Quantum Brownian motion: Transition from monotonic to oscillatory behaviour

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Outline

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- Quantum Brownian motion
  - **Linear response theory**
  - **Fluctuation dissipation theorem**
  - **Mean square displacement**

- Quantum Brownian motion in a magnetic field: Transition from monotonic to oscillatory behaviour
  - **Independent Oscillator model**
  - **Generalized quantum Langevin equation**
  - **Mean square displacement**

- Quantum Brownian motion

  - ➢ **Linear response theory**
  - ➢ **Fluctuation dissipation theorem**
  - ➢ **Mean square displacement**

- Quantum Brownian motion in a magnetic field: Transition from monotonic to oscillatory behaviour

  - ➢ **Independent Oscillator model**
  - ➢ **Generalized quantum Langevin equation**
  - ➢ **Mean square displacement**

Brownian motion

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- Brownian motion is the random motion of large particles suspended in a large bath.

$$m\frac{d^2x}{dt^2} = -m\gamma\frac{dx}{dt} + f(t) \longrightarrow \text{Langevin equation}$$

- Fluctuation dissipation theorem relates the two force components.

$$D = \frac{k_B T}{m\gamma} \longrightarrow \text{Einstein's relation}$$

$$\langle \Delta x^2 \rangle = 2Dt$$

$$\langle \Delta x^2 \rangle = \frac{2k_B T t}{m\gamma}$$

How does the Brownian particle diffuse in quantum regime?

Linear response theory

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$H = H_0 + H^F(t)$ where, $H^F(t) = -Af(t)$ is the perturbation.

ρ is equilibrium density matrix.
ρ' is non equilibrium density matrix.

$$\frac{d}{dt}\rho'(t) = \frac{1}{i\hbar}[H, \rho'(t)]$$

$\rho'(-\infty) = \rho$ is the initial condition.
$f(-\infty) = 0.$

$$\langle \Delta B(t) \rangle = \langle B(t) \rangle_{ne} - \langle B(t) \rangle_{eq}$$

$$= Tr\,\rho'(t)B - Tr\,\rho B = Tr\,\Delta\rho(t)B = -\frac{1}{i\hbar}Tr\int_{-\infty}^{t} dt'[A,\rho]B(t-t')f(t')$$

$$= \int_{-\infty}^{t} dt' R(t-t')f(t')$$

$$\boxed{R(t-t') = -\frac{1}{i\hbar}\,Tr\,[A,\rho]B(t-t') = \frac{1}{i\hbar}\,Tr\,\rho[A, B(t-t')] = \frac{1}{i\hbar}\,\langle [A, B(t-t')] \rangle}$$

$$R(t-t') = 0, t < t'$$

$$R(t-t') = \frac{1}{i\hbar}\,\langle [A, B(t-t')] \rangle \theta(t-t')$$

Ryogo Kubo, J. Phys. Soc. Jpn. **12**, pp. 570-586 (1957).

Fluctuation dissipation theorem

A general relationship between the response of a given system to an external disturbance and the internal fluctuation of the system in the absence of the disturbance is fluctuation dissipation theorem.

$$R(t) = \frac{1}{i\hbar} \langle [A, B(t)] \rangle \theta(t)$$

$$C(t) = \frac{1}{2} \langle \{A, B(t)\} \rangle$$

$$\tilde{R}(\omega) = \int_{0}^{\infty} dt\, e^{i\omega t} R(t)$$

$$\tilde{C}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} C(t)$$

$$\tilde{R}(\omega) = \int_0^\infty dt\, e^{i\omega t} R(t)$$

$$= \frac{1}{i\hbar} \int_0^\infty dt\, e^{i\omega t}\ Tr\, \rho[A, B(t)]$$

$$= \frac{1}{i\hbar}\frac{1}{Z} \sum_n \sum_m e^{-\beta E_n}\left(1 - e^{\beta\hbar\omega_{nm}}\right) \int_0^\infty dt\ e^{i\omega t} e^{-it\omega_{nm}} A_{nm} B_{mn}$$

$$= \frac{1}{i\hbar}\frac{1}{Z} \sum_n \sum_m e^{-\beta E_n}\left(1 - e^{\beta\hbar\omega}\right) \pi\delta(\omega - \omega_{nm}) A_{nm} B_{mn}$$

$$\tilde{C}(\omega) = \int_{-\infty}^\infty dt\, e^{i\omega t} C(t)$$

$$= \frac{1}{2} \int_{-\infty}^\infty dt\, e^{i\omega t}\ Tr\, \rho\{A, B(t)\}$$

$$= \frac{1}{2}\frac{1}{Z} \sum_n \sum_m e^{-\beta E_n}\left(1 + e^{\beta\hbar\omega_{nm}}\right) \int_{-\infty}^\infty dt\ e^{i\omega t} e^{-it\omega_{nm}} A_{nm} B_{mn}$$

$$= \frac{1}{Z} \sum_n \sum_m e^{-\beta E_n}\left(1 + e^{\beta\hbar\omega}\right) \pi\delta(\omega - \omega_{nm}) A_{nm} B_{mn}$$

$$\frac{Im\left(\tilde{R}(\omega)\right)}{\tilde{C}(\omega)} = -\frac{1}{\hbar}\frac{\left(1 - e^{\beta\hbar\omega}\right)}{\left(1 + e^{\beta\hbar\omega}\right)} = \frac{1}{\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)$$

$$\tilde{C}(\nu) = \hbar \coth(\pi\beta\hbar\nu) Im\, \tilde{R}(\nu)$$

R. Balescu, Equilibrium and Non-Equilibrium Statistical Mechanics (John Wiley & Sons, 1975) pp 663-669.

Fluctuation dissipation theorem

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A general relationship between the response of a given system to an external disturbance and the internal fluctuation of the system in the absence of the disturbance is fluctuation dissipation theorem.

$$\boxed{\tilde{C}(\nu) = \hbar \coth(\pi\beta\hbar\nu) Im\, \tilde{R}(\nu)}$$

$$\boxed{C(t) = \frac{1}{2\beta}\int_0^\infty dt'\, R(t') \left[\coth\left(\frac{t'-t}{t_{th}}\right) + \coth\left(\frac{t'+t}{t_{th}}\right)\right]}$$

$$t_{th} = \frac{\beta\hbar}{\pi}, \text{ is thermal time}$$

Mean square displacement

$$\langle \Delta x^2 \rangle = \langle [x(t) - x(0)]^2 \rangle = \langle x(t)^2 \rangle + \langle x(0)^2 \rangle - \langle \{x(t), x(0)\} \rangle \;\; = 2C(0) - 2C(t)$$

$$\boxed{\langle \Delta x^2 \rangle = \frac{1}{\beta}\int_0^\infty dt'\, R(t') \left[2\coth\left(\frac{t'}{t_{th}}\right) - \coth\left(\frac{t'-t}{t_{th}}\right) - \coth\left(\frac{t'+t}{t_{th}}\right)\right]}$$

Supurna Sinha and Rafael D. Sorkin, *Phys. Rev. B*. **45**, 8123 (1992).

Step function R(t)

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Supurna Sinha and Rafael D. Sorkin, *Phys. Rev. B*. **45**, 8123 (1992).

$$R(t) = \mu\Theta(t-\tau)$$

μ is mobility

τ is relaxation time

$$\langle \Delta x^2 \rangle = \frac{1}{\beta}\int_0^\infty dt'\, R(t')\left[2\coth\left(\frac{t'}{t_{th}}\right) - \coth\left(\frac{t'-t}{t_{th}}\right) - \coth\left(\frac{t'+t}{t_{th}}\right)\right]$$

$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta} t_{th}\left\{ ln\left[\frac{\sqrt{\sinh\left(\frac{|t-\tau|}{t_{th}}\right)\sinh\left(\frac{|t+\tau|}{t_{th}}\right)}}{\sinh\left(\frac{\tau}{t_{th}}\right)}\right]\right\}, \qquad t \gg \tau$$

For $t \to \tau$, L.H.S is +ve but R.H.S is –ve.

Does not satisfy positivity conditions.

Positivity conditions

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1. **Wightman positivity**
2. **Passivity**

Wightman positivity

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Two point Wightman function: $W(t) = \langle x(t)x(0)\rangle$

Wightman positivity $\boxed{\tilde{W}(\nu) \geq 0}$

$$W(t) - W(-t) = -i\hbar\check{R}(t), \;\; W(-t) = \langle x(0)x(t)\rangle$$

Equivalent odd Response function:

$$\check{R}(t) = \mathrm{sgn}(t)R(|t|)$$

$$\tilde{\check{R}}(\nu) = 2iIm\,\tilde{R}(\nu)$$

$$\tilde{W}(-\nu) = e^{2\pi\beta\hbar\nu}\tilde{W}(\nu)$$

$$\boxed{\tilde{W}(\nu) = \frac{i\hbar}{1 - e^{2\pi\beta\hbar\nu}}\tilde{\check{R}}(\nu)}$$ Iff R.H.S is positive.

Passivity

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The work done on the system:

$$\boxed{\bar{W} \geq 0}$$

$$\bar{W} = \int_{-\infty}^{\infty} dt\, f(t)\langle \dot{x}(t)\rangle$$

Using Linear Response Theory, $\langle x(t)\rangle - \langle x(0)\rangle = \int_{-\infty}^{t} dt'\, R(t-t')f(t')$

$$\langle \dot{x}(t)\rangle = \int_{-\infty}^{t} dt'\, K(t-t')f(t') \qquad K(t) = \frac{dR(t)}{dt}$$

$$\bar{W} = \int_{-\infty}^{\infty} dt\, f(t) \int_{-\infty}^{t} dt'\, K(t-t')f(t')$$

$$= \int_{-\infty}^{\infty} d\nu\, \tilde{K}(\nu)\left|\tilde{f}(\nu)\right|^2$$

$$= 2\int_{0}^{\infty} d\nu\, Re\, \tilde{K}(\nu)\left|\tilde{f}(\nu)\right|^2$$

$$\boxed{Re\, \tilde{K}(\nu) \geq 0}$$

Positivity conditions

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1. **Wightman positivity** $\boxed{\tilde{W}(\nu) \geq 0}$
2. **Passivity** $\boxed{Re\ \tilde{K}(\nu) \geq 0}$

$$Im\ \tilde{R}(\nu) = \frac{1}{\hbar}\tanh(\pi\beta\hbar\nu)\tilde{C}(\nu)$$

$$Re\ \tilde{K}(\nu) = \frac{\pi\nu}{\hbar}\left(e^{2\pi\beta\hbar\nu} - 1\right)\tilde{W}(\nu)$$

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$$R(t) = \mu\left(1 - e^{-t/\tau}\right)\theta(t)$$

μ is mobility, τ is relaxation time

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Positivity conditions: $R(t) = \mu\left(1 - e^{-t/\tau}\right)\theta(t)$

μ is mobility, τ is relaxation time

Wightman positivity

$$\widetilde{W}(\nu) = \frac{i\hbar}{1 - e^{2\pi\beta\hbar\nu}}\tilde{\tilde{R}}(\nu) \geq 0$$

$$i\tilde{\tilde{R}}(\nu) = \mu\left[-2\, Im\int_{-\infty}^{\infty} dt\, e^{2\pi i\nu t} + 2\, Im\int_{-\infty}^{\infty} dt\, e^{2\pi i\nu t} e^{-t/\tau}\right]$$

$$= \frac{-\frac{\mu}{\tau^2}}{\pi\nu\left(\frac{1}{\tau^2} + (2\pi\nu)^2\right)}$$

$$\widetilde{W}(\nu) = \frac{\frac{\hbar\mu}{\tau^2}}{\pi\nu\left(\frac{1}{\tau^2} + (2\pi\nu)^2\right)\left(e^{2\pi\beta\hbar\nu} - 1\right)} \geq 0$$

Passivity

$$Re\,\widetilde{K}(\nu) \geq 0$$

$$K(t) = \frac{dR(t)}{dt}$$

$$\widetilde{K}(\nu) = \frac{\frac{\mu}{\tau}}{\frac{1}{\tau} - 2\pi i\nu}$$

$$Re\,\widetilde{K}(\nu) = \frac{\frac{\mu}{\tau^2}}{\left(\frac{1}{\tau^2} + (2\pi\nu)^2\right)} \geq 0$$

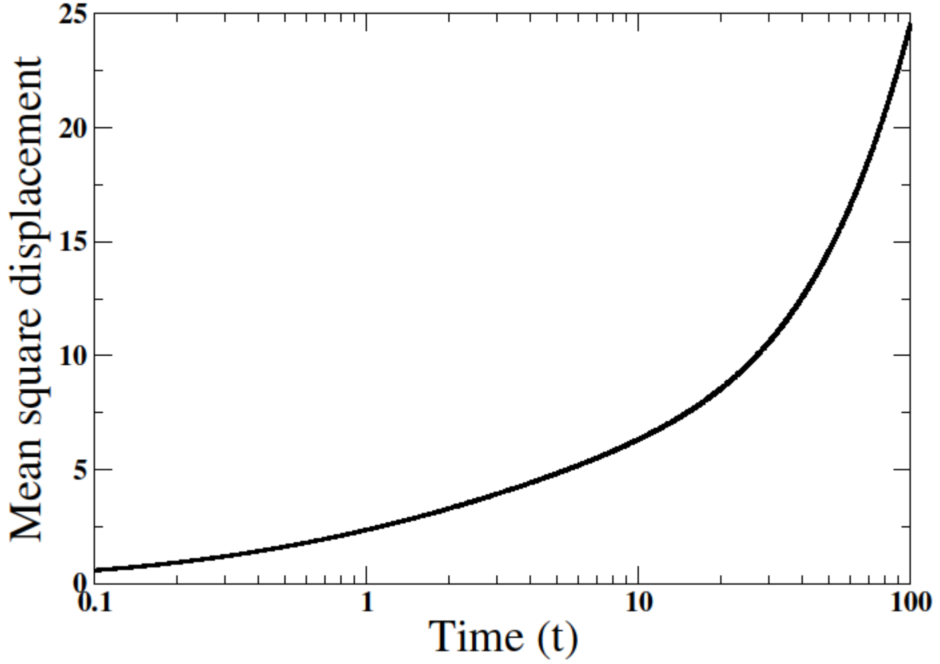
Urbashi Satpathi, Supurna Sinha and Rafael D. Sorkin, J. Stat. Mech. **2017**, 123105

Mean square displacement

$$R(t) = \mu\left(1 - e^{-t/\tau}\right)\theta(t)$$

$$\langle \Delta x^2 \rangle = \frac{1}{\beta}\int_0^\infty dt'\, R(t') \left[2\coth\left(\frac{t'}{t_{th}}\right) - \coth\left(\frac{t'-t}{t_{th}}\right) - \coth\left(\frac{t'+t}{t_{th}}\right)\right]$$

$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta} t_{th} \left\{ ln\left[2\sinh\left(\frac{t}{t_{th}}\right)\right] + \psi^0\left(1 + \frac{t_{th}}{2\tau}\right) + \gamma + \frac{2\tau}{t_{th}}\left[\ {}_2F_1\left(1, \frac{t_{th}}{2\tau}, 1 + \frac{t_{th}}{2\tau}, e^{-\frac{2t}{t_{th}}}\right)\right] - 1\right\}$$



ψ_0 is polygamma function.
γ is Euler-Macheroni constant.
${}_2F_1$ is hypergeometric function.

t : observation time
τ : relaxation time
t_{th} : thermal time

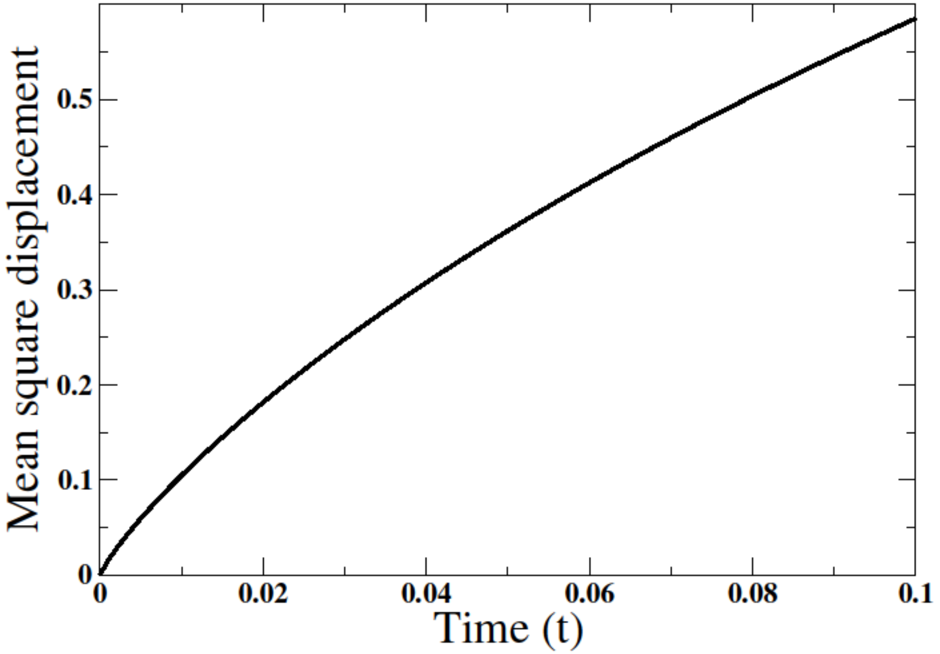
Limiting Cases

$$\left.\begin{array}{l} \text{Case 1: } t \ll \tau \ll t_{th} \\ \text{Case 2: } t \ll t_{th} \ll \tau \\ \text{Case 3: } t_{th} \ll t \ll \tau \end{array}\right\} \quad t \ll \tau$$

$$\left.\begin{array}{l} \text{Case 4: } \tau \ll t \ll t_{th} \\ \text{Case 5: } \tau \ll t_{th} \ll \text{t} \\ \text{Case 6: } t_{th} \ll \tau \ll t \end{array}\right\} \quad t \gg \tau$$

Mean square displacement : Case 1

$\boxed{t \ll \tau \ll t_{th}}$



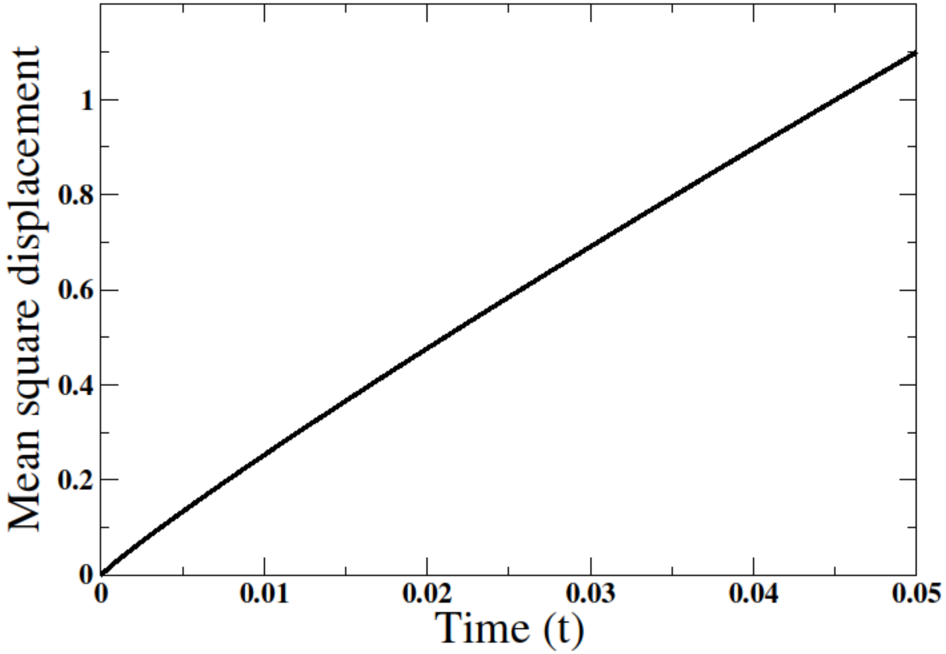
$$\langle \Delta x^2 \rangle = \frac{2\mu\hbar}{\pi}\left\{\frac{t}{\tau}\left[1 - ln\left(\frac{t}{\tau}\right) - \gamma + \frac{\tau}{t_{th}}\right]\right\}$$

$$\tau = 10\ \mu s$$
$$t \sim ns, T \sim \mu K$$

Chu S, Hollberg L, Bjorkholm J E, Cable A and Ashkin A 1985 *Phys. Rev. Lett.* **55** 48.

Mean square displacement : Case 2

$\boxed{t \ll t_{th} \ll \tau}$

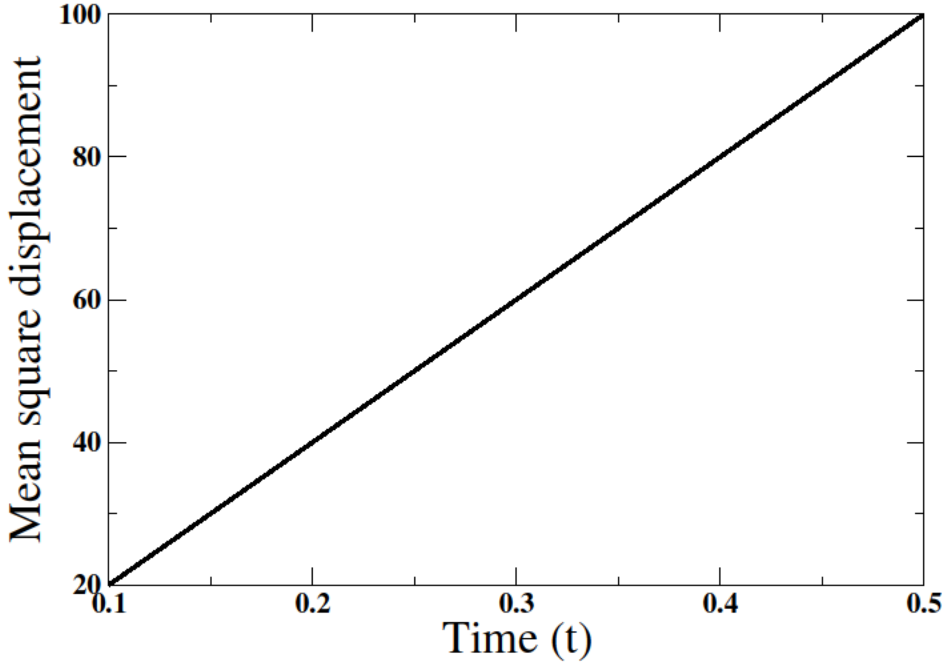


$$\langle \Delta x^2 \rangle = \frac{2\mu\hbar}{\pi}\left\{\frac{t}{\tau}\left[1 - ln\left(\frac{2t}{t_{th}}\right) - \frac{\pi^2}{12}\frac{t_{th}}{\tau} + \frac{\tau}{t_{th}}\right]\right\}$$

$\tau = 10\ \mu s$
$t \sim ns$ or less
$T \sim \mu K$ to mK

Mean square displacement : Case 3

$\boxed{t_{th} \ll t \ll \tau}$



$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta}\left\{t + \frac{\pi^2}{12}\frac{t_{th}^2}{\tau}\right\}$$

$\tau = 10\ \mu s$

$t \sim \mu s$

$T \sim mK$

$t \gg \tau$

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$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta} t_{th} \left\{ ln \left[2 \sinh\left(\frac{t}{t_{th}}\right) \right] + \psi^0 \left(1 + \frac{t_{th}}{2\tau}\right) + \gamma + \frac{2\tau}{t_{th}} \left[{}_2F_1 \left(1, \frac{t_{th}}{2\tau}, 1 + \frac{t_{th}}{2\tau}, e^{-\frac{2t}{t_{th}}}\right) \right] - 1 \right\}$$

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$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta} t_{th} \left\{ ln \left[\frac{\sqrt{\sinh\left(\frac{|t-\tau|}{t_{th}}\right) \sinh\left(\frac{|t+\tau|}{t_{th}}\right)}}{\sinh\left(\frac{\tau}{t_{th}}\right)} \right] \right\}$$

For, $R(t) = \mu\Theta(t - \tau)$

Supurna Sinha and Rafael D. Sorkin, *Phys. Rev. B*. **45**, 8123 (1992).

Mean square displacement : Quantum

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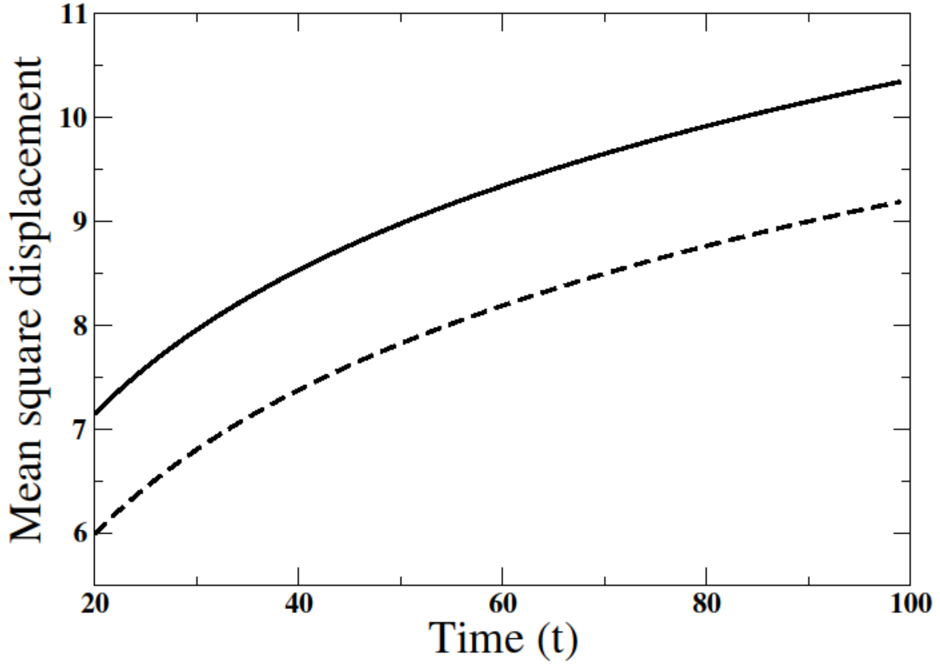
$\boxed{\tau \ll t \ll t_{th}}$



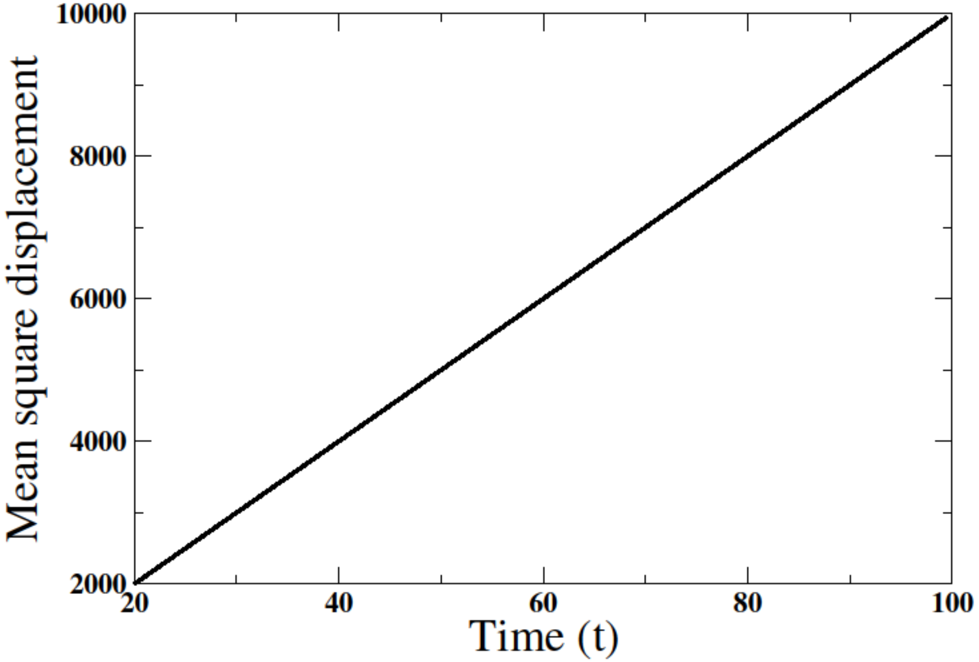
$$\langle \Delta x^2 \rangle = \frac{2\mu\hbar}{\pi}\left\{ln\left(\frac{t}{\tau}\right) + \gamma\right\}$$

$$\langle \Delta x^2 \rangle = \frac{2\mu\hbar}{\pi} ln\left(\frac{t}{\tau}\right)$$

$$\tau = 10\ \mu s$$
$$t \sim ms$$
$$T \sim nK$$

Mean square displacement : Classical

$\boxed{t_{th} \ll \tau \ll t}$



$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta} t$$

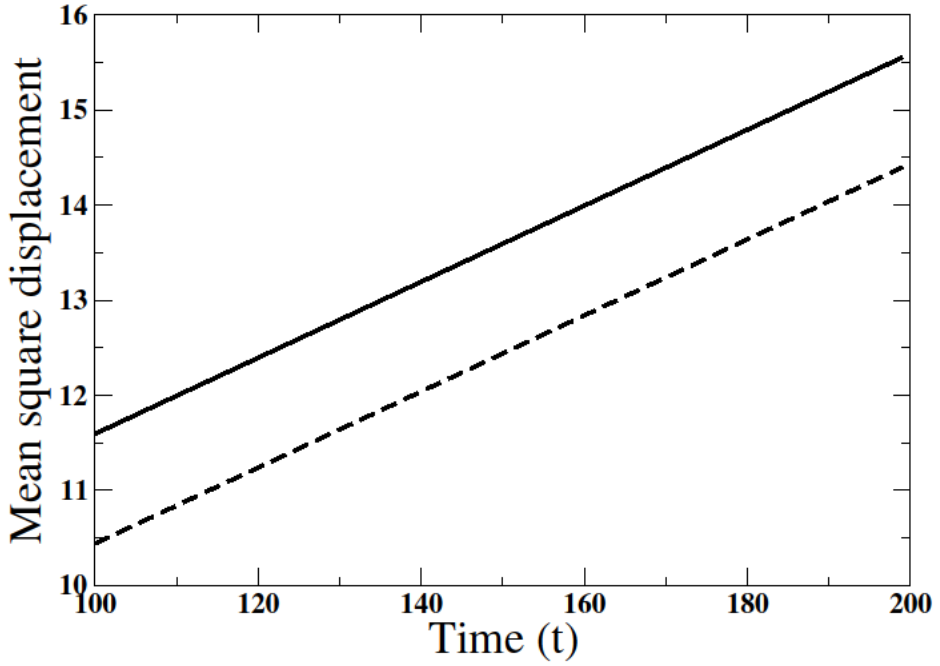
$\tau = 10\ \mu s$

$t \sim s$

$T \sim mK$ to K

Mean square displacement : Intermediate

$\boxed{\tau \ll t_{th} \ll t}$



$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta}\left\{t + t_{th}\left[ln\left(\frac{t_{th}}{2\tau}\right) + \gamma\right]\right\}$$

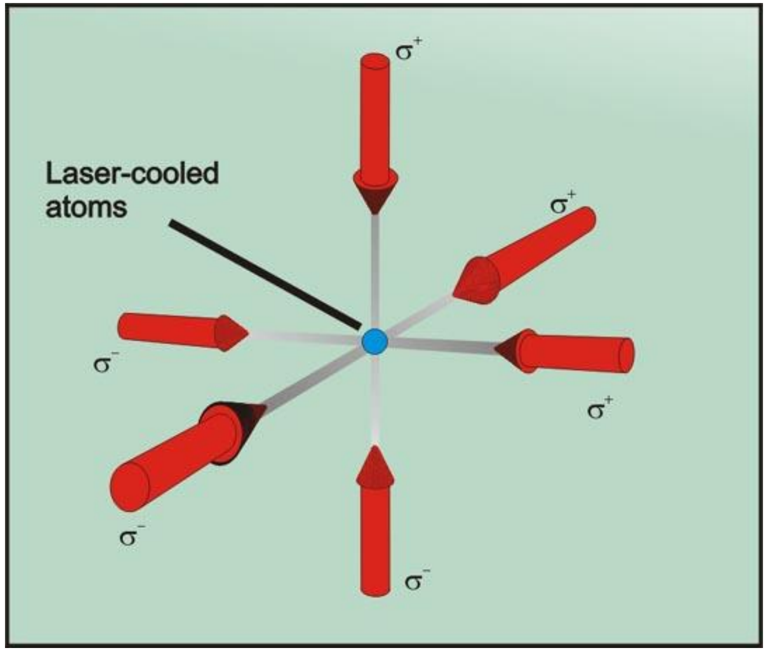
$$\langle \Delta x^2 \rangle = \frac{2\mu}{\beta}\left\{t + t_{th} ln\left(\frac{t_{th}}{2\tau}\right)\right\}$$

$\tau = 10\ \mu s$

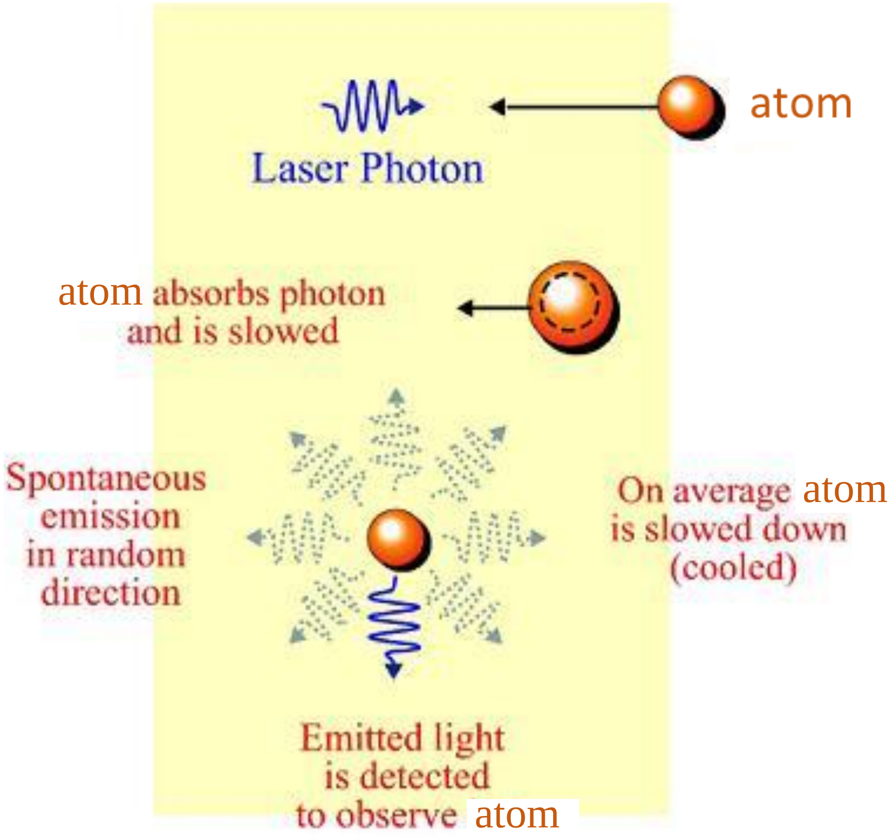
$t \sim ms$

$T \sim \mu K$

Experimental implication



Optical molasses



Collaboration with Dr. Sanjukta Roy, LAMP, RRI

Conclusions

- On the basis of FDT and a choice of RF, MSD temporal growth is analysed.
- $R(t) = \mu\left(1 - e^{-t/\tau}\right)\theta(t)$, satisfies positivity requirements: Wightman positivity and linear order passivity.
- Probe the short time regime: $t \ll \tau$.
- The logarithmic behaviour is robust.
- The law of diffusion: can be tested in cold atom laboratory experiments.

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  - Linear response theory
  - Fluctuation dissipation theorem
  - Mean square displacement

- Quantum Brownian motion in a magnetic field: Transition from monotonic to oscillatory behaviour
  - **Independent Oscillator (IO) model**
  - **Generalized quantum Langevin equation**
  - **Mean square displacement**

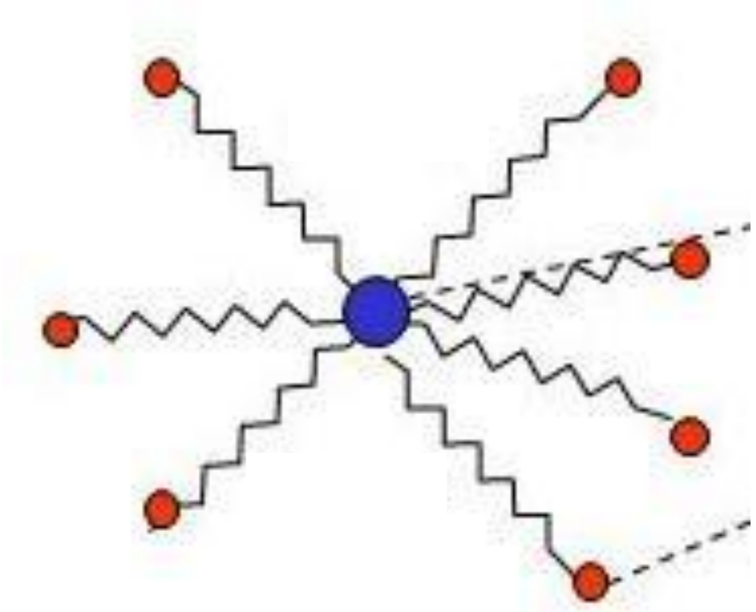
The IO model Hamiltonian

$$H = H_S + H_B + H_I$$

$$H_S = \frac{p^2}{2M} + V(q)$$

$$H_B = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 q_i^2 \right)$$

$$H_I = -q \sum_{i=1}^{N} c_i q_i + q^2 \sum_{i=1}^{N} \frac{c_i^2}{2 m_i \omega_i^2}$$



$$\boxed{H = \frac{p^2}{2M} + V(q) + \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + \frac{m_i \omega_i^2}{2} \left(q_i - \frac{c_i}{m_i \omega_i^2} q \right)^2} \qquad (1)$$

Generalized Langevin equation

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Equation of motion for system degrees of freedom $$M\ddot{q} = -\frac{dV}{dq} + \sum_{i=1}^{N} c_i \left(q_i - \frac{c_i}{m_i \omega_i^2} q \right) \qquad (2)$$

Equation of motion for bath degrees of freedom $$m_i \ddot{q}_i = -m_i \omega_i^2 q_i + c_i q \qquad (3)$$

$$\boxed{q_i(t) = q_i(t_0)\cos\big(\omega_i(t-t_0)\big) + \frac{p_i(t_0)}{m_i}\sin\big(\omega_i(t-t_0)\big) + \frac{c_i}{m_i\omega_i}\int_{t_0}^{t} ds\, \sin\big(\omega_i(s-t_0)\big)\, q(s)} \qquad (4)$$

$$\boxed{M\ddot{q} = -\frac{dV}{dq} - \sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i^2} q + \sum_{i=1}^{N} \frac{c_i^2}{m_i\omega_i}\int_{t_0}^{t} ds\, \sin\big(\omega_i(s-t_0)\big)\, q(s) + \sum_{i=1}^{N} c_i \left(q_i(t_0)\cos\big(\omega_i(t-t_0)\big) + \frac{p_i(t_0)}{m_i}\sin\big(\omega_i(t-t_0)\big) \right)}$$

Generalized Langevin equation

$$\boxed{M\ddot{q} = -\frac{dV}{dq} - M\int_{t_0}^{t} ds\, \mu(s - t_0)\, \dot{q}(s) + F(t)} \qquad (5)$$

$$\mu(t) = \frac{1}{M}\sum_{i=1}^{N}\frac{c_i^2}{m_i\omega_i^2}\cos(\omega_i t)\theta(t) \longrightarrow \text{Memory friction kernel}$$

$$\text{Random force} \longrightarrow F(t) = \sum_{i=1}^{N} c_i\left(\left(q_i(t_0) - \frac{c_i}{m_i\omega_i^2}q(t_0)\right)\cos\big(\omega_i(t - t_0)\big) + \frac{p_i(t_0)}{m_i}\sin\big(\omega_i(t - t_0)\big)\right)$$

$$\langle F(t)\rangle = 0$$

$$\langle\{F(t), F(t')\}\rangle = \hbar\sum_{i}\frac{c_i^2}{m_i\omega_i}\coth\left(\frac{\beta\hbar\omega_i}{2}\right)\cos(\omega_i(t - t'))$$

Generalized Langevin equation in the presence of magnetic field

$$\boxed{M\ddot{\vec{r}}(t) = -\frac{dV}{dq} - M\int_{t_0}^{t} \mu(t-t')\,\dot{\vec{r}}(t')dt' + \frac{q}{c}\left(\dot{\vec{r}}(t) \times \vec{B}\right) + \vec{F}(t)}$$

$$\mu(t) = \frac{1}{M}\sum_{i=1}^{N}\frac{c_i^2}{m_i\omega_i^2}\cos(\omega_i t)\theta(t)$$

$$F(t) = \sum_{i=1}^{N} c_i\left(\left(q_i(t_0) - \frac{c_i}{m_i\omega_i^2}q(t_0)\right)\cos\big(\omega_i(t-t_0)\big) + \frac{p_i(t_0)}{m_i}\sin\big(\omega_i(t-t_0)\big)\right)$$

$$\langle F(t)\rangle = 0$$

$$\left\langle\left\{F_\alpha(t), F_\beta(t')\right\}\right\rangle = \hbar\delta_{\alpha\beta}\sum_i \frac{c_i^2}{m_i\omega_i}\coth\left(\frac{\beta\hbar\omega_i}{2}\right)\cos(\omega_i(t-t'))$$

Generalized Langevin equation in the presence of magnetic field

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$$\boxed{M\ddot{\vec{r}}(t) = -\frac{dV}{dq} - M\int_{t_0}^{t}\mu(t-t')\,\dot{\vec{r}}(t')dt' + \frac{q}{c}\left(\dot{\vec{r}}(t)\times\vec{B}\right) + \vec{F}(t)}$$

$$\mu(t) = \frac{1}{M}\sum_{i=1}^{N}\frac{c_i^2}{m_i\omega_i^2}\cos(\omega_i t)\theta(t)$$

$$F(t) = \sum_{i=1}^{N} c_i\left(\left(q_i(t_0) - \frac{c_i}{m_i\omega_i^2}q(t_0)\right)\cos\left(\omega_i(t-t_0)\right) + \frac{p_i(t_0)}{m_i}\sin\left(\omega_i(t-t_0)\right)\right)$$

$$\langle F(t)\rangle = 0$$

$$\left\langle\left\{F_\alpha(t), F_\beta(t')\right\}\right\rangle = \frac{\delta_{\alpha\beta}}{2\pi}\int_0^\infty d\omega\,\hbar\omega\coth\left(\frac{\hbar\omega}{2k_BT}\right)\,Re[\mu(\omega)]\cos(\omega(t-t'))$$

X.L. Li, G.W. Ford, R.F. O'Connell, Phys. Rev. A **41** (1990) 5287, X.L. Li, G.W. Ford, R.F. O'Connell, Phys. Rev. A **42** (1990) 4519.

Quantum Brownian motion in a magnetic field

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Quantum Langevin equation of a charged particle in the presence of a magnetic field:

$$m\ddot{\vec{r}}(t) = -\int_{-\infty}^{t} \mu(t-t')\dot{\vec{r}}(t')dt' + \frac{q}{c}\left(\dot{\vec{r}}(t)\times\vec{B}\right) + \vec{F}(t) \qquad (1)$$

$$\ddot{\vec{r}}(t) = -\int_{-\infty}^{t} K(t-t')\dot{\vec{r}}(t')dt' + \frac{q}{mc}\left(\dot{\vec{r}}(t)\times\vec{B}\right) + \frac{\vec{F}(t)}{m} \quad , \qquad K(t) = \frac{\mu(t)}{m}$$

$$\langle F(t)\rangle = 0$$

$$\langle\{F_\alpha(t), F_\beta(t')\}\rangle = \frac{\delta_{\alpha\beta}}{2\pi}\int_{-\infty}^{\infty} d\omega\, m\,\hbar\omega \coth\left(\frac{\hbar\omega}{2k_BT}\right) Re[K(\omega)]\ e^{-i\omega(t-t')}$$

Urbashi Satpathi and Supurna Sinha, Physica A **506,** 692 (2018).

Quantum Brownian motion in a magnetic field

Let $\vec{B} = (0, 0, B)$, then,

$$\ddot{x}(t) = -\int_{-\infty}^{t} K(t-t')\dot{x}(t')dt' + \omega_c \dot{y}(t) + \frac{F_x(t)}{m} \qquad (2a)$$

$$\ddot{y}(t) = -\int_{-\infty}^{t} K(t-t')\dot{y}(t')dt' - \omega_c \dot{x}(t) + \frac{F_y(t)}{m} \qquad (2b)$$

$\omega_c = \dfrac{qB}{mc}$ is the cyclotron frequency

$$x(\omega) = \frac{1}{m}\frac{i\omega_c F_y(\omega) - \big(\omega - iK(\omega)\big)F_x(\omega)}{\omega[\omega^2 - {\omega_c}^2 - K(\omega)^2 - 2i\omega K(\omega)]} \qquad (3a)$$

$$y(\omega) = \frac{1}{m}\frac{-i\omega_c F_x(\omega) - \big(\omega - iK(\omega)\big)F_y(\omega)}{\omega[\omega^2 - {\omega_c}^2 - K(\omega)^2 - 2i\omega K(\omega)]} \qquad (3b)$$

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Mean square displacement

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$$C_x(t) = \frac{1}{2}\langle\{x(t), x(0)\}\rangle, \qquad C_y(t) = \frac{1}{2}\langle\{y(t), y(0)\}\rangle$$

$$C_x(t) = C_y(t) = \frac{\hbar}{2\pi m}\int_{-\infty}^{\infty} d\omega \frac{Re[K(\omega)] \coth\left(\frac{\hbar\omega}{2k_B T}\right) [(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2] \; e^{-i\omega t}}{\omega\{[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]^2 - [2\omega_c(\omega + Im[K(\omega)])]^2\}}$$

$$\langle \Delta x^2 \rangle = 2\big(C_x(0) - C_x(t)\big)$$

$$\langle \Delta x^2 \rangle = \langle \Delta y^2 \rangle = \frac{\hbar}{\pi m}\int_{-\infty}^{\infty} d\omega \frac{Re[K(\omega)] \coth\left(\frac{\hbar\omega}{2k_B T}\right) [(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2] \; (1 - e^{-i\omega t})}{\omega\{[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]^2 - [2\omega_c(\omega + Im[K(\omega)])]^2\}}$$

$$\boxed{\langle \Delta r^2 \rangle = \frac{2\hbar}{\pi m}\int_{-\infty}^{\infty} d\omega \frac{Re[K(\omega)] \coth\left(\frac{\hbar\omega}{2k_B T}\right) [(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2] \; (1 - e^{-i\omega t})}{\omega\{[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]^2 - [2\omega_c(\omega + Im[K(\omega)])]^2\}}}$$

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Ohmic bath

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$$K(t) = 2\gamma\Theta(t)\delta(t)$$

$$K(\omega) = \gamma, Re[K(\omega)] = \gamma, Im[K(\omega)] = 0$$

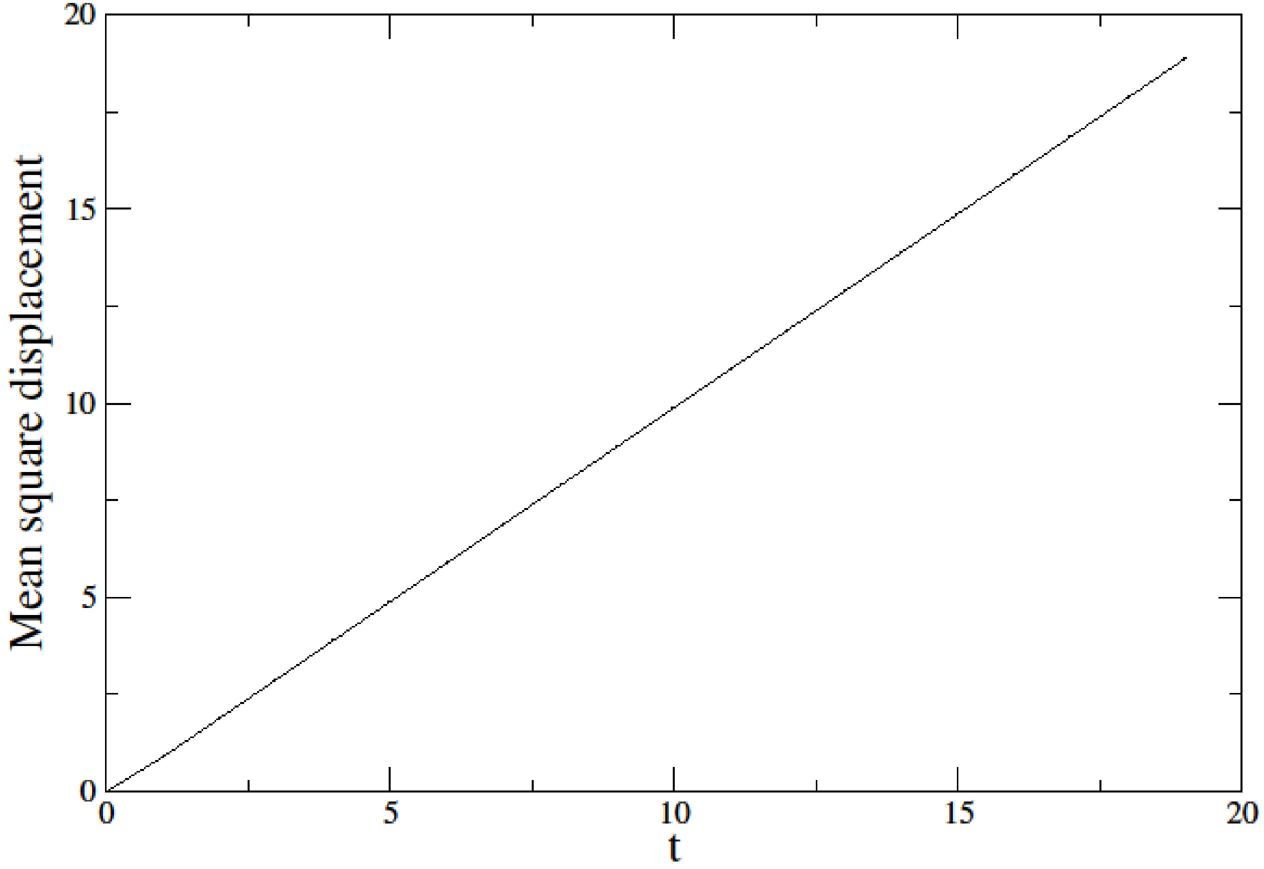
This is memoryless kernel

$$\langle \Delta r^2 \rangle = \frac{2\hbar}{\pi m}\int_{-\infty}^{\infty} d\omega \frac{Re[K(\omega)]\coth\left(\frac{\hbar\omega}{2k_BT}\right)[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]\ \ (1 - e^{-i\omega t})}{\omega\{[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]^2 - [2\omega_c(\omega + Im[K(\omega)])]^2\}}$$

In the high temperature limit:

$$\langle \Delta r^2 \rangle = \frac{4k_BT}{m}\left\{\frac{\gamma t}{\gamma^2 + \omega_c^2} - \frac{\gamma^2 - \omega_c^2}{(\gamma^2 + \omega_c^2)^2} + \frac{\gamma^2 - \omega_c^2}{(\gamma^2 + \omega_c^2)^2}\cos(\omega_c t)e^{-\gamma t} + \frac{2\gamma\omega_c}{(\gamma^2 + \omega_c^2)^2}\sin(\omega_c t)e^{-\gamma t}\right\}$$

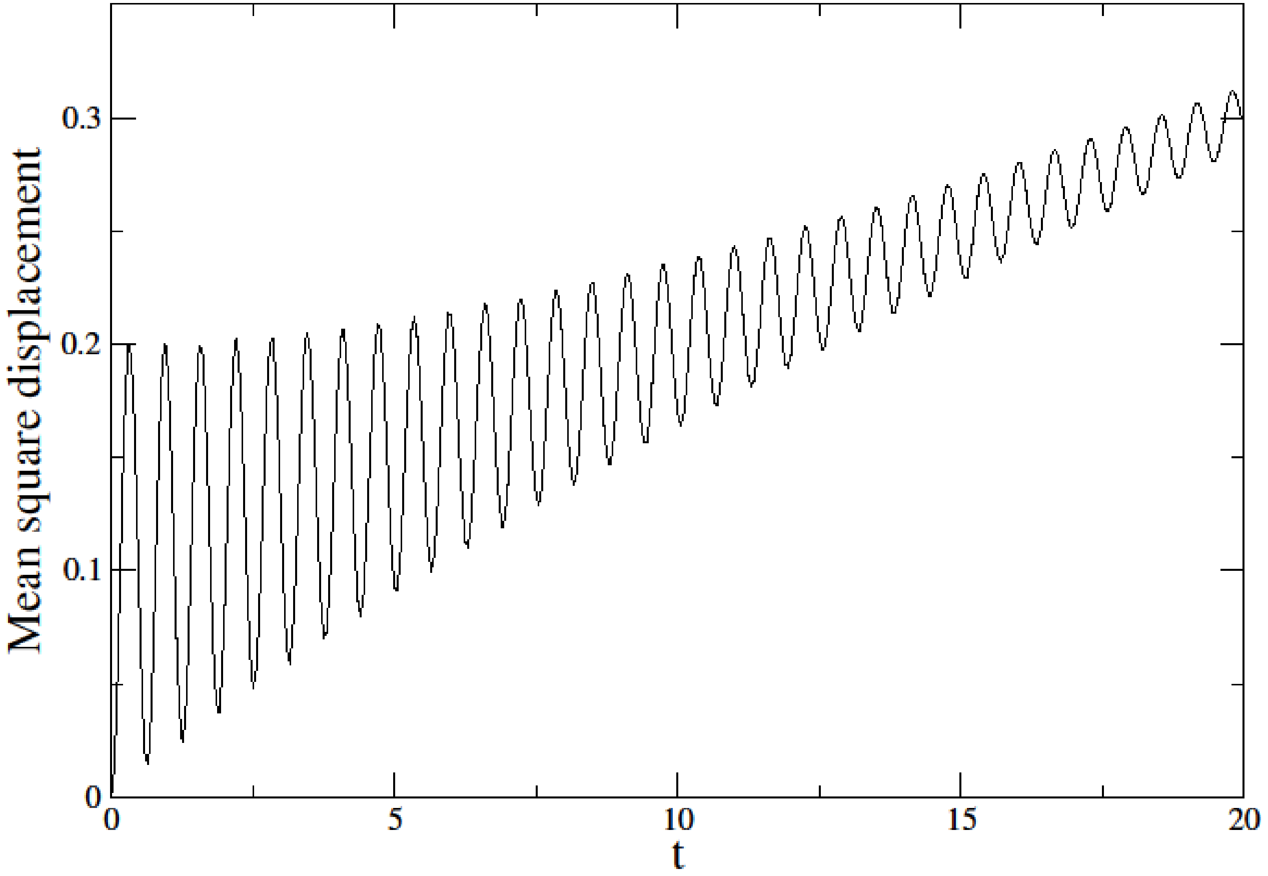
Mean square displacement



High temperature
$\gamma \gg \omega_c$

Urbashi Satpathi and Supurna Sinha, Physica A **506,** 692 (2018).

Mean square displacement



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$$K(t) = 2\gamma\Theta(t)\delta(t)$$

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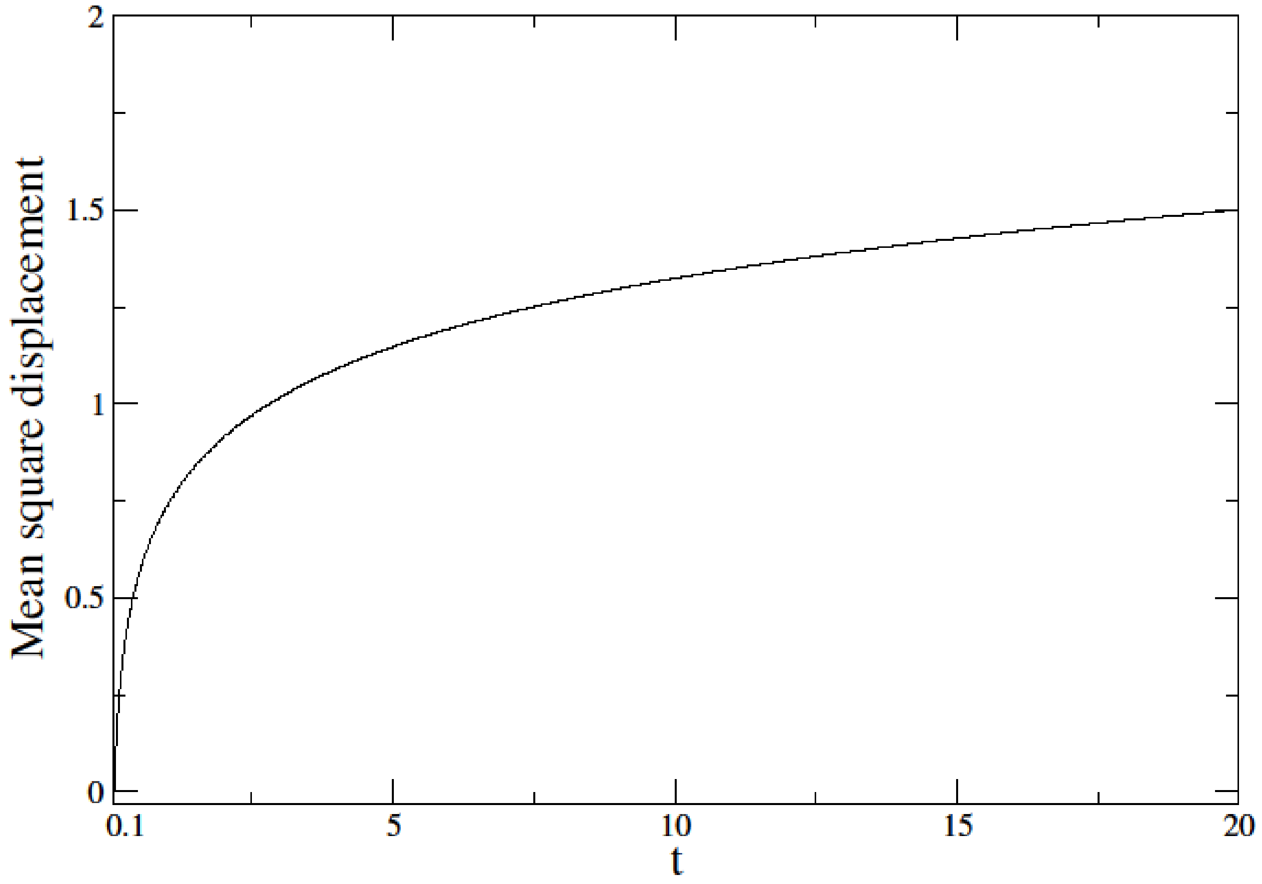
$$\boxed{\langle \Delta r^2 \rangle = \frac{2\hbar}{\pi m}\int_{-\infty}^{\infty} d\omega \frac{Re[K(\omega)] \coth\left(\frac{\hbar\omega}{2k_B T}\right)[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]\ (1 - e^{-i\omega t})}{\omega\{[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]^2 - [2\omega_c(\omega + Im[K(\omega)])]^2\}}}$$

In the low temperature limit:

$$\boxed{\langle \Delta r^2 \rangle = \frac{2\gamma\hbar}{m\pi(\gamma^2 + \omega_c^2)}\left\{2\ln\left(t\sqrt{\gamma^2 + \omega_c^2}\right) + 2\gamma_0 + \frac{\pi\omega_c}{\gamma} - \pi e^{-\gamma t}\left[\frac{\omega_c}{\gamma}\cos(\omega_c t) + \sin(\omega_c t)\right]\right\}}$$

Mean square displacement



Low temperature
$\gamma \gg \omega_c$

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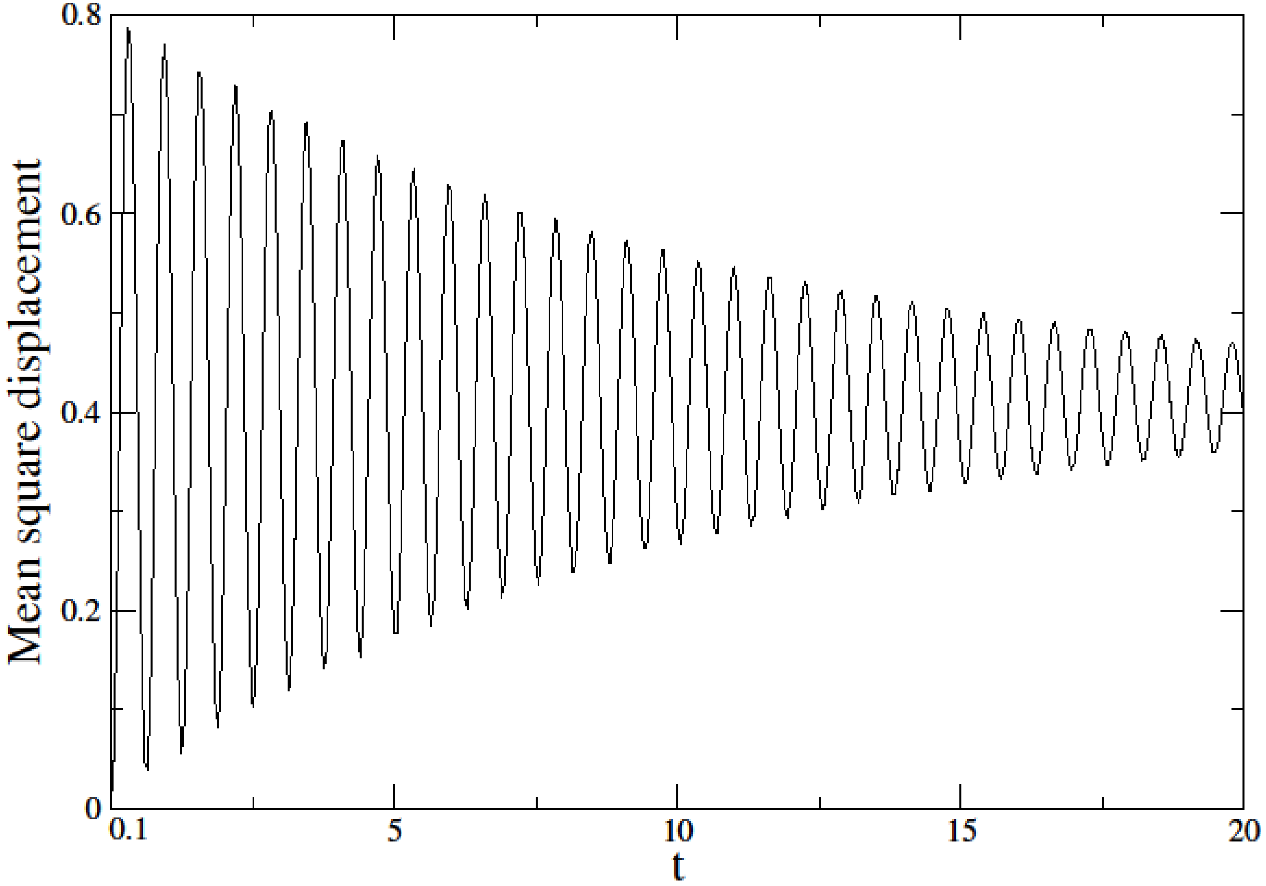
Mean square displacement



Low temperature
$\omega_c \gg \gamma$

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Single relaxation time bath

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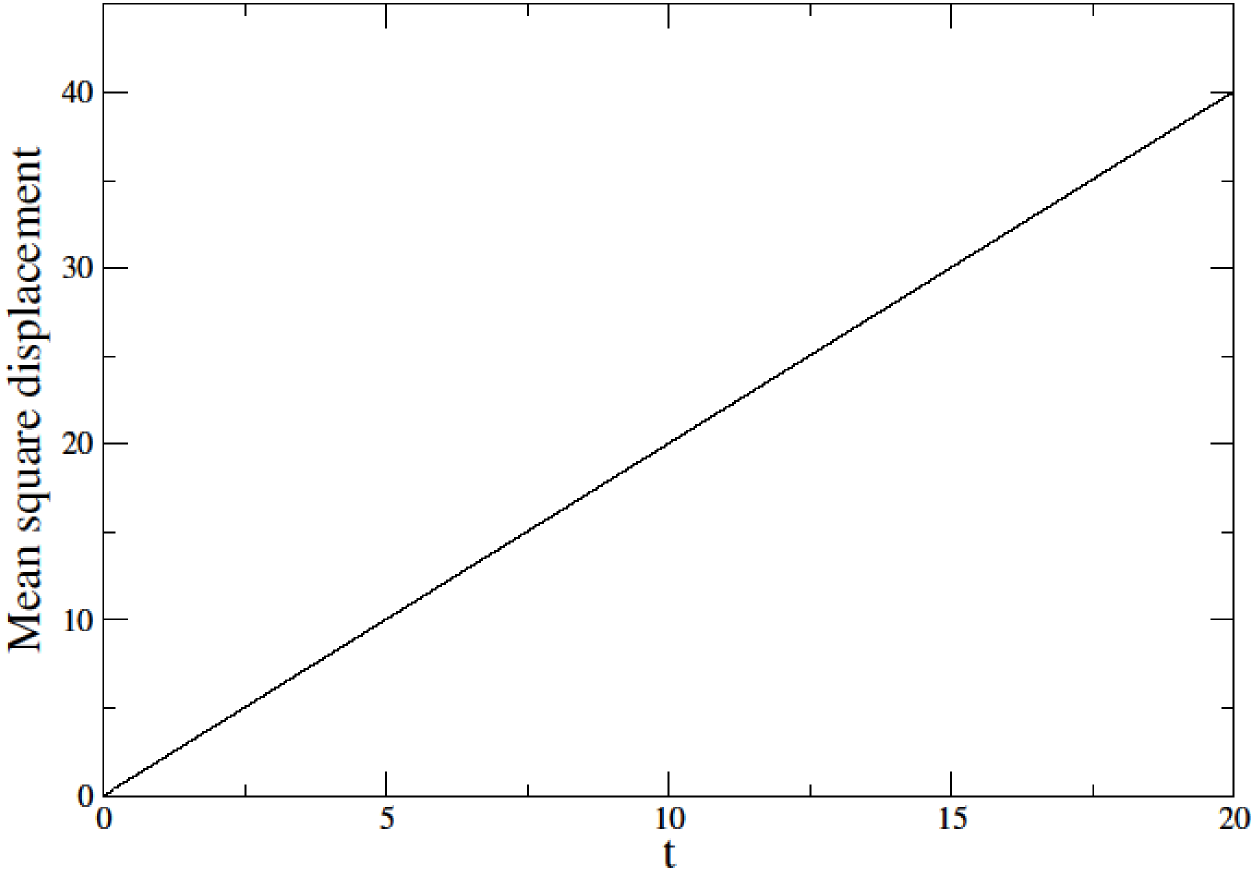
$$K(t) = \frac{\gamma}{\tau}\Theta(t)e^{-\frac{t}{\tau}}$$

$$K(\omega) = \frac{\gamma}{1 - i\omega\tau}, Re[K(\omega)] = \frac{\gamma}{1 + \omega^2\tau^2}, Im[K(\omega)] = \frac{\omega\tau\gamma}{1 + \omega^2\tau^2}$$

This kernel has inbuild memory through relaxation time

$$\boxed{\langle \Delta r^2 \rangle = \frac{2\hbar}{\pi m}\int_{-\infty}^{\infty} d\omega \frac{Re[K(\omega)] \coth\left(\frac{\hbar\omega}{2k_B T}\right)[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]\ (1 - e^{-i\omega t})}{\omega\{[(\omega + Im[K(\omega)])^2 + \omega_c^2 + Re[K(\omega)]^2]^2 - [2\omega_c(\omega + Im[K(\omega)])]^2\}}}$$

Mean square displacement



High temperature
$\gamma \gg \omega_c$

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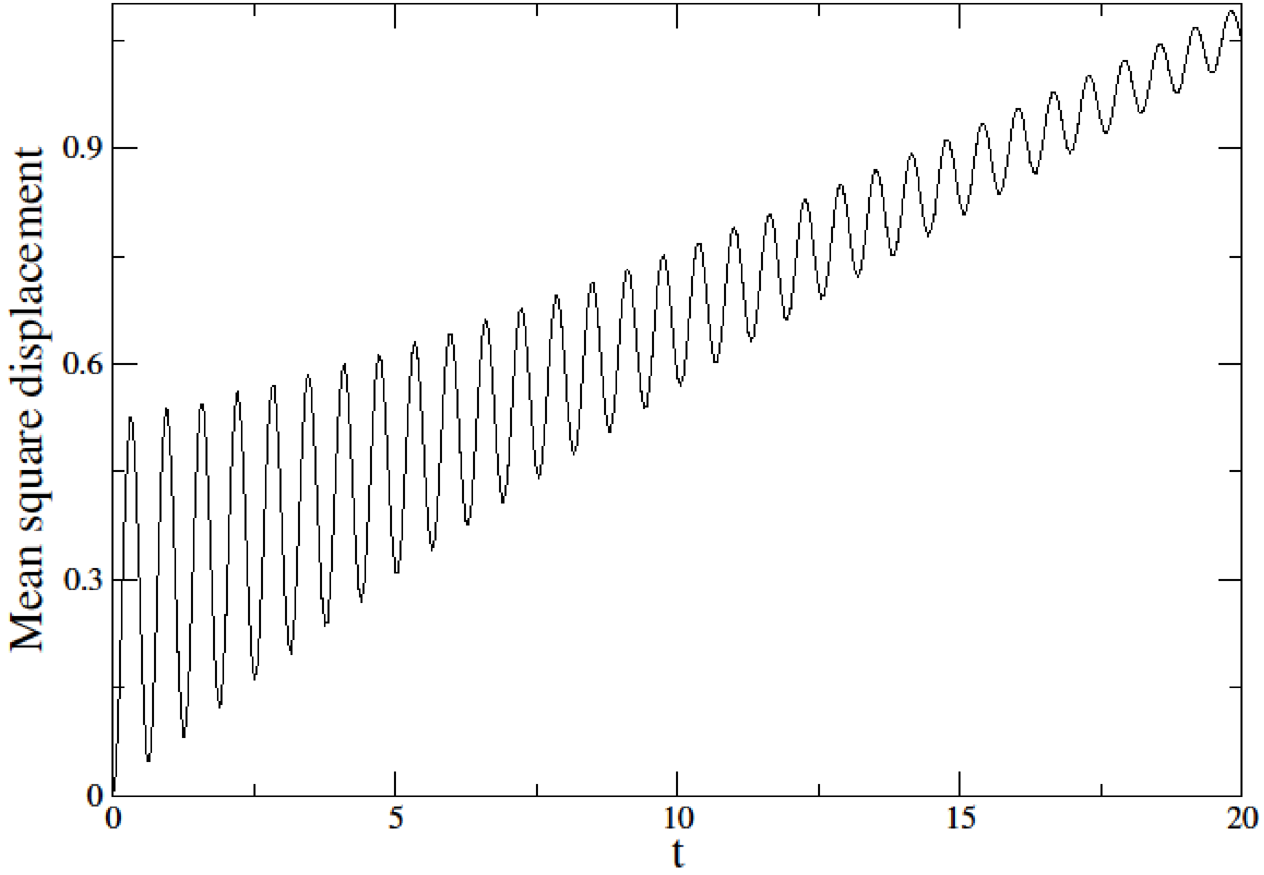
Mean square displacement



High temperature
$\omega_c \gg \gamma$

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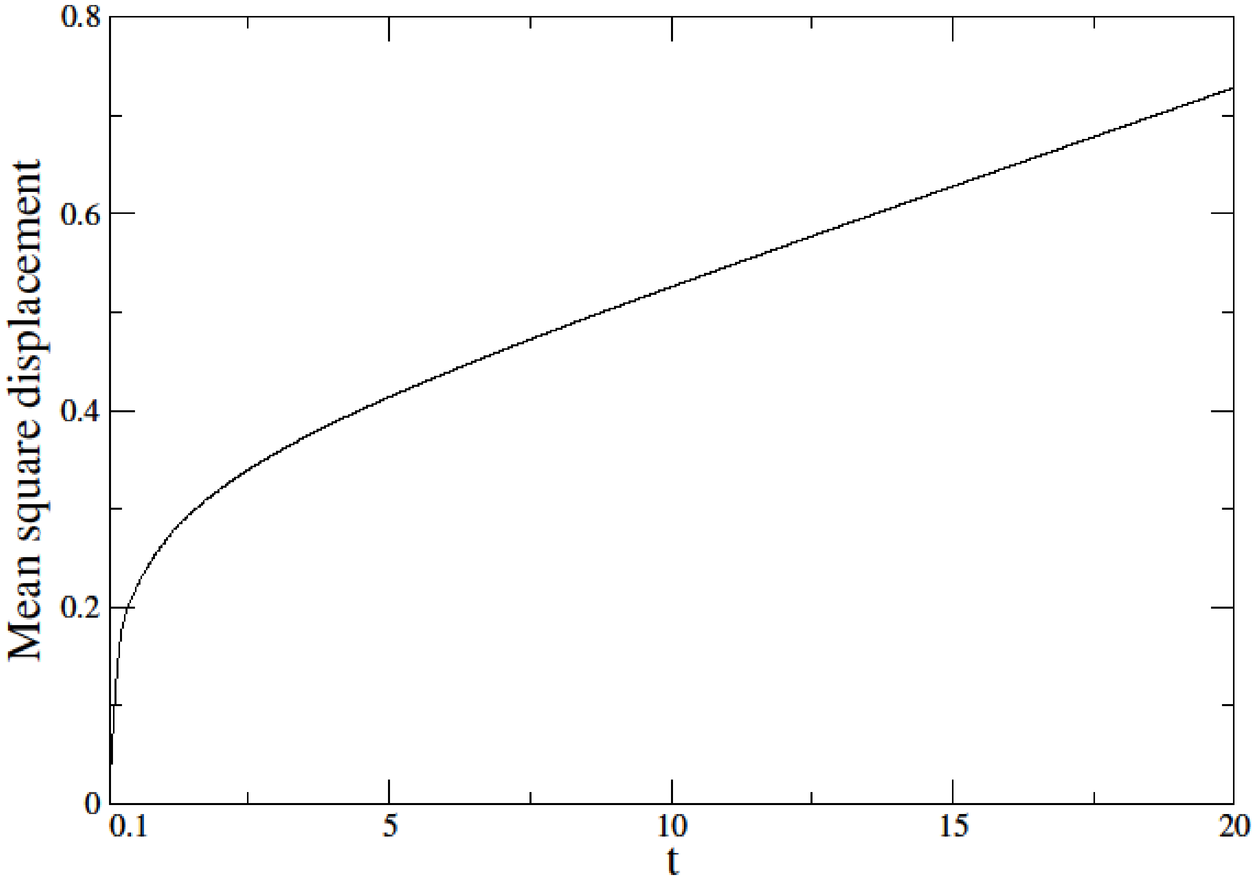
Mean square displacement



Low temperature
$\gamma \gg \omega_c$

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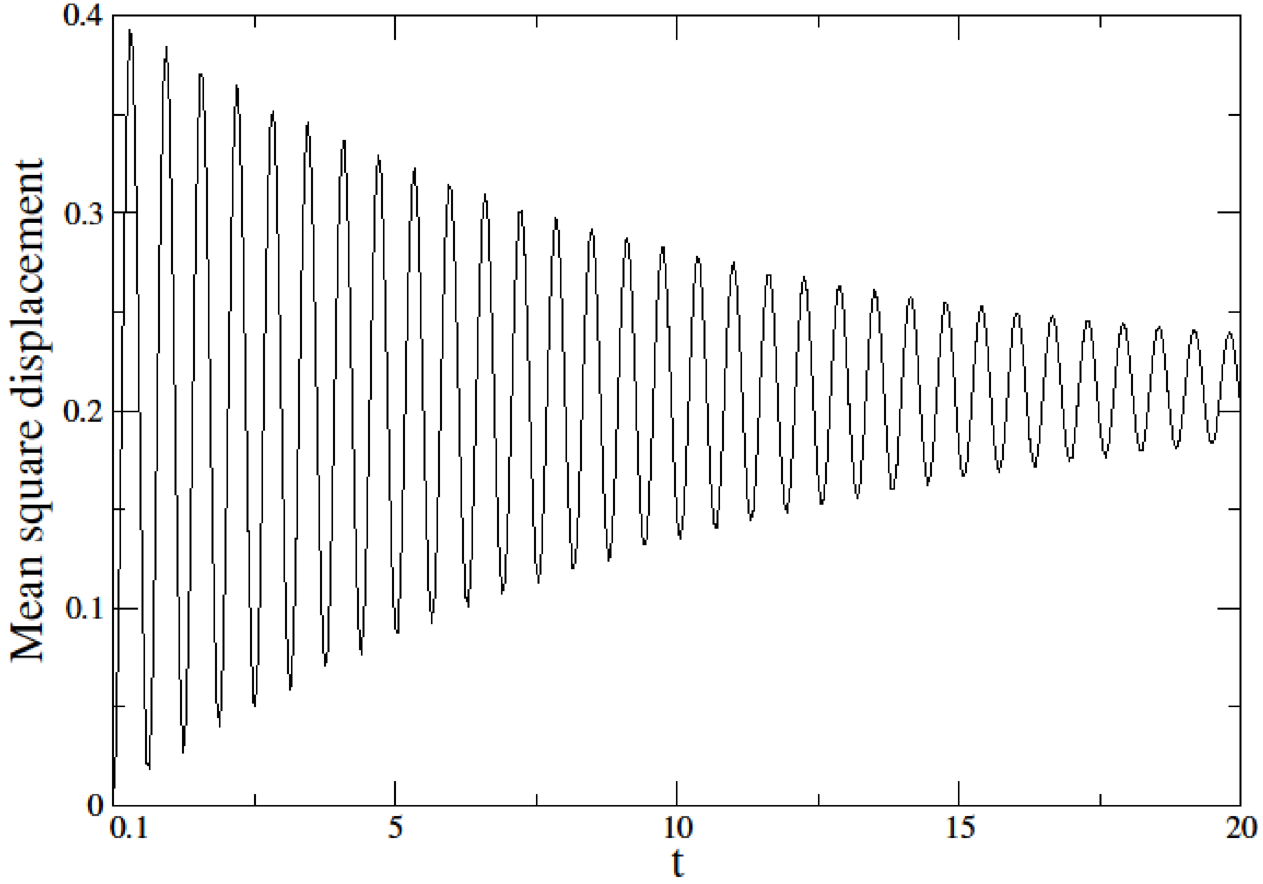
Mean square displacement



Low temperature
$\omega_c \gg \gamma$

Urbashi Satpathi and Supurna Sinha, Physica A **506,** 692 (2018).

Conclusions

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- In the magnetic field dominated regime for both classical and quantum cases, oscillatory behaviour is observed.
- A transition from a monotonic behaviour to a damped oscillatory behaviour, as the magnetic field strength is increased.
- Qualitative features of the mean square displacement are essentially the same for an ohmic dissipation model and a single relaxation time model for the memory kernel.

Future interests

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- ✓ The time dependent orbital diamagnetic moment of a charged particle in a magnetic field in a viscous medium via the Quantum Langevin Equation.
- ✓ Check the Bohr-Van Leeuwen theorem.
- ✓ Check the competition between different time scales.

Thank you

$$\langle \Delta r^2 \rangle = \frac{\hbar}{\pi m(\gamma^2+\omega_c^2)}\left\{(\gamma+i\omega_c)\left(H_{\frac{\gamma-i\omega_c}{\pi\Omega_{th}}}+H_{-\frac{\gamma-i\omega_c}{\pi\Omega_{th}}}\right)+(\gamma-i\omega_c)\left(H_{\frac{\gamma+i\omega_c}{\pi\Omega_{th}}}+H_{-\frac{\gamma+i\omega_c}{\pi\Omega_{th}}}\right)+\right\}$$

$$\begin{aligned}
\langle \Delta \mathrm{r}^2 \rangle \; = \; & \frac{\hbar}{\pi m\left(\gamma^2+\omega_c^2\right)}\left\{(\gamma+i\omega_c)\left[H_{\frac{\gamma-i\omega_C}{\pi\Omega_{th}}}+H_{-\frac{\gamma-i\omega_C}{\pi\Omega_{th}}}\right]+(\gamma-i\omega_c)\left[H_{-\frac{\gamma+i\omega_C}{\pi\Omega_{th}}}+H_{\frac{\gamma+i\omega_C}{\pi\Omega_{th}}}\right]\right. \\
& + e^{-\pi t\Omega_{th}}(\gamma+i\omega_c)\left[\Phi\left(e^{-\pi t\Omega_{th}},1,\frac{\gamma+\pi\Omega_{th}-i\omega_c}{\pi\Omega_{th}}\right)+\Phi\left(e^{-\pi t\Omega_{th}},1,\frac{-\gamma+\pi\Omega_{th}+i\omega_c}{\pi\Omega_{th}}\right)\right] \\
& + e^{-\pi t\Omega_{th}}(\gamma-i\omega_c)\left[\Phi\left(e^{-\pi t\Omega_{th}},1,\frac{\gamma+\pi\Omega_{th}+i\omega_c}{\pi\Omega_{th}}\right)+\Phi\left(e^{-\pi t\Omega_{th}},1,-\frac{\gamma-\pi\Omega_{th}+i\omega_c}{\pi\Omega_{th}}\right)\right] \\
& + 2\gamma\left[\pi t\Omega_{th}+2\ln\left(1-e^{-\pi t\Omega_{th}}\right)\right]+\pi(i\gamma+\omega_c)\left(1-e^{-t(\gamma+i\omega_C)}\right)\coth\left(\frac{\omega_c-i\gamma}{\Omega_{th}}\right) \\
& \left. + \; \pi(-i\gamma+\omega_c)\left(1-e^{-\gamma t+it\omega_c}\right)\coth\left(\frac{\omega_c+i\gamma}{\Omega_{th}}\right)\right\}
\end{aligned}$$

Linear response theory

$H = H_0 + H^F(t)$ where, $H^F(t) = -Af(t)$ is the perturbation.

$$\frac{d}{dt}\rho'(t) = \frac{1}{i\hbar}[H, \rho'(t)]$$

$\rho'(-\infty) = \rho$ is the initial condition.

$f(-\infty) = 0.$

$$\rho'(t) = \rho + \Delta\rho(t)$$

$$\Delta\rho(t) = \int_{-\infty}^{t} dt' e^{-i(t-t')H/\hbar}[A, \rho]e^{i(t-t')H/\hbar} f(t')$$

Ryogo Kubo, J. Phys. Soc. Jpn. **12**, pp. 570-586 (1957).

$$\tilde{R}(\omega) = \int_0^\infty dt\, e^{i\omega t} R(t)$$

$$= \frac{1}{i\hbar}\int_0^\infty dt\, e^{i\omega t}\; Tr\, \rho[A, B(t)]$$

$$= \frac{1}{i\hbar}\frac{1}{Z}\int_0^\infty dt\, e^{i\omega t} \sum_n \sum_m e^{-\beta E_n} \begin{pmatrix} A_{nm} e^{i(E_m - E_n)\frac{t}{\hbar}} B_{mn} \\ -e^{i(E_n - E_m)\frac{t}{\hbar}} B_{nm} A_{mn} \end{pmatrix}$$

$$= \frac{1}{i\hbar}\frac{1}{Z}\sum_n \sum_m e^{-\beta E_n}\left(1 - e^{\beta\hbar\omega_{nm}}\right) \int_0^\infty dt\;\; e^{i\omega t} e^{-it\omega_{nm}} A_{nm} B_{mn}$$

$$= \frac{1}{i\hbar}\frac{1}{Z}\sum_n \sum_m e^{-\beta E_n}\left(1 - e^{\beta\hbar\omega}\right) \pi\delta(\omega - \omega_{nm}) A_{nm} B_{mn}$$

$$\tilde{C}(\omega) = \int_{-\infty}^\infty dt\, e^{i\omega t} C(t)$$

$$= \frac{1}{2}\int_{-\infty}^\infty dt\, e^{i\omega t}\; Tr\, \rho\{A, B(t)\}$$

$$= \frac{1}{2}\frac{1}{Z}\int_{-\infty}^\infty dt\, e^{i\omega t} \sum_n \sum_m e^{-\beta E_n} \begin{pmatrix} A_{nm} e^{i(E_m - E_n)\frac{t}{\hbar}} B_{mn} \\ +e^{i(E_n - E_m)\frac{t}{\hbar}} B_{nm} A_{mn} \end{pmatrix}$$

$$= \frac{1}{2}\frac{1}{Z}\sum_n \sum_m e^{-\beta E_n}\left(1 + e^{\beta\hbar\omega_{nm}}\right) \int_{-\infty}^\infty dt\;\; e^{i\omega t} e^{-it\omega_{nm}} A_{nm} B_{mn}$$

$$= \frac{1}{Z}\sum_n \sum_m e^{-\beta E_n}\left(1 + e^{\beta\hbar\omega}\right) \pi\delta(\omega - \omega_{nm}) A_{nm} B_{mn}$$

$$\frac{Im\left(\tilde{R}(\omega)\right)}{\tilde{C}(\omega)} = -\frac{1}{\hbar}\frac{\left(1 - e^{\beta\hbar\omega}\right)}{\left(1 + e^{\beta\hbar\omega}\right)} = \frac{1}{\hbar}\tanh\left(\frac{\beta\hbar\omega}{2}\right)$$

$$\tilde{C}(\nu) = \hbar\coth(\pi\beta\hbar\nu) Im\, \tilde{R}(\nu)$$

R. Balescu, Equilibrium and Non-Equilibrium Statistical Mechanics (John Wiley & Sons, 1975) pp 663-669.