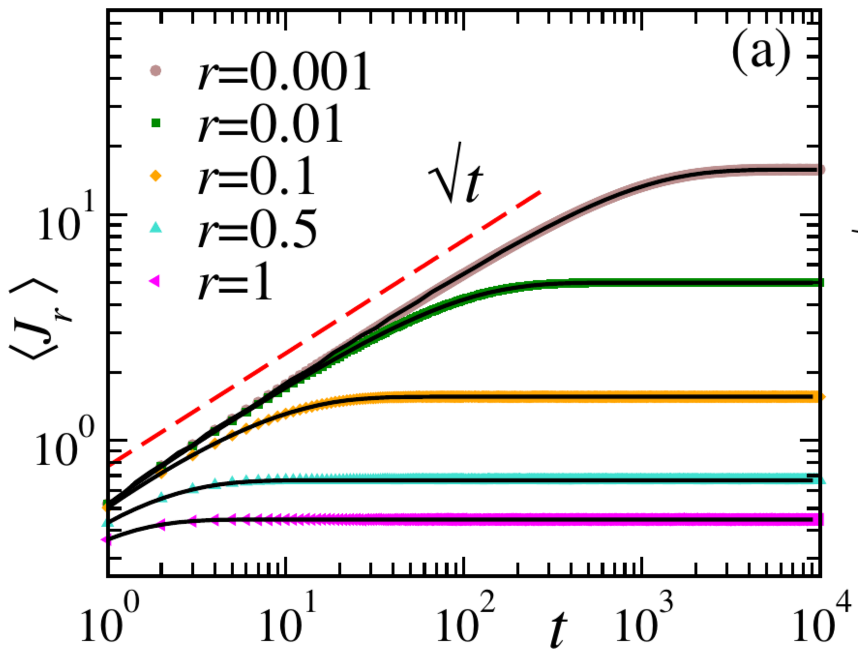
$$P_r(J_r,t) = e^{-rt}P_0(J_r,t) + r\int_0^t \mathrm{d}s\ e^{-rs}P_0(J_r,s)$$

- Reaches a stationary state
- Moments satisfy same renewal equation

Moments of J_r

- Average total current (for r << 1)

$$\langle J_r(t)\rangle = \frac{1}{2\sqrt{r}}\mathrm{erf}(\sqrt{rt})$$



- Stationary value (for any r) is $\frac{1}{\sqrt{r(r+4)}}$

- Variance shows similar behaviour

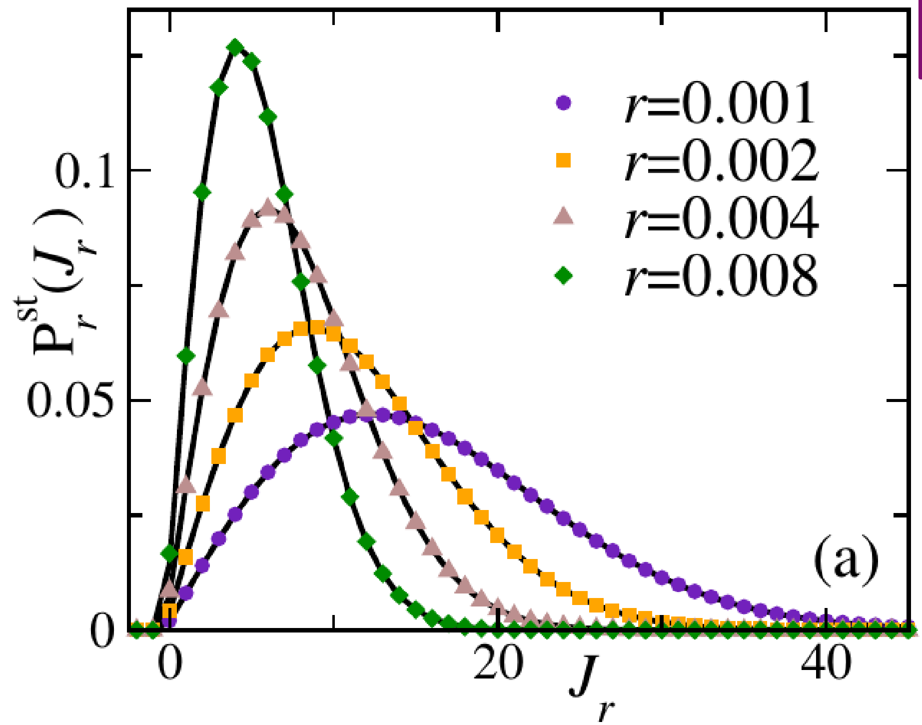
Distribution of total current

- Stationary distribution

For small r

$$P_r^{\rm st}(J_r) = r\int_0^\infty {\rm d}\tau\, \frac{e^{-r\tau}}{\sqrt{2\pi\sigma_\tau^2}}\exp\left[-\frac{(J_r-\mu_\tau)^2}{2\sigma_\tau^2}\right]$$

Gaussian distribution for SEP with r=0



$$P_r^{\rm st}(J_r) = \frac{2\sqrt{2}r}{\pi^{1/4}\sqrt{b}}\exp\left(\frac{J_r}{b}\right) \times \sum_{n=0}^\infty \frac{(-r)^n}{n!}\left(\sqrt{\pi}J_r\right)^{2n+\frac{3}{2}} K_{2n+\frac{3}{2}}\left(\frac{J_r}{b}\right)$$

★ Strongly non-gaussian skewed distribution

Conclusions

- Behaviour of Symmetric Exclusion process under stochastic resetting
- Inhomogeneous stationary density profile
- Diffusive current and its variance grows *linearly* with time in contrast to the $\sqrt{t}$ behaviour for ordinary SEP
- Total current reaches a stationary limit
- Non-gaussian fluctuations of the total current

Thank you!

Resetting current

- Net contribution from resetting events
- Negative
- Negatively correlated with diffusive current